String cobordism at the prime 3

Vitaly Lorman
Swarthmore College

Carl McTague
Boston College

Doug Ravenel
University of Rochester

UCLA Topology Seminar
May 17, 2021
String cobordism or $MString$ is Haynes Miller’s name for the spectrum also known as $MO\langle 8 \rangle$. Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971. It is known to admit a map to $tmf$ (the spectrum for topological modular forms) that is surjective in mod 2 homology. At each prime larger than 3, it is known to split as a wedge of suspensions of the Brown-Peterson spectrum $BP$. There is some subtlety in its multiplicative structure, which is the subject of a 2008 paper by Mark Hovey.
What is string cobordism?

String cobordism or $MString$ is Haynes Miller’s name for the spectrum also known as $MO\langle 8 \rangle$, the Thom spectrum associated with the $BO\langle 8 \rangle$, the 7-connected cover of the space $BO$. Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971. It is known to admit a map to $tmf$ (the spectrum for topological modular forms) that is surjective in mod 2 homology. At each prime larger than 3, it is known to split as a wedge of suspensions of the Brown-Peterson spectrum $BP$. There is some subtlety in its multiplicative structure, which is the subject of a 2008 paper by Mark Hovey.
String cobordism or $\textit{MString}$ is Haynes Miller’s name for the spectrum also known as $\textit{MO}\langle 8 \rangle$, the Thom spectrum associated with the $\textit{BO}\langle 8 \rangle$, the 7-connected cover of the space $\textit{BO}$.

Its homotopy type at the prime 2 is quite complicated and still not fully understood.
What is string cobordism?

String cobordism or $MString$ is Haynes Miller’s name for the spectrum also known as $MO\langle 8 \rangle$, the Thom spectrum associated with the $BO\langle 8 \rangle$, the 7-connected cover of the space $BO$.

Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971.
What is string cobordism?

String cobordism or $\text{MString}$ is Haynes Miller’s name for the spectrum also known as $\text{MO}(8)$, the Thom spectrum associated with the $\text{BO}(8)$, the 7-connected cover of the space $\text{BO}$.

Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971. It is known to admit a map to $\text{tmf}$ (the spectrum for topological modular forms) that is surjective in mod 2 homology.
What is string cobordism?

String cobordism or $MString$ is Haynes Miller’s name for the spectrum also known as $MO\langle 8 \rangle$, the Thom spectrum associated with the $BO\langle 8 \rangle$, the 7-connected cover of the space $BO$.

Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971. It is known to admit a map to $tmf$ (the spectrum for topological modular forms) that is surjective in mod 2 homology.

At each prime larger than 3, it is known to split as a wedge of suspensions of the Brown-Peterson spectrum $BP$. 

String cobordism at the prime 3

Carl McTague
Vitaly Lorman
Doug Ravenel

Introduction
$MSU$ at $p = 2$
Wilson spaces and Hopf rings
$H_* BO\langle 8 \rangle$ and $H_* MO\langle 8 \rangle$
Two change of rings isomorphisms
The Adams spectral sequence for $MO\langle 8 \rangle$
String cobordism or $MString$ is Haynes Miller’s name for the spectrum also known as $MO\langle 8 \rangle$, the Thom spectrum associated with the $BO\langle 8 \rangle$, the 7-connected cover of the space $BO$.

Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971. It is known to admit a map to $tmf$ (the spectrum for topological modular forms) that is surjective in mod 2 homology.

At each prime larger than 3, it is known to split as a wedge of suspensions of the Brown-Peterson spectrum $BP$. There is some subtlety in its multiplicative structure,
String cobordism or $MString$ is Haynes Miller’s name for the spectrum also known as $MO\langle 8 \rangle$, the Thom spectrum associated with the $BO\langle 8 \rangle$, the 7-connected cover of the space $BO$.

Its homotopy type at the prime 2 is quite complicated and still not fully understood. It was first studied by Vince Giambalvo in 1971. It is known to admit a map to $tmf$ (the spectrum for topological modular forms) that is surjective in mod 2 homology.

At each prime larger than 3, it is known to split as a wedge of suspensions of the Brown-Peterson spectrum $BP$. There is some subtlety in its multiplicative structure, which is the subject of a 2008 paper by Mark Hovey.
What is string cobordism? (continued)

Our goal is to study $MO\langle 8 \rangle$ at the prime 3.
Our goal is to study \( MO\langle 8 \rangle \) at the prime 3. This is the sweet spot in that its homotopy type is both interesting and accessible.
Our goal is to study $MO\langle 8 \rangle$ at the prime 3. This is the sweet spot in that its homotopy type is both interesting and accessible. It is the subject of a 1995 paper by Hovey and the third author.

Abstract. In this paper, we study the cobordism spectrum $MO\langle 8 \rangle$ at the prime 3. This spectrum is important because it is conjectured to play the role for elliptic cohomology that Spin cobordism plays for real $K$-theory. We show that the torsion is all killed by 3, and that the Adams-Novikov spectral sequence collapses after only 2 differentials. Many of our methods apply more generally.
Some informative history

It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2.
Some informative history

It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2. $MSO$ is the subject of 1960 paper by Terry Wall.
Some informative history

It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2. $MSO$ is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra $A_\ast$, $H_\ast MSO$ splits as a direct sum of suspensions of two types:
It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2. $MSO$ is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra $A_*$, $H_\ast MSO$ splits as a direct sum of suspensions of two types:

- $A_* = P(\zeta_1, \zeta_2, \ldots)$ with $\zeta_i = 2^i - 1$. 

It is useful to compare this problem with the study of \( MSO \) (oriented cobordism) and \( MSU \) (special unitary cobordism) at the prime 2. \( MSO \) is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra \( A_* \), \( H_*MSO \) splits as a direct sum of suspensions of two types:

- \( A_* = P(\zeta_1, \zeta_2, \ldots) \) with \( \zeta_i = 2^i - 1 \). This is the homology of the mod 2 Eilenberg-Mac Lane spectrum \( H\mathbb{Z}/2 \).
It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2. $MSO$ is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra $A_*$, $H_*MSO$ splits as a direct sum of suspensions of two types:

- $A_* = P(\zeta_1, \zeta_2, \ldots)$ with $\zeta_i = 2^i - 1$. This is the homology of the mod 2 Eilenberg-Mac Lane spectrum $HZ/2$.
- $(A//A(0))_* = P(\zeta_1^2, \zeta_2, \zeta_3, \ldots)$. 

Some informative history

- $MSU$ at $p = 2$
- Wilson spaces and Hopf rings
- $H_*BO(8)$ and $H_*MO(8)$
- Two change of rings isomorphisms
- The Adams spectral sequence for $MO(8)$
Some informative history

It is useful to compare this problem with the study of \( MSO \) (oriented cobordism) and \( MSU \) (special unitary cobordism) at the prime 2. \( MSO \) is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra \( \mathcal{A}_* \), \( H_* MSO \) splits as a direct sum of suspensions of two types:

- \( \mathcal{A}_* = P(\zeta_1, \zeta_2, \ldots) \) with \( \zeta_i = 2^i - 1 \). This is the homology of the mod 2 Eilenberg-Mac Lane spectrum \( H\mathbb{Z}/2 \).
- \( (\mathcal{A}/\mathcal{A}(0))_* = P(\zeta_1^2, \zeta_2, \zeta_3, \ldots) \). This is the homology of the integer Eilenberg-Mac Lane spectrum \( H\mathbb{Z} \).
It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2. $MSO$ is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra $\mathcal{A}_*$, $H_*MSO$ splits as a direct sum of suspensions of two types:

- $\mathcal{A}_* = P(\zeta_1, \zeta_2, \ldots)$ with $\zeta_i = 2^i - 1$. This is the homology of the mod 2 Eilenberg-Mac Lane spectrum $HZ/2$.

- $(\mathcal{A}/\mathcal{A}(0))_* = P(\zeta_1^2, \zeta_2, \zeta_3, \ldots)$. This is the homology of the integer Eilenberg-Mac Lane spectrum $HZ$. There is one such summand for each monomial in the graded ring $P(x_4, x_8, x_{12}, \ldots)$. 
It is useful to compare this problem with the study of $MSO$ (oriented cobordism) and $MSU$ (special unitary cobordism) at the prime 2. $MSO$ is the subject of 1960 paper by Terry Wall.

As a comodule over the dual Steenrod algebra $A_*$, $H_* MSO$ splits as a direct sum of suspensions of two types:

- $A_* = P(\zeta_1, \zeta_2, \ldots)$ with $\zeta_i = 2^i - 1$. This is the homology of the mod 2 Eilenberg-Mac Lane spectrum $HZ/2$.
- $(A//A(0))_* = P(\zeta_1^2, \zeta_2, \zeta_3, \ldots)$. This is the homology of the integer Eilenberg-Mac Lane spectrum $HZ$. There is one such summand for each monomial in the graded ring $P(x_4, x_8, x_{12}, \ldots)$.

There is a corresponding splitting of the spectrum $MSO_{(2)}$ into a wedge of integer and mod 2 Eilenberg-Mac Lane spectra. The Adams spectral sequence for $MSO$ collapses from $E_2$. 
Some informative history: \textit{MSU} at the prime 2


\(H_\ast \text{MSU}\) is the “double” of \(H_\ast \text{MSO}\).

$H_*\ MSU$ is the “double” of $H_*\ MSO$. This means that as a comodule over the dual mod 2 Steenrod algebra $A_*$, $H_*\ MSO$ splits as a direct sum of suspensions of two types:

$H_\ast MSU$ is the “double” of $H_\ast MSO$. This means that as a comodule over the dual mod 2 Steenrod algebra $A_\ast$, $H_\ast MSO$ splits as a direct sum of suspensions of two types:

- The double of $A_\ast$, $P(\zeta_1^2, \zeta_2^2, \ldots)$ with $\zeta_i = 2^i - 1$.

Some informative history: $MSU$ at the prime 2
Some informative history: $MSU$ at the prime 2


$H_* MSU$ is the “double” of $H_* MSO$. This means that as a comodule over the dual mod 2 Steenrod algebra $A_*$, $H_* MSO$ splits as a direct sum of suspensions of two types:

- The double of $A_*, P(\zeta_1^2, \zeta_2^2, \ldots)$ with $\zeta_i = 2^i - 1$. This is the homology of the spectrum $BP$.
The 2-primary homotopy type of $\text{MSU}$ is the subject of David Pengelley’s thesis, published in 1982.

$H_* \text{MSU}$ is the “double” of $H_* \text{MSO}$. This means that as a comodule over the dual mod 2 Steenrod algebra $A_*$, $H_* \text{MSO}$ splits as a direct sum of suspensions of two types:

- The double of $A_*$, $P(\zeta_1^2, \zeta_2^2, \ldots)$ with $\zeta_i = 2^i - 1$. This is the homology of the spectrum $BP$.
- The double of $(A//A(0))_*$, $P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots)$.
Some informative history: *MSU* at the prime 2

The 2-primary homotopy type of *MSU* is the subject of David Pengelley’s thesis, published in 1982.

\( H_* MSU \) is the “double” of \( H_* MSO \). This means that as a comodule over the dual mod 2 Steenrod algebra \( \mathcal{A}_* \), \( H_* MSO \) splits as a direct sum of suspensions of two types:

- The double of \( \mathcal{A}_*, P(\zeta_1^2, \zeta_2^2, \ldots) \) with \( \zeta_i = 2^i - 1 \). This is the homology of the spectrum \( BP \).
- The double of \( (\mathcal{A}/\mathcal{A}(0))_*, P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \). You might think this is the homology of a new spectrum \( X \).
Some informative history: \textit{MSU} at the prime 2


\(H_\ast MSU\) is the “double” of \(H_\ast MSO\). This means that as a comodule over the dual mod 2 Steenrod algebra \(\mathcal{A}_\ast\), \(H_\ast MSO\) splits as a direct sum of suspensions of two types:

- The double of \(\mathcal{A}_\ast\), \(P(\zeta_1^2, \zeta_2^2, \ldots)\) with \(\zeta_i = 2^i - 1\). This is the homology of the spectrum \(BP\).

- The double of \((\mathcal{A}/\mathcal{A}(0))_\ast\), \(P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots)\). You might think this is the homology of a new spectrum \(X\). There is one such summand for each monomial in the graded ring \(P(y_8, y_{16}, y_{24}, \ldots)\).
Some informative history: \textit{MSU} (continued)

It is easy to work out the Adams spectral sequence for the hypothetical spectrum $X$ with

$$H_* X = P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots).$$
Some informative history: \textit{MSU} (continued)

It is easy to work out the Adams spectral sequence for the hypothetical spectrum \( X \) with

\[ H_{\ast}X = P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots). \]

We find that

\[ \pi_{\ast}X \cong \pi_{\ast}bo \otimes P(v_2, v_3, \ldots), \]
It is easy to work out the Adams spectral sequence for the hypothetical spectrum $X$ with

$$H_* X = P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots).$$

We find that

$$\pi_* X \cong \pi_* bo \otimes P(v_2, v_3, \ldots),$$

where $v_n \in \pi_{2(2^n - 1)}$ (in Adams filtration 1) is related to the generator of $\pi_* BP$ of the same name.
It is easy to work out the Adams spectral sequence for the hypothetical spectrum $X$ with

$$H_* X = P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots).$$

We find that

$$\pi_* X \cong \pi_* bo \otimes P(v_2, v_3, \ldots),$$

where $v_n \in \pi_{2(2^n-1)}$ (in Adams filtration 1) is related to the generator of $\pi_* BP$ of the same name. Recall that $\pi_* bo$ has torsion in dimensions congruent to 1 and 2 modulo 8.
Some informative history: \( MSU \) (continued)

Here is the Adams \( E_2 \) page for the hypothetical summand \( X \) of \( MSU_{(2)} \).
Here is the Adams $E_2$ page for the hypothetical summand $X$ of $MSU_{(2)}$. 

Some informative history: $MSU$ (continued)

In 1966 Pierre Conner and Ed Floyd proved that the torsion in $\pi_\ast MSU$ is also confined to dimensions congruent to 1 and 2 modulo 8. This means $\eta v_2$ must be killed by an Adams differential.
Here is the Adams $E_2$ page for the hypothetical summand $X$ of $MSU_{(2)}$.

<table>
<thead>
<tr>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
</table>

In 1966 Pierre Conner and Ed Floyd proved that the torsion in $\pi_* MSU$ is also confined to dimensions congruent to 1 and 2 modulo 8.
Some informative history: \textit{MSU} (continued)

Here is the Adams $E_2$ page for the hypothetical summand $X$ of $\text{MSU}_{(2)}$.

In 1966 Pierre Conner and Ed Floyd proved that the torsion in $\pi_*\text{MSU}$ is also confined to dimensions congruent to 1 and 2 modulo 8. This means $\eta v_2$ must be killed by an Adams differential.
Some informative history: \( MSU \) (continued)

We have seen that \( H_* MSU \) has an \( A_* \)-comodule summand isomorphic to

\[
P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes P(y_8, y_{16}, y_{24}, \ldots) \subset H_* MSU.
\]
We have seen that $H_\ast MSU$ has an $A_\ast$-comodule summand isomorphic to

$$P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes P(y_8, y_{16}, y_{24}, \ldots) \subset H_\ast MSU.$$  

The Conner-Floyd theorem leads to Adams differentials

$$d_2(y_{2n+1}) = \eta v_n \quad \text{for } n \geq 2,$$

Some informative history: *MSU* (continued)
We have seen that $H_* MSU$ has an $A_*$-comodule summand isomorphic to

$$P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes P(y_8, y_{16}, y_{24}, \ldots) \subset H_* MSU.$$  

The Conner-Floyd theorem leads to Adams differentials

$$d_2(y_{2n+1}) = \eta v_n \quad \text{for } n \geq 2,$$

which we call Pengelley differentials.
Some informative history: MSU (continued)

We have seen that $H_\ast MSU$ has an $A_\ast$-comodule summand isomorphic to

$$P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes P(y_8, y_{16}, y_{24}, \ldots) \subset H_\ast MSU.$$

The Conner-Floyd theorem leads to Adams differentials

$$d_2(y_{2n+1}) = \eta v_n \quad \text{for } n \geq 2,$$

which we call Pengelley differentials.

This means that MSU does not split as expected into a wedge of suspensions of $X$ and $BP$. 
Some informative history: MSU (continued)

We have seen that $H_*MSU$ has an $A_*$-comodule summand isomorphic to

$$P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes P(y_8, y_{16}, y_{24}, \ldots) \subset H_*MSU.$$ 

The Conner-Floyd theorem leads to Adams differentials

$$d_2(y_{2n+1}) = \eta v_n \quad \text{for } n \geq 2,$$

which we call Pengelley differentials.

This means that MSU does not split as expected into a wedge of suspensions of $X$ and $BP$. Instead of $X$, Pengelley gets a spectrum $BoP$ with an additive $A_*$-comodule isomorphism

$$H_*BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$
Indeed of $X$, Pengelley gets a spectrum $BoP$ with an additive isomorphism

$$H_* BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$
Some informative history: \textit{MSU} (continued)

Instead of \(X\), Pengelley gets a spectrum \(BoP\) with an additive isomorphism

\[ H_* BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots). \]

\(BoP\) is not known to be a ring spectrum,
Instead of $X$, Pengelley gets a spectrum $BoP$ with an additive isomorphism

$$H_* BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$

$BoP$ is not known to be a ring spectrum, but it is known to support a map to $bo$ inducing an isomorphism of torsion in $\pi_*$. 

Some informative history: $MSU$ (continued)
Some informative history: MSU (continued)

Instead of $X$, Pengelley gets a spectrum $BoP$ with an additive isomorphism

$$H_* BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$

$BoP$ is not known to be a ring spectrum, but it is known to support a map to $bo$ inducing an isomorphism of torsion in $\pi_*$.  

Pengelley shows that $MSU_{(2)}$ is equivalent to a wedge of suspensions of $BoP$ and $BP$.  

**String cobordism at the prime 3**

Carl McTague
Vitaly Lorman
Doug Ravenel

Introduction

$MSU$ at $p = 2$

Wilson spaces and Hopf rings
$H_* BO\langle 8 \rangle$ and $H_* MO\langle 8 \rangle$

Two change of rings isomorphisms

The Adams spectral sequence for $MO\langle 8 \rangle$
Some informative history: \textit{MSU} (continued)

Instead of $X$, Pengelley gets a spectrum $BoP$ with an additive isomorphism

$$H_* BoP \cong P(\zeta_4^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$

$BoP$ is not known to be a ring spectrum, but it is known to support a map to $bo$ inducing an isomorphism of torsion in $\pi_*$. Pengelley shows that $MSU_{(2)}$ is equivalent to a wedge of suspensions of $BoP$ and $BP$.

\textbf{Spoiler:} Our goal is to prove a similar statement about $MO \langle 8 \rangle_{(3)}$. 
Some informative history: *MSU* (continued)

Instead of $X$, Pengelley gets a spectrum $BoP$ with an additive isomorphism

$$H_* BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$

$BoP$ is not known to be a ring spectrum, but it is known to support a map to $bo$ inducing an isomorphism of torsion in $\pi_*$. 

Pengelley shows that $MSU(2)$ is equivalent to a wedge of suspensions of $BoP$ and $BP$.

**Spoiler:** Our goal is to prove a similar statement about $MO\langle 8 \rangle(3)$. Our analog of $BoP$ supports a map to $tmf$ (instead of $bo$) inducing an isomorphism of torsion in $\pi_*$. 
Instead of $X$, Pengelley gets a spectrum $BoP$ with an additive isomorphism

$$H_* BoP \cong P(\zeta_1^4, \zeta_2^2, \zeta_3^2, \ldots) \otimes E(y_8, y_{16}, y_{32}, \ldots).$$

$BoP$ is not known to be a ring spectrum, but it is known to support a map to $bo$ inducing an isomorphism of torsion in $\pi_*$. Pengelley shows that $MSU_{(2)}$ is equivalent to a wedge of suspensions of $BoP$ and $BP$.

**Spoiler:** Our goal is to prove a similar statement about $MO_{(3)}$. Our analog of $BoP$ supports a map to $tmf$ (instead of $bo$) inducing an isomorphism of torsion in $\pi_*$. Hence we call it $BmP$. 

Some informative history: $MSU$ (continued)
The space $BO\langle 8\rangle_{(3)}$ is a Wilson space,
The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it has both torsion-free homology and torsion-free homotopy.
The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper.
The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that is has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.
More history: Wilson spaces and Hopf rings

The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$,
The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum.
The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it is has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$. 

More history: Wilson spaces and Hopf rings
More history: Wilson spaces and Hopf rings

The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that is has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$.

Let $e_n = (p^{n+1} - 1)/(p - 1) = 1 + p + \cdots + p^n$. 

...
More history: Wilson spaces and Hopf rings

The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that is has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$.

Let $e_n = (p^{n+1} - 1)/(p - 1) = 1 + p + \cdots + p^n$.

Then Wilson shows the following:
The space $BO\langle 8 \rangle(3)$ is a Wilson space, meaning that is has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$.

Let $e_n = (p^{n+1} - 1)/(p - 1) = 1 + p + \cdots + p^n$.

Then Wilson shows the following:

- $BP_k$ is a Wilson space for each $k$. 

More history: Wilson spaces and Hopf rings
The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$. Let $e_n = (p^{n+1} - 1)/(p - 1) = 1 + p + \cdots + p^n$.

Then Wilson shows the following:

- $BP_k$ is a Wilson space for each $k$.
- $BP\langle n \rangle_k$ is one for $k \leq 2e_n$. 
More history: Wilson spaces and Hopf rings

The space $BO\langle 8 \rangle_{(3)}$ is a Wilson space, meaning that it has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$. Let $e_n = \frac{(p^{n+1} - 1)}{(p - 1)} = 1 + p + \cdots + p^n$.

Then Wilson shows the following:

- $BP_k$ is a Wilson space for each $k$.
- $BP\langle n \rangle_k$ is one for $k \leq 2e_n$.
- Every Wilson space is equivalent to a product of these spaces.
The space $BO\langle 8 \rangle^{(3)}$ is a Wilson space, meaning that it has both torsion free homology and torsion free homotopy. Such spaces are classified by Steve Wilson in a 1973 paper. Their homology groups are described in the 1977 “Hopf ring” paper of Wilson and the third author.

Given a spectrum $E$, let $E_k$ denote the $k$th space in its $\Omega$-spectrum. We are interested in the spectra $BP$ and $BP\langle n \rangle$. Let $e_n = (p^{n+1} - 1)/(p - 1) = 1 + p + \cdots + p^n$. Then Wilson shows the following:

- $BP_k$ is a Wilson space for each $k$.
- $BP\langle n \rangle_k$ is one for $k \leq 2e_n$.
- Every Wilson space is equivalent to a product of these spaces.
- In particular, for such $k$, $BP\langle n \rangle_k$ is a factor of $BP_k$ and of $BP\langle n' \rangle_k$ for each $n' > n$. 
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n\rangle$),
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space,
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n\rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_\ast E_k$ (with field coefficients) is a Hopf algebra.
More history: Wilson spaces and Hopf rings (continued)

Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_* E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_* E_k$, we denote their product by $x \star y$, the star product.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_\ast E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_\ast E_k$, we denote their product by $x \star y$, the star product.
- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_\ast E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_\ast E_k$, we denote their product by $x \star y$, the star product.

- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$. Given $x \in H_\ast E_k$ and $y \in H_\ast E_\ell$, the image of $x \otimes y$ in $H_\ast E_{k+\ell}$ is denoted by $x \circ y$, the circle product. It plays nicely with the Hopf algebra coproduct.

- These two products make the graded space $E_\bullet$ into a graded ring object in the category of coalgebras, a Hopf ring. The star and circle products are related by the Hopf ring distributive law, in which they correspond respectively to addition and multiplication.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_*E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_*E_k$, we denote their product by $x \ast y$, the **star product**.

- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$. Given $x \in H_*E_k$ and $y \in H_*E_\ell$, the image of $x \otimes y$ in $H_*E_{k+\ell}$ is denoted by $x \circ y$, the **circle product**.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_*E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_*E_k$, we denote their product by $x \star y$, the star product.

- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$. Given $x \in H_*E_k$ and $y \in H_*E_\ell$, the image of $x \otimes y$ in $H_*E_{k+\ell}$ is denoted by $x \circ y$, the circle product. It plays nicely with the Hopf algebra coproduct.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_* E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_* E_k$, we denote their product by $x \star y$, the **star product**.

- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$. Given $x \in H_* E_k$ and $y \in H_* E_\ell$, the image of $x \otimes y$ in $H_* E_{k+\ell}$ is denoted by $x \circ y$, the **circle product**. It plays nicely with the Hopf algebra coproduct.

- These two products make the graded space $E_\ast$ into a graded ring object in the category of coalgebras,
More history: Wilson spaces and Hopf rings (continued)

Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_*E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_*E_k$, we denote their product by $x \star y$, the star product.

- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$. Given $x \in H_*E_k$ and $y \in H_*E_\ell$, the image of $x \otimes y$ in $H_*E_{k+\ell}$ is denoted by $x \circ y$, the circle product. It plays nicely with the Hopf algebra coproduct.

- These two products make the graded space $E_\bullet$ into a graded ring object in the category of coalgebras, a Hopf ring.
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_* E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_* E_k$, we denote their product by $x \star y$, the star product.

- The multiplication in $E$ induces maps $E_k \times E_\ell \to E_{k+\ell}$. Given $x \in H_* E_k$ and $y \in H_* E_\ell$, the image of $x \otimes y$ in $H_* E_{k+\ell}$ is denoted by $x \circ y$, the circle product. It plays nicely with the Hopf algebra coproduct.

- These two products make the graded space $E_\bullet$ into a graded ring object in the category of coalgebras, a Hopf ring. The star and circle products are related by the Hopf ring distributive law,
Given a homotopy commutative ring spectrum $E$ (such as $BP$ or $BP\langle n \rangle$), let $E_k$ denote the $k$th space in its $\Omega$-spectrum. Then

- $E_k$ is an infinite loop space, so $H_* E_k$ (with field coefficients) is a Hopf algebra. Given $x, y \in H_* E_k$, we denote their product by $x \ast y$, the **star product**.

- The multiplication in $E$ induces maps $E_k \times E_{\ell} \to E_{k+\ell}$. Given $x \in H_* E_k$ and $y \in H_* E_{\ell}$, the image of $x \otimes y$ in $H_* E_{k+\ell}$ is denoted by $x \circ y$, the **circle product**. It plays nicely with the Hopf algebra coproduct.

- These two products make the graded space $E_\ast$ into a graded ring object in the category of coalgebras, a **Hopf ring**. The star and circle products are related by the Hopf ring distributive law, in which they correspond respectively to addition and multiplication.
For $x \in \pi_m E$, we get an element

$$[x] \in H_0 E_{-m},$$

the Hurewicz image of $x \in \pi_0 E_{-m}$. 
For \( x \in \pi_m E \), we get an element

\[
[x] \in H_0 E_{-m},
\]

the Hurewicz image of \( x \in \pi_0 E_{-m} \).

When \( E \) is complex oriented, we get a map \( \mathbb{C}P^\infty \to E_2 \),
More history: Wilson spaces and Hopf rings (continued)

For \( x \in \pi_m E \), we get an element

\[ [x] \in H_0 E_{-m}, \]

the Hurewicz image of \( x \in \pi_0 E_{-m} \).

When \( E \) is complex oriented, we get a map \( CP^\infty \to E_2 \), under which we have

\[ H_{2k} CP^\infty \ni \beta_k \mapsto b_k \in H_{2k} E_2. \]

where \( \beta_k \) is the usual generator of \( H_{2k} CP^\infty \).
More history: Wilson spaces and Hopf rings (continued)

For $x \in \pi_m E$, we get an element

$$[x] \in H_0 E_{-m},$$

the Hurewicz image of $x \in \pi_0 E_{-m}$.

When $E$ is complex oriented, we get a map $CP^\infty \rightarrow E_2$, under which we have

$$H_{2k} CP^\infty \ni \beta_k \mapsto b_k \in H_{2k} E_2.$$

where $\beta_k$ is the usual generator of $H_{2k} CP^\infty$. $b_k$ is known to be decomposable under the star product when $k$ is not a power of $p$. 
We are interested in elements of the form

\[ \left[ v^I \right] b^J = [v_1^{i_1} \ldots v_n^{i_n}] b_1^{j_0} b_p^{j_1} \cdots \in H_{2m} BP\langle n\rangle_{2k} \]
We are interested in elements of the form

\[ [v^J]b^J = [v_1^{i_1} \ldots v_n^{i_n}]b_0^{j_0}b_p^{j_1} \cdots \in H_{2m}BP\langle n\rangle_{2k} \]

where the multiplication is the circle product,
More history: Wilson spaces and Hopf rings (continued)

We are interested in elements of the form

\[ [v^I] b^J = [v_1^{i_1} \ldots v_n^{i_n}] b_0^j b_1^{j_1} \ldots \in H_{2m}BP\langle n \rangle_{2k} \]

where the multiplication is the circle product,

\[ m = ||J|| := j_0 + j_1 p + j_2 p^2 + \ldots \]

and
We are interested in elements of the form
\[ [v^l] b^J = [v_1^{i_1} \ldots v_n^{i_n}] b_{p_0}^{j_0} b_{p_1}^{j_1} \cdots \in H_{2m}BP\langle n\rangle_{2k} \]
where the multiplication is the circle product,
\[ m = ||J|| := j_0 + j_1 p + j_2 p^2 + \ldots \]
and
\[ k = |l| - ||l|| + |J| \]
\[ = i_1 + \cdots + i_n - (i_1 p + \cdots + i_n p^n) + j_0 + j_1 + j_2 + \ldots \]
We are interested in elements of the form

\[ [v^I] b^J = [v_1^{i_1} \ldots v_n^{i_n}] b_0^{j_0} b_1^{j_1} \cdots \in H_{2m} BP\langle n\rangle_{2k} \]

where the multiplication is the circle product,

\[ m = \|J\| := j_0 + j_1 p + j_2 p^2 + \ldots \]

and

\[ k = |I| - ||I|| + |J| \]
\[ = i_1 + \cdots + i_n - (i_1 p + \cdots + i_n p^n) + j_0 + j_1 + j_2 + \ldots \]

It is known that \( H_*BP\langle n\rangle_{2k} \) for \( k \leq e_n \) is generated by such elements as a ring under the star product,
More history: Wilson spaces and Hopf rings (continued)

We are interested in elements of the form

\[ [\nu^i] b^J = [\nu_1^{i_1} \ldots \nu_n^{i_n}] b_0^{j_0} b_1^{j_1} \cdots \in H_{2m} BP \langle n \rangle_{2k} \]

where the multiplication is the circle product,

\[ m = ||J|| := j_0 + j_1 p + j_2 p^2 + \ldots \]

and

\[ k = ||l|| - ||l|| + |J| \]

\[ = i_1 + \cdots + i_n - (i_1 p + \cdots + i_n p^n) + j_0 + j_1 + j_2 + \ldots \]

It is known that \( H_* BP \langle n \rangle_{2k} \) for \( k \leq e_n \) is generated by such elements as a ring under the star product, subject to the Hopf ring relation,
We are interested in elements of the form

\[ [v^I]b^J = [v_1^{i_1} \ldots v_n^{i_n}]b_0^{j_0}b_1^{j_1} \cdots \in H_{2m}BP\langle n\rangle_{2k} \]

where the multiplication is the circle product,

\[ m = ||J|| := j_0 + j_1p + j_2p^2 + \ldots \]

and

\[ k = |I| - ||I|| + |J| \]
\[ = i_1 + \cdots + i_n - (i_1p + \cdots + i_np^n) + j_0 + j_1 + j_2 + \cdots \]

It is known that \( H_\ast BP\langle n\rangle_{2k} \) for \( k \leq e_n \) is generated by such elements as a ring under the star product, subject to the Hopf ring relation, which is related to the formal group law.
More history: Wilson spaces and Hopf rings (continued)

We are interested in elements of the form

\[ [v^l]b^J = [v_1^{i_1} \ldots v_n^{i_n}]b_1^{j_0}b_p^{j_1} \cdots \in H_{2m}\text{BP} \langle n \rangle_{2k} \]

where the multiplication is the circle product,

\[ m = ||J|| := j_0 + j_1 p + j_2 p^2 + \ldots \]

and

\[ k = |I| - ||I|| + |J| = i_1 + \cdots + i_n - (i_1 p + \cdots + i_n p^n) + j_0 + j_1 + j_2 + \ldots \]

It is known that \( H_*\text{BP} \langle n \rangle_{2k} \) for \( k \leq e_n \) is generated by such elements as a ring under the star product, subject to the Hopf ring relation, which is related to the formal group law. For example, it implies that for each \( t \geq 0 \),

\[ [v_1]b_p^{\rho t} = -b_p^{\rho t} \in H_{2\rho t+1}\text{BP} \langle n \rangle_2. \]
More history: Wilson spaces and Hopf rings (continued)

We will refer to computations with the elements \([v^I b^J]\)
using the Hopf ring distributive law and the Hopf ring relation,
as bee keeping.
We will refer to computations with the elements $[v^I]b^J$, 

More history: Wilson spaces and Hopf rings (continued)

Introduction

$MSU$ at $p = 2$

Wilson spaces and Hopf rings

$H_\ast BO\langle 8 \rangle$ and $H_\ast MO\langle 8 \rangle$

Two change of rings isomorphisms

The Adams spectral sequence for $MO\langle 8 \rangle$
More history: Wilson spaces and Hopf rings (continued)

We will refer to computations with the elements \([v^I]b^J\), using the Hopf ring distributive law and the Hopf ring relation,
We will refer to computations with the elements $[\nu^I] b^J$, using the Hopf ring distributive law and the Hopf ring relation, as bee keeping.
It is known that $H_*BP\langle n \rangle_{2k}$ is a polynomial algebra under the star product when $k < e_n$. 

It is known that $H_* BP\langle n \rangle_{2k}$ is a polynomial algebra under the star product when $k < e_n$, but not for the borderline case $k = e_n$. 
It is known that $H_* BP \langle n \rangle_{2k}$ is a polynomial algebra under the star product when $k < e_n$, but not for the borderline case $k = e_n$. Recall that $e_1 = 1 + p$. 
It is known that $H_* \text{BP} \langle n \rangle_{2k}$ is a polynomial algebra under the star product when $k < e_n$, but not for the borderline case $k = e_n$. Recall that $e_1 = 1 + p$.

At $p = 3$, $BO\langle 8 \rangle$ is the borderline Wilson space $BP\langle 1 \rangle_8$. 
String cobordism at the prime 3

Carl McTague
Vitaly Lorman
Doug Ravenel

Introduction

$H_\ast BO\langle 8 \rangle$ and $H_\ast MO\langle 8 \rangle$

It is known that $H_\ast BP\langle n \rangle_{2k}$ is a polynomial algebra under the star product when $k < e_n$, but not for the borderline case $k = e_n$. Recall that $e_1 = 1 + p$.

At $p = 3$, $BO\langle 8 \rangle$ is the borderline Wilson space $BP\langle 1 \rangle_8$. Its homology has a polynomial factor and a truncated polynomial factor of height 3.
String cobordism at the prime 3
Carl McTague
Vitaly Lorman
Doug Ravenel
Introduction

MSU at $p = 2$

Wilson spaces and Hopf rings

$H_* BO\langle 8 \rangle$ and $H_* MO\langle 8 \rangle$

Two change of rings isomorphisms

The Adams spectral sequence for $MO\langle 8 \rangle$

$H_* BO\langle 8 \rangle$ and $H_* MO\langle 8 \rangle$

H is known that $H_* BP\langle n \rangle_{2k}$ is a polynomial algebra under the star product when $k < e_n$, but not for the borderline case $k = e_n$. Recall that $e_1 = 1 + p$.

At $p = 3$, $BO\langle 8 \rangle$ is the borderline Wilson space $BP\langle 1 \rangle_8$. Its homology has a polynomial factor and a truncated polynomial factor of height 3. Its first few generators are

$$y_8 = b_1^4$$
with $y_8^3 = 0$

$$x_{12} = b_1^3 b_3$$
$$x_{16} = b_1^2 b_3^2$$

$$y_{20} = b_1 b_3^3$$
with $y_{20}^3 = 0$

$$x_{24} = b_1^3 b_9$$
$$y_{24} = b_3^4 - b_1^3 b_9$$
with $y_{24}^3 = 0$

$$x_{28} = b_1^2 b_3 b_9$$
$$x_{32} = b_1 b_3^2 b_9$$

$$\vdots$$

$$x_{52} = [v_1] b_1^2 b_3^2 b_9^2$$, the first appearance of $[v_1]$
We find that

\[ H_\ast BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \]
\[ \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]
We find that

\[ H_\ast BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]

where \( \Gamma(y) \) denotes the divided power algebra on \( y \),
We find that

\[ H_\ast BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]

where \( \Gamma(y) \) denotes the divided power algebra on \( y \), which is dual to the polynomial algebra on the dual of \( y \).
We find that

\[ H_\ast BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \]
\[ \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]

where \( \Gamma(y) \) denotes the divided power algebra on \( y \), which is dual to the polynomial algebra on the dual of \( y \). For example,

\[ \Gamma(y_8) \cong P(y_8, y_{24}, y_{72}, \ldots) / (y_8^3), \]
We find that

\[ H_* BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]

where \( \Gamma(y) \) denotes the divided power algebra on \( y \), which is dual to the polynomial algebra on the dual of \( y \). For example,

\[ \Gamma(y_8) \cong P(y_8, y_{24}, y_{72}, \ldots) / (y_8^{3^i}), \]

and the Verschiebung map \( V \), the dual of the \( p \)th power map, divides each subscript by 3.
We find that

\[ H_\ast BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \]
\[ \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]

where \( \Gamma(y) \) denotes the divided power algebra on \( y \), which is dual to the polynomial algebra on the dual of \( y \). For example,

\[ \Gamma(y_8) \cong P(y_8, y_{24}, y_{72}, \ldots) / (y_8^3), \]

and the Verschiebung map \( V \), the dual of the \( p \)th power map, divides each subscript by 3.

It is not hard to work out the right action of the mod 3 Steenrod algebra \( A \) on \( H_\ast BO\langle 8 \rangle \).
We find that

\[ H_\ast BO\langle 8 \rangle \cong P(x_{4m} : m \geq 3, 2m \neq 1 + 3^n) \]
\[ \otimes \Gamma(y_{2(1+3^n)} : n \geq 0), \]

where \( \Gamma(y) \) denotes the divided power algebra on \( y \), which is dual to the polynomial algebra on the dual of \( y \). For example,

\[ \Gamma(y_8) \cong P(y_8, y_{24}, y_{72}, \ldots )/(y_8^{3}3^i), \]

and the Verschiebung map \( V \), the dual of the \( p \)th power map, divides each subscript by 3.

It is not hard to work out the right action of the mod 3 Steenrod algebra \( \mathcal{A} \) on \( H_\ast BO\langle 8 \rangle \), and on the Thom isomorphic ring \( H_\ast MO\langle 8 \rangle \).
Two change of rings isomorphisms

We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. 
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. 
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. The dual of the subalgebra $P \subseteq A$ generated by the Steenrod reduced power operations is

$$P_* \cong P(\zeta_1, \zeta_2, \ldots).$$
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. The dual of the subalgebra $P \subseteq A$ generated by the Steenrod reduced power operations is

$$P_* \cong P(\zeta_1, \zeta_2, \ldots).$$

$A$ has a subalgebra $E$ with

$$E_* \cong E(\tau_0, \tau_1, \ldots).$$
Two change of rings isomorphisms

We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. The dual of the subalgebra $\mathcal{P} \subseteq A$ generated by the Steenrod reduced power operations is

$$\mathcal{P}_* \cong P(\zeta_1, \zeta_2, \ldots).$$

$A$ has a subalgebra $\mathcal{E}$ with

$$\mathcal{E}_* \cong E(\tau_0, \tau_1, \ldots).$$

and

$$\text{Ext}_{\mathcal{E}_*}(\mathbb{Z}/3, \mathbb{Z}/3) \cong P(a_0, a_1, \ldots) =: V.$$
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. The dual of the subalgebra $\mathcal{P} \subseteq A$ generated by the Steenrod reduced power operations is

$$\mathcal{P}_* \cong P(\zeta_1, \zeta_2, \ldots).$$

$A$ has a subalgebra $\mathcal{E}$ with

$$\mathcal{E}_* \cong E(\tau_0, \tau_1, \ldots).$$

and

$$\text{Ext}_{\mathcal{E}_*}(\mathbb{Z}/3, \mathbb{Z}/3) \cong P(a_0, a_1, \ldots) =: V.$$
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. The dual of the subalgebra $P \subseteq A$ generated by the Steenrod reduced power operations is

$$P_* \cong P(\zeta_1, \zeta_2, \ldots).$$

$A$ has a subalgebra $E$ with

$$E_* \cong E(\tau_0, \tau_1, \ldots).$$

and

$$\text{Ext}_{E_*}(Z/3, Z/3) \cong P(a_0, a_1, \ldots) =: V.$$
We want to study the 3-primary Adams spectral sequence for $MO\langle 8 \rangle$. Recall that

$$A_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\zeta_1, \zeta_2, \ldots),$$

with $|\tau_n| = 2 \cdot 3^n - 1$ and $|\zeta_n| = 2 \cdot 3^n - 2$. The dual of the subalgebra $P \subseteq A$ generated by the Steenrod reduced power operations is

$$P_* \cong P(\zeta_1, \zeta_2, \ldots).$$

$A$ has a subalgebra $E$ with

$$E_* \cong E(\tau_0, \tau_1, \ldots).$$

and

$$\text{Ext}_{E_*}(\mathbb{Z}/3, \mathbb{Z}/3) \cong P(a_0, a_1, \ldots) =: V.$$

Here $a_n$ corresponds to $v_n \in \pi_* BP$, where $v_0 = 3$. It has Adams filtration 1 and topological dimension $2(3^n - 1)$. 

Two change of rings isomorphisms
There is a Cartan-Eilenberg spectral sequence converging to our Adams $E_2$-page with

$$E_1^{*,*,*} \cong \Ext_{P_*} \left( \mathbb{Z}/3, \Ext_{E_*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \right) \right)$$

$$\cong \Ext_{P_*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \otimes V \right).$$

(1)
There is a Cartan-Eilenberg spectral sequence converging to our Adams $E_2$-page with

$$E_1^{*,*,*} \cong \text{Ext}_{P_*} \left( \mathbb{Z}/3, \text{Ext}_{E_*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \right) \right)$$

$$\cong \text{Ext}_{P_*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \otimes V \right).$$

The coaction of $E_*$ on $H_*MO\langle 8 \rangle$ is trivial since the latter is concentrated in even dimensions.
Two change of rings isomorphisms (continued)

There is a Cartan-Eilenberg spectral sequence converging to our Adams $E_2$-page with

$$
E_1^{*,*,*} \cong \text{Ext}_{P_*} \left( \mathbb{Z}/3, \text{Ext}_{E_*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \right) \right)
\cong \text{Ext}_{P_*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \otimes V \right).
$$

The coaction of $E_*$ on $H_*MO\langle 8 \rangle$ is trivial since the latter is concentrated in even dimensions. This leads to the second isomorphism of (1).
Two change of rings isomorphisms (continued)

Let

\[ J = (x_{12}^3, x_{16}^3, x_{52}, x_{160}, \ldots) \subseteq H_* MO\langle 8 \rangle, \]
Let
\[ J = (x_{12}^3, x_{16}^3, x_{52}, x_{160}, \ldots) \subseteq H_\ast MO\langle 8 \rangle, \]
the change of rings ideal.
Two change of rings isomorphisms (continued)

Let

\[ J = (x_1^{3}, x_{12}^{3}, x_{16}, x_{52}, x_{160}, \ldots) \subseteq H_* MO\langle 8 \rangle, \]

the change of rings ideal. One can show that

\[ \text{Ext}_{P_*} (\mathbb{Z}/3, H_* MO\langle 8 \rangle) \cong \text{Ext}_{P(1)_*} (\mathbb{Z}/3, H_* MO\langle 8 \rangle / J), \]

the first change of rings isomorphism,
Two change of rings isomorphisms (continued)

Let

\[ J = (x_{12}^3, x_{16}^3, x_{52}, x_{160}, \ldots) \subseteq H_* \text{MO}(8), \]

the change of rings ideal. One can show that

\[ \text{Ext}_{P_*} (\mathbb{Z}/3, H_* \text{MO}(8)) \cong \text{Ext}_{P(1)_*} (\mathbb{Z}/3, H_* \text{MO}(8)/J), \]

the first change of rings isomorphism, where

\[ 36 \ 48 \ 52 \ 160 \]

\[ P(1)_* = P_*/(\zeta_1^9, \zeta_2^3, \zeta_3, \zeta_4, \ldots) \]

\[ = P(\zeta_1, \zeta_2)/(\zeta_1^9, \zeta_2^3) \]
Two change of rings isomorphisms (continued)

Let

\[ J = (x_1^3, x_5, x_{16}, x_{160}, \ldots) \subseteq H_* \text{MO}\langle 8 \rangle, \]

the change of rings ideal. One can show that

\[
\text{Ext}_{P} (\mathbb{Z}/3, H_* \text{MO}\langle 8 \rangle) \cong \text{Ext}_{P(1)} (\mathbb{Z}/3, H_* \text{MO}\langle 8 \rangle/J),
\]

the first change of rings isomorphism, where

\[
P(1)_* = P_*/(\zeta_1^9, \zeta_2^3, \zeta_3, \zeta_4, \ldots)
\]

is dual to the subalgebra \(P(1) \subseteq P\) generated by the Steenrod operations \(P^1\) and \(P^3\).
Two change of rings isomorphisms (continued)

Let

\[ J = (x_{12}^3, x_{16}^3, x_{52}, x_{160}, \ldots) \subseteq H_*MO\langle 8 \rangle, \]

the change of rings ideal. One can show that

\[ \text{Ext}_{P_*} (\mathbb{Z}/3, H_*MO\langle 8 \rangle) \cong \text{Ext}_{P(1)_*} (\mathbb{Z}/3, H_*MO\langle 8 \rangle/J), \]

the first change of rings isomorphism, where

\[
\begin{align*}
36 & 48 & 52 & 160 \\
\mathcal{P}(1)_* = & P_*/(\zeta_1^9, \zeta_2^3, \zeta_3, \zeta_4, \ldots) \\
= & P(\zeta_1, \zeta_2)/(\zeta_1^9, \zeta_2^3)
\end{align*}
\]

is dual to the subalgebra \( \mathcal{P}(1) \subseteq \mathcal{P} \) generated by the Steenrod operations \( P^1 \) and \( P^3 \). This is a major simplification.
Recall

\[ \text{Ext}_{P_*}(\mathbb{Z}/3, H_*MO\langle 8 \rangle) \cong \text{Ext}_{P(1)_*}(\mathbb{Z}/3, L), \]

where \( L = H_*MO\langle 8 \rangle/J \) and \( P(1)_* = P(\zeta_1, \zeta_2)/(\zeta_1^9, \zeta_2^3). \)
Recall

\[
\text{Ext}_{\mathcal{P}}^* (\mathbb{Z}/3, H_* MO\langle 8 \rangle) \cong \text{Ext}_{\mathcal{P}(1)}^* (\mathbb{Z}/3, L),
\]

where \( L = H_* MO\langle 8 \rangle / J \) and \( \mathcal{P}(1)_* = P(\zeta_1, \zeta_2) / (\zeta_1^9, \zeta_2^3) \).

The algebra \( \mathcal{P}(1) \) is noncommutative, has rank 27 and has a complicated Ext group.
Recall

$$\text{Ext}_{P^*}(\mathbb{Z}/3, H_* MO\langle 8 \rangle) \cong \text{Ext}_{P(1)^*}(\mathbb{Z}/3, L),$$

where $L = H_* MO\langle 8 \rangle / J$ and $P(1)^* = P(\zeta_1, \zeta_2)/(\zeta_1^9, \zeta_2^3)$.

The algebra $P(1)$ is noncommutative, has rank 27 and has a complicated Ext group. The dual of $\zeta_2$ is

$$Q := [P^3, P^1] = P^3 P^1 - P^4 \quad \text{with} \quad Q^3 = 0.$$
Recall

\[ \text{Ext}_{\mathcal{P}_*}(\mathbb{Z}/3, H_* MO\langle 8 \rangle) \cong \text{Ext}_{\mathcal{P}(1)_*}(\mathbb{Z}/3, L) , \]

where \( L = H_* MO\langle 8 \rangle / J \) and \( \mathcal{P}(1)_* = P(\zeta_1, \zeta_2)/(\zeta_9^9, \zeta_3^3) \).

The algebra \( \mathcal{P}(1) \) is noncommutative, has rank 27 and has a complicated Ext group. The dual of \( \zeta_2 \) is

\[ Q := [P^3, P^1] = P^3 P^1 - P^4 \quad \text{with} \quad Q^3 = 0. \]

The \( \mathcal{P}(1) \)-module \( L \) is free over the subalgebra \( T \) generated by \( Q \).
Recall

\[ \text{Ext}_{P^*} \left( \mathbb{Z}/3, H_* \text{MO}(8) \right) \cong \text{Ext}_{P(1)^*} \left( \mathbb{Z}/3, L \right), \]

where \( L = H_* \text{MO}(8)/J \) and \( P(1)^* = P(\zeta_1, \zeta_2)/(\zeta_9^9, \zeta_3^3) \).

The algebra \( P(1) \) is noncommutative, has rank 27 and has a complicated Ext group. The dual of \( \zeta_2 \) is

\[ Q := [P^3, P^1] = P^3 P^1 - P^4 \quad \text{with} \quad Q^3 = 0. \]

The \( P(1) \)-module \( L \) is free over the subalgebra \( T \) generated by \( Q \). This gives the second change of rings isomorphism

\[ \text{Ext}_{P(1)^*} \left( \mathbb{Z}/3, L \right) \cong \text{Ext}_{P(1)^*} \left( \mathbb{Z}/3, L' \right), \]
Recall

\[ \text{Ext}_{P_*} \left( \mathbb{Z}/3, H_* \text{MO}(8) \right) \cong \text{Ext}_{P(1)_*} \left( \mathbb{Z}/3, L \right), \]

where \( L = H_* \text{MO}(8)/J \) and \( P(1)_* = P(\zeta_1, \zeta_2)/(\zeta_9^1, \zeta_3^3) \).

The algebra \( P(1) \) is noncommutative, has rank 27 and has a complicated Ext group. The dual of \( \zeta_2 \) is

\[ Q := [P^3, P^1] = P^3 P^1 - P^4 \quad \text{with} \quad Q^3 = 0. \]

The \( P(1) \)-module \( L \) is free over the subalgebra \( T \) generated by \( Q \). This gives the second change of rings isomorphism

\[ \text{Ext}_{P(1)_*} \left( \mathbb{Z}/3, L \right) \cong \text{Ext}_{P(1)'_*} \left( \mathbb{Z}/3, L' \right), \]

where \( P(1)' = P(1)/T \) is commutative with dual

\[ P(1)'_* = P(\zeta_1)/(\zeta_9^1), \]
Two change of rings isomorphisms (continued)

Recall

\[ \text{Ext}_{\mathcal{P}_*} (\mathbb{Z}/3, H_* MO \langle 8 \rangle) \cong \text{Ext}_{\mathcal{P}(1)_*} (\mathbb{Z}/3, L), \]

where \( L = H_* MO \langle 8 \rangle / J \) and \( \mathcal{P}(1)_* = P(\zeta_1, \zeta_2)/(\zeta_9, \zeta_3). \)

The algebra \( \mathcal{P}(1) \) is noncommutative, has rank 27 and has a complicated Ext group. The dual of \( \zeta_2 \) is

\[ Q := [P^3, P^1] = P^3 P^1 - P^4 \quad \text{with} \quad Q^3 = 0. \]

The \( \mathcal{P}(1) \)-module \( L \) is free over the subalgebra \( T \) generated by \( Q \). This gives the second change of rings isomorphism

\[ \text{Ext}_{\mathcal{P}(1)_*} (\mathbb{Z}/3, L) \cong \text{Ext}_{\mathcal{P}(1)'_*} (\mathbb{Z}/3, L'), \]

where \( \mathcal{P}(1)' = \mathcal{P}(1) / T \) is commutative with dual

\[ \mathcal{P}(1)'_* = P(\zeta_1)/(\zeta_9^9), \]

and \( L' \subseteq L \) is the subring on which \( Q \) acts trivially.
Similarly in the Adams spectral sequence for $MO\langle 8 \rangle$, 

$$E_2 = \text{Ext}_{P*} \left( \mathbb{Z}/3, H_*MO\langle 8 \rangle \otimes V \right)$$

$$\cong \text{Ext}_{P(1)*} \left( \mathbb{Z}/3, L \otimes V \right)$$

$$\cong \text{Ext}_{P(1)'*} \left( \mathbb{Z}/3, (L \otimes V)' \right)$$
Similarly in the Adams spectral sequence for $MO\langle 8 \rangle$, we have:

$$E_2 = \text{Ext}_{P_\ast} \left( \mathbb{Z}/3, H_\ast MO\langle 8 \rangle \otimes V \right)$$

$$\cong \text{Ext}_{P(1)_\ast} \left( \mathbb{Z}/3, L \otimes V \right)$$

$$\cong \text{Ext}_{P(1)'_\ast} \left( \mathbb{Z}/3, (L \otimes V)' \right)$$

where $P(1)'_\ast = P(\zeta_1)/\zeta_1^9$ and

$$(L \otimes V)' := \ker Q \subseteq L \otimes V.$$
Here is the first $P(1)'$-summand of $L'$.

\[
\begin{array}{ccc}
0 & 12 & 24 \\
\downarrow & \downarrow & \downarrow \\
1 & x_{12} & x_{12}^2 + \bar{y}_{24} \\
\downarrow & \downarrow & \downarrow \\
y_8 & \bar{y}_{20} - y_8 x_{12} & \bar{y}_{20} + y_8 (x_{12}^2 - \bar{y}_{24}) \\
8 & 20 & 32 \\
\end{array}
\]
Here is the first $P(1)'$-summand of $L'$.

\[
\begin{array}{c|cc}
0 & 12 & 24 \\
1 & \xleftarrow{\begin{smallmatrix} -1 \\ P^3 \end{smallmatrix}} x_{12} & \xleftarrow{\begin{smallmatrix} -1 \\ P^3 \end{smallmatrix}} x_{12}^2 + y_{24} \\
& \xrightarrow{P^1} & \xrightarrow{P^1} \\
y_8 & \xleftarrow{\begin{smallmatrix} -1 \\ P^3 \end{smallmatrix}} y_{20} - y_8 x_{12} & \xleftarrow{\begin{smallmatrix} -1 \\ P^3 \end{smallmatrix}} x_{12} y_{20} + y_8 (x_{12}^2 - y_{24}), \\
8 & 20 & 32 \\
\end{array}
\]

where $y_{20} = y_{20} + y_8 x_{12}$, and $y_{24} = y_{24} - y_8 x_{16}$.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

Here is the first $P(1)'$-summand of $L'$.

\[
\begin{array}{ccc}
0 & 12 & 24 \\
1 & \leftarrow^{P^3} & \leftarrow^{P^3} \frac{-1}{P^3} x_{12} \leftarrow_{P^3} x_{12}^2 + y_{24} \\
& \downarrow_{P^1} & \downarrow_{P^1} \\
& y_8 \leftarrow_{P^3} \overline{y}_{20} - y_8 x_{12} \leftarrow_{P^3} \frac{-1}{P^3} x_{12} \overline{y}_{20} + y_8 (x_{12}^2 - \overline{y}_{24}), \\
& 8 & 20 \downarrow_{P^1} \downarrow_{P^1} \leftarrow_{P^3} y_{20} \leftarrow_{P^3} \frac{-1}{P^3} \overline{y}_{20} - y_8 x_{12} y_{24} \\
& 16 & 28 \downarrow_{P^1} \downarrow_{P^1} \leftarrow_{P^3} y_{20}^2 \leftarrow_{P^3} \frac{-1}{P^3} y_{8} \overline{y}_{20} + y_8^2 x_{12} \leftarrow_{P^3} y_8 x_{12} \overline{y}_{20} + y_8^2 (x_{12}^2 - \overline{y}_{24}) \\
& 32 & 40 \end{array}
\]

where $\overline{y}_{20} = y_{20} + y_8 x_{12}$, and $\overline{y}_{24} = y_{24} - y_8 x_{16}$. Here is the next one, which is free.

\[
\begin{array}{ccc}
24 & 36 & 48 \\
\overline{y}_{24} & \leftarrow_{P^3} \frac{-1}{P^3} x_{12} \overline{y}_{24} + y_8^2 \overline{y}_{20} \leftarrow_{P^3} \frac{-1}{P^3} x_{12}^2 \overline{y}_{24} \\
& \downarrow \downarrow \downarrow \\
\overline{y}_{20} & \leftarrow_{P^3} \frac{-1}{P^3} x_{12} \overline{y}_{20} + y_8 \overline{y}_{24} \leftarrow_{P^3} x_{12}^2 \overline{y}_{20} - y_8 x_{12} \overline{y}_{24} \\
& \downarrow \downarrow \downarrow \\
y_8^2 & \leftarrow_{P^3} \frac{-1}{P^3} - y_8 \overline{y}_{20} + y_8^2 x_{12} \leftarrow_{P^3} y_8 x_{12} \overline{y}_{20} + y_8^2 (x_{12}^2 - \overline{y}_{24}) \\
& 16 & 28 \downarrow \downarrow \downarrow \\
& 24 & 36 \downarrow \downarrow \downarrow \\
& 32 & 40 \end{array}
\]
Here is a third one.

\[
\begin{array}{ccc}
32 & 44 & 56 \\
\downarrow & \& \downarrow \\
\downarrow & \& \downarrow \\
28 & \& \& 64 \\
\end{array}
\]

\[\begin{array}{cccc}
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\end{array}\]

This one is isomorphic to the first one tensored with a rank 2 module in the first column.
Here is a third one.

\[ y_8 y_{24} \quad 44 \quad 56 \]

\[ y_8 y_{20} \quad 28 \]

\[ y_8 y_{20} \quad 36 \quad 48 \quad 60 \]

This one is isomorphic to the first one tensored with a rank 2 module in the first column.
Here is a third one.

\[
\begin{array}{ccc}
32 & 44 & 56 \\
\downarrow & \downarrow & \downarrow \\
\bar{y}_8\bar{y}_{24} & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bar{y}_8\bar{y}_{20} & \bullet & \bullet & 64 \\
\downarrow & \downarrow & \downarrow \\
28 & \bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
36 & 48 & 60 \\
\end{array}
\]

This one is isomorphic to the first one tensored with a rank 2 module in the first column.

In each case the Ext group is easy to compute.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

Here is a third one.

$$\begin{array}{ccc}
32 & 44 & 56 \\
y_8\bar{y}_{24} & \bullet & \bullet \\
\downarrow & & \downarrow \\
y_8\bar{y}_{20} & \bullet & \bullet & 64 \\
28 & \bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow \\
36 & 48 & 60 \\
\end{array}$$

This one is isomorphic to the first one tensored with a rank 2 module in the first column.

In each case the Ext group is easy to compute. It turns out that both $L'$ and $(L \otimes V)'$ decompose as a direct sum of $P(1)'$-modules of these three types.
Here is a third one.

\[
\begin{array}{ccc}
32 & 44 & 56 \\
y_8 y_{24} \downarrow & \bullet & \bullet \\
y_8 y_{20} & \bullet & \bullet & 64 \\
28 & \bullet & \bullet & \bullet \\
36 & 48 & 60
\end{array}
\]

This one is isomorphic to the first one tensored with a rank 2 module in the first column.

In each case the Ext group is easy to compute. It turns out that both $L'$ and $(L \otimes V)'$ decompose as a direct sum of $P(1)'$-modules of these three types. Each free summand of $L'$ corresponds to summand of the spectrum $MO\langle 8 \rangle$ equivalent to a suspension of $BP$. 
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

$E_1$ page

This chart shows Adams $d_1$ and $d_2$ in for the subalgebra of $L'$ generated by $y_8$, $x_{12}$, $y_{20}$ and $y_{24}$. The 48-dimensional class $a_3$ is excluded to avoid clutter.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

This chart shows Adams $d_1$s and $d_2$s in for the subalgebra of $L'$ generated by $y_8$, $x_{12}$, $\bar{y}_{20}$ and $\bar{y}_{24}$.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

This chart shows Adams $d_1$s and $d_2$s in for the subalgebra of $L'$ generated by $y_8, x_{12}, \bar{y}_{20}$ and $\bar{y}_{24}$. The 48-dimensional class $\bar{a}_2^3$ is excluded to avoid clutter.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

$E_3$ page

This chart shows the resulting $E_3$ page with torsion elements shown in blue.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

This chart shows the resulting $E_3$ page with torsion elements shown in blue.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

$E_3$ page

$w_{24,1}, a_2^3, \overline{a_2^6}$

This is the previous chart with $a_3^2$ tensored in. It shows a larger range of dimensions with higher Toda type differentials, with more elements removed to avoid clutter.
String cobordism at the prime 3

Carl McTague
Vitaly Lorman
Doug Ravenel

Introduction

MSU at $p = 2$

Wilson spaces and Hopf rings

$H_\ast BO\langle 8 \rangle$ and $H_\ast MO\langle 8 \rangle$

Two change of rings isomorphisms

The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

$E_3$ page

This is the previous chart with $a_2^3$ tensored in.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

This is the previous chart with $\overline{a}_2^3$ tensored in. It shows a larger range of dimensions with higher Toda type differentials, with more elements removed to avoid clutter.
The Adams spectral sequence for $\text{MO}(8)$ (continued)

$E_7$ page

Thus shows the resulting $E_\infty$ page with torsion elements in blue.

They coincide with Dominic Culver's 2019 description of the 3-primary torsion in $\pi_{\ast} \text{tmf}$, which is 144-dimensional periodic.
Thus shows the resulting $E_\infty$ page with torsion elements in blue.
The Adams spectral sequence for $MO\langle 8 \rangle$ (continued)

Thus shows the resulting $E_\infty$ page with torsion elements in blue. They coincide with Dominic Culver’s 2019 description of the 3-primary torsion in $\pi_* tmf$, which is 144-dimensional periodic.
Introduction

MSU at $p = 2$

Wilson spaces and Hopf rings

$H_* BO \langle 8 \rangle$ and $H_* MO \langle 8 \rangle$

Two change of rings isomorphisms

The Adams spectral sequence for $MO \langle 8 \rangle$