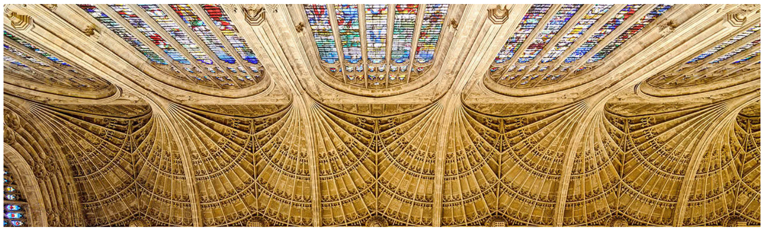

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Model category structures for equivariant spectra
Mike Hopkins Harvard University
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Conference on Equivariant and Motivic Homotopy Theory Isaac Newton Institute, Cambridge, UK

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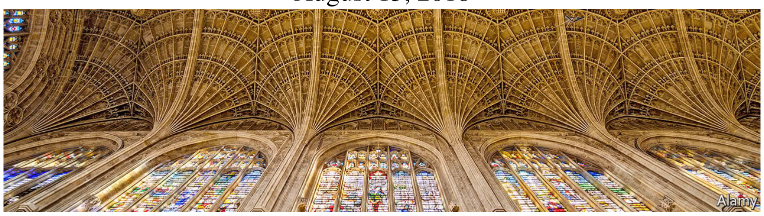
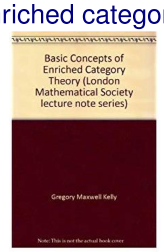


1.1

1 Enriched category theory

Enriched category theory



Equivariant spectra are defined in terms of enriched category theory. In an enriched category, instead of morphism sets we have morphism objects that live a symmetric monoidal category $(\mathscr{V}, \otimes, \mathbf{1})$. The monoidal structure is needed to define the enriched analog of composition of morphisms.

Given objects X, Y and Z in an **ordinary category** $\mathscr{C}$, one has composition morphism

$$c_{X,Y,Z} : \mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \to \mathscr{C}(X,Z),$$

which is a map of sets with suitable properties. In a category enriched over $\mathscr{V}$, instead of morphism sets we have morphism objects in $\mathscr{V}$, and the above is replaced by a composition morphism in $\mathscr{V}$,

$$c_{X,Y,Z} : \mathscr{C}(Y,Z) \otimes \mathscr{C}(X,Y) \to \mathscr{C}(X,Z).$$

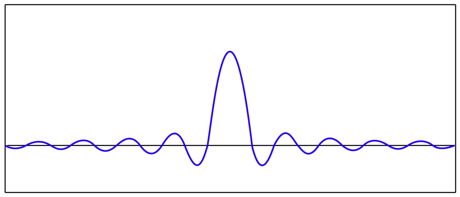
1.2

Enriched category theory (continued)

Usually $\mathscr{V}$ will be some variant of $\mathscr{T} = (\mathscr{T}_0, \wedge, S^0)$, the category of pointed topological spaces under smash product. A closed symmetric monoidal category such as $\mathscr{T}$ is enriched over itself.

There are notions of enriched functors and enriched natural transformations. If $\mathscr{C}$ and $\mathscr{D}$ are categories enriched over $\mathscr{V}$, we denote by $[\mathscr{C}, \mathscr{D}]$ the category of enriched functors $\mathscr{C} \to \mathscr{D}$.

Day Convolution Theorem (1970). *Let* $(\mathscr{J}, \oplus, 0)$ *be a small symmetric monoidal category enriched over a cocomplete closed symmetric monoidal category* $\mathscr{V}$. *Then the enriched functor category* $[\mathscr{J}, \mathscr{V}]$ *is closed symmetric monoidal.*

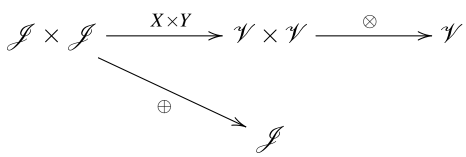


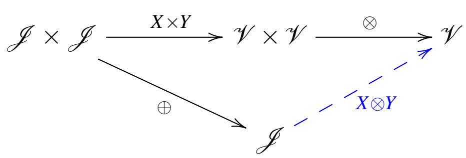
1.3

Enriched category theory (continued)

Day Convolution Theorem (1970). *Let* $(\mathscr{J}, \oplus, 0)$ *be a small symmetric monoidal category enriched over a cocomplete closed symmetric monoidal category* $(\mathscr{V}, \otimes, 1)$. *Then the enriched functor category* $[\mathscr{J}, \mathscr{V}]$ *is closed symmetric monoidal.*

To define this monoidal structure, suppose we have two functors $X, Y : \mathscr{J} \to \mathscr{V}$. Consider the diagram





The functor $X \otimes Y$ is the left Kan extension of the composite $\otimes(X \times Y)$ along $\oplus$. It exists because $\mathscr{J} \times \mathscr{J}$ is small and $\mathscr{V}$ is cocomplete. 1.4

2 Some equivariant homotopy theory

Some equivariant homotopy theory

For a finite group G, let $\mathscr{T}^G$ be the category of pointed G-spaces and equivariant maps. In the Bredon model structure a map $f : X \to Y$ is a fibration or a weak equivalence if the map $f^H : X^H \to Y^H$ of fixed point sets is one for each subgroup H. Cofibrations are defined in terms of left lifting properties.

For each subgroup $H \subseteq G$, there is a pair of adjoint functors

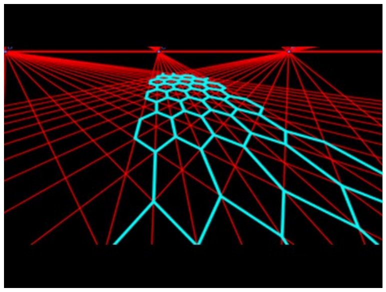
$$G_+ \underset{H}{\wedge} (-) : \mathscr{T}^H \leftrightarrows \mathscr{T}^G : i_H^G,$$

where i_H^G is the forgetful functor and $G_+ \underset{H}{\wedge} (-)$ is the induction functor. Both categories have a Bredon model structure. The above is known to be a Quillen adjunction, which is very convenient. This means that the left (right) functor preserves weak equivalences and cofibrations (fibrations).

We use the term equifibrant to describe this happy state of affairs. We need an equifibrant model structure on the category of G-spectra. 1.5

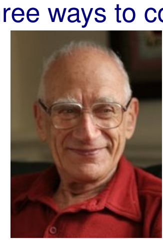
3 Three constructions of new model categories

Three ways to construct new model categories from old ones



1. Given a model category $\mathscr{M}$ and a small category J, we define the projective model structure on the functor category $\mathscr{M}^J$ as follows. A map (aka natural transformation) $f : X \to Y$ between functors is a weak equivalence or a fibration if $f_j : X_j \to Y_j$ is one for each object j in J. Cofibrations are defined in terms of left lifting properties. 1.6

Three ways to construct new model categories from old ones (continued)


Dan Kan 1928-2013 2. Given a model category $\mathscr{M}$ and a pair of adjoint functors

$$F : \mathscr{M} \leftrightarrows \mathscr{N} : U,$$

the Kan transfer theorem says that under certain conditions there is model structure on $\mathscr{N}$ that makes the above a Quillen adjunction. A morphism in $\mathscr{N}$ is a weak equivalence or a fibration iff its image under U is one. 1.7

Three ways to construct new model categories from old ones (continued)


Pete Bousfield

3. Bousfield localization. Given a model category $\mathscr{M}$ satisfying certain conditions, we can define a new model structure $\mathscr{M}'$ with the same underlying category as follows. $\mathscr{M}'$ has the same cofibrations as $\mathscr{M}$, but more weak equivalences and hence more trivial cofibrations. Fibrations are maps having the right lifting property with respect to all trivial cofibrations, so there are fewer of them. This means that fibrant replacement is more interesting in $\mathscr{M}'$ than in $\mathscr{M}$. 1.8

4 The main construction

The main construction

Suppose we have a diagram of small categories enriched over $\mathscr{T}^G$ (to be named later),

$$\begin{array}{ccc} \mathscr{J}_G^+ & \xrightarrow{k_+} & \widetilde{\mathscr{J}}_G^+ \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \widetilde{i}} \\ \mathscr{J}_G & \xrightarrow{k} & \widetilde{\mathscr{J}}_G \end{array}$$

Then we get a diagram of enriched functor categories

$$\begin{array}{ccccc} & & [\mathscr{J}_G^+, \mathscr{T}^G] & \xleftarrow{k_+^*} & [\widetilde{\mathscr{J}}_G^+, \mathscr{T}^G] \\ & & {\scriptstyle i_!}\downarrow & & \downarrow{\scriptstyle \widetilde{i}_!} \\ \mathscr{S}p^G & = & [\mathscr{J}_G, \mathscr{T}^G] & \xleftarrow{k^*} & [\widetilde{\mathscr{J}}_G, \mathscr{T}^G] \end{array}$$

where k^* and k_+^* are induced by precomposition, and $i_!$ and $\widetilde{i}_!$ are induced by left Kan extension. The category $\mathscr{J}_G$ is chosen so that the functor category $[\mathscr{J}_G, \mathscr{T}^G]$ is that of orthogonal G-spectra and equivariant maps. 1.9

The main construction (continued)

$$\begin{array}{ccccc} [\mathscr{J}_G^+, \mathscr{T}^G] & & \xleftarrow{k_+^*} & & [\widetilde{\mathscr{J}}_G^+, \mathscr{T}^G] \\ & \mathscr{J}_G^+ & \xrightarrow{k_+} & \widetilde{\mathscr{J}}_G^+ & \\ {\scriptstyle i_!}\downarrow & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \widetilde{i}} & \downarrow{\scriptstyle \widetilde{i}_!} \\ & \mathscr{J}_G & \xrightarrow{k} & \widetilde{\mathscr{J}}_G & \\ \mathscr{S}p^G = [\mathscr{J}_G, \mathscr{T}^G] & & \xleftarrow{k^*} & & [\widetilde{\mathscr{J}}_G, \mathscr{T}^G] \end{array}$$

Now we proceed as follows.

(i) Start with the projective model structure on $[\widetilde{\mathscr{J}}_G^+, \mathscr{T}^G]$. It is equifibrant, while the projective model structure on $[\mathscr{J}_G, \mathscr{T}^G]$ is not.

(ii) The composite functor $i_! k_+^* = k^* \widetilde{i}_!$ is a left adjoint, so we can use the Kan transfer theorem to get a model structure on $\mathscr{S}p^G$. This transferred model structure is also equifibrant.

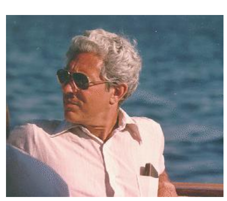
(iii) Expand the transferred class of weak equivalences on $\mathscr{S}p^G$ to that of stable equivalences and apply Bousfield localization.

1.10

5 Defining the four small categories

Defining the four small categories

Mike Mandell



Peter May

$\mathscr{J}_G$ is the Mandell-May category. Its objects are finite dimensional orthogonal representations V of G. The morphism space $\mathscr{J}_G(V,W)$ is the Thom space of the following vector bundle.

Let $O(V,W)$ be the (possibly empty) Stiefel manifold of isometric embeddings (which need not be equivariant) of V into W. For each such embedding $f : V \hookrightarrow W$ one has the orthogonal compliment $V^{\perp}$ of $f(V)$ in W, which is the fiber of our vector bundle over $O(V,W)$. 1.11

Defining the four small categories (continued)

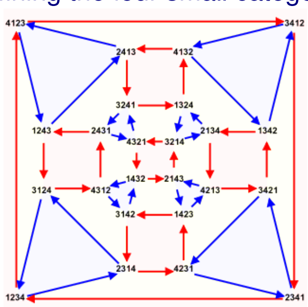
The morphism space $\mathscr{J}_G(V,W)$ is the Thom space of a certain vector bundle over the embedding space $O(V,W)$.

The Mandell-May category is symmetric monoidal under direct sum. This means that the functor category $\mathscr{S}p^G = [\mathscr{J}_G, \mathscr{T}^G]$, our category of equivariant spectra, is closed symmetric monoidal by the Day Convolution Theorem.

The projective model structure on $\mathscr{S}p^G$ is not equifibrant.

The positive Mandell-May category $\mathscr{J}_G^+$ is the full subcategory of representations V for which the invariant subspace V^G is nontrivial. 1.12

Defining the four small categories (continued)



$\widetilde{\mathscr{J}}_G$ is the equifibrant Mandell-May category. Its objects are finite dimensional orthogonal representations of finite G-sets. For a G-set T there is a category $\mathscr{B}_T G$ whose objects are the elements of T, and for each $(t,\gamma) \in T \times G$ there is a morphism that sends t to γt. This category is a split groupoid.

A representation V of T is a functor from $\mathscr{B}_T G$ to the category of finite dimensional real orthogonal vector spaces.

If $T = G/H$, such a functor is equivalent to an orthogonal representation of H. In general for each orbit of T we get a representation of its isotropy group. 1.13

Defining the four small categories (continued)

Recall that Mandell-May morphism objects involved orthogonal embeddings $V \hookrightarrow W$. An orthogonal embedding $f : (S,V) \to (T,W)$ consists of the following data.

- For each $t \in T$ an element $\overline{f}(t) \in S$ such that $\dim V_{\overline{f}(t)} \le \dim W_t$.
- For each $t \in T$ an orthogonal embedding $f_t : V_{\overline{f}(t)} \hookrightarrow W_t$.

We call the map $\overline{f} : T \to S$ a choice. It need not be equivariant. We say the embedding f is chosen by $\overline{f}$. For a given (S,V) and (T,W), there may be no choices.

Such orthogonal embeddings can be composed in an obvious way.

We denote the space of all such embeddings chosen by $\overline{f}$ by

$$O((S,V),(T,W))_{\overline{f}}.$$

It is a product of ordinary Stiefel manifolds. 1.14

Defining the four small categories (continued)

Given an orthogonal embedding

$$(S,V) \xrightarrow{\ f\ } (T,W),$$

the orthogonal complement $f^{\perp}$ of f is the direct sum of the orthogonal complements of $f_t(V_{\overline{f}(t)})$ in W_t. Using these direct sums as fibers, we get a vector bundle over the space $O((S,V),(T,W))_{\overline{f}}$ of embeddings chosen by $\overline{f}$. We denote its Thom space by

$$\widetilde{\mathscr{J}}_G((S,V),(T,W))_{\overline{f}}.$$

It is a smash product of ordinary Mandell-May morphism spaces.

The morphism object in $\widetilde{\mathscr{J}}_G$ is

$$\widetilde{\mathscr{J}}_G((S,V),(T,W)) := \bigvee_{\overline{f}:T\to S} \widetilde{\mathscr{J}}_G((S,V),(T,W))_{\overline{f}},$$

the one point union over all possible choices $\overline{f}$. 1.15

Defining the four small categories (continued)

The morphism object in $\widetilde{\mathscr{J}}_G$ is

$$\widetilde{\mathscr{J}}_G((S,V),(T,W)) := \bigvee_{\overline{f}:T\to S} \widetilde{\mathscr{J}}_G((S,V),(T,W))_{\overline{f}},$$

the one point union over all possible choices.

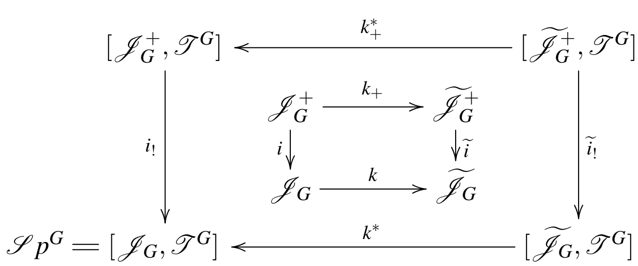
This category is symmetric monoidal under Cartesian product, so the functor category $[\widetilde{\mathscr{J}}_G, \mathscr{T}^G]$ is closed symmetric monoidal by the Day Convolution Theorem.

The ordinary Mandell-May category $\mathscr{J}_G$ is the full subcategory of $\widetilde{\mathscr{J}}_G$ with objects of the form $(G/G,V)$.

The positive equifibrant Mandell-May category $\widetilde{\mathscr{J}}_G^+$ is the full subcategory with objects (T,V) in which the representation for each orbit of T has a nontrivial invariant vector. 1.16

6 Summary

The main construction again



(i) Start with the projective model structure on $[\widetilde{\mathscr{J}}_G^+, \mathscr{T}^G]$.

(ii) Use Kan's theorem to transfer it to a model structure on $\mathscr{S}p^G$. This is the positive equifibrant model structure.

(iii) Expand the class of weak equivalences on $\mathscr{S}p^G$ to that of stable equivalences and apply Bousfield localization. The result is the positive stable equifibrant model structure. The positivity condition enables us to define a model structure on the category of equivariant commutative ring spectra.

1.17

THANK YOU!

1.18