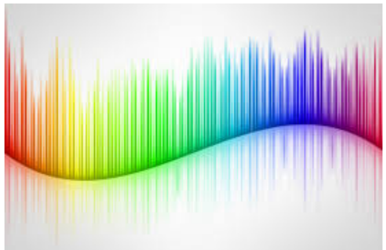
What is a G-spectrum?

Lehigh University Geometry
and Topology Conference

May 22, 2015



Mike Hill
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1.1

1 Introduction

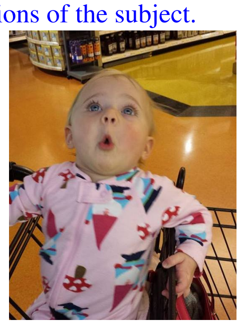
Introduction

Algebraic topologists have been studying spectra for over 50 years and G-spectra for over 30 years.

The basic definitions have changed several times, yet our intuition about spectra has not.

We have made extensive calculations with them from the very beginning. None of these have been affected in the least by the changing foundations of the subject.

This is a peculiar state of affairs!



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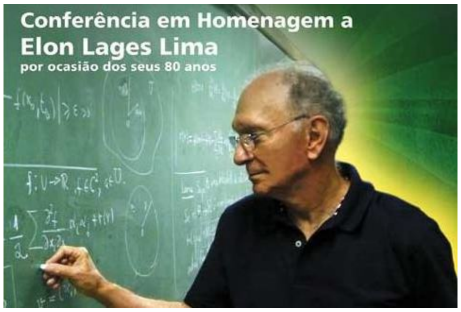
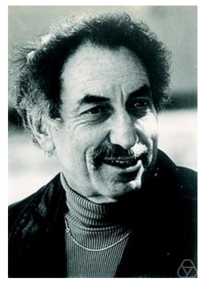
Introduction (continued)



Spectra were first defined in a 1959 paper of Lima, who is now a very prominent mathematician in Brazil. He was a student of Spanier at the University of Chicago.



Ed Spanier
1921-1996

Introduction (continued)

George Whitehead
1918-2004

GENERALIZED HOMOLOGY THEORIES(1)

BY

GEORGE W. WHITEHEAD

Here is the original definition in a 1962 paper by Whitehead, the earliest online reference I could find.

4. **Spectra**(2). A *spectrum* E is a sequence(3) $\{E_n \mid n \in Z\}$ of spaces together with a sequence of maps

$$\epsilon_n : SE_n \to E_{n+1}.$$

If E, E' are spectra, a map $f: E \to E'$ is a sequence of maps

$$f_n : E_n \to E_n'$$

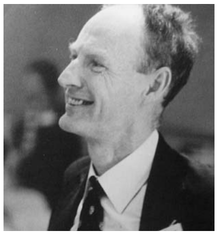
such that the diagrams

$$\begin{array}{ccc} SE_n & \xrightarrow{\epsilon_n} & E_{n+1} \\ Sf_n \downarrow & & \downarrow f_{n+1} \\ SE_n' & \xrightarrow{\epsilon_n'} & E_{n+1}' \end{array}$$

(3) By a *sequence* we shall always mean a function on *all* the integers.

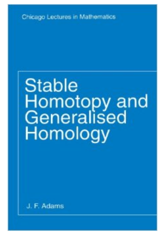
Introduction (continued)

This definition was adequate for many calculations over the next 20 years.



Frank Adams
1930-1989

It was used by Adams in his "blue book" of 1974.



The definition led to a lot of technical problems especially in connection with smash products. The definition we use today is more categorical. 1.5

Introduction (continued)

Some words you will not hear again in this talk:

- up to homotopy
- simplicial
- operad
- universe
- ∞-category
- chromatic
- Mackey functor
- slice spectral sequence

1.6

2 Categorical notions

2.1 Enrichment I

Some categorical notions: Enrichment, I

In a (locally small) category $\mathscr{C}$, for each pair of object X and Y, one has a set of morphisms $\mathscr{C}(X,Y)$. It sometimes happens that this set has a richer structure. Here are two examples.

(i) Let $\mathscr{A}b$ be the category of abelian groups. Then for abelian groups A and B, the set $\mathscr{A}b(A,B)$ of homomorphisms $A \to B$, is itself an abelian group. Composition of morphisms $A \to B \to C$ induces a map $\mathscr{A}b(B,C) \otimes \mathscr{A}b(A,B) \to \mathscr{A}b(A,C)$.

(ii) Let $\mathscr{T}$ be the category of pointed compactly generated weak Hausdorff spaces. Then for such spaces X and Y, the set $\mathscr{T}(X,Y)$ of pointed continuous maps $X \to Y$, is itself a pointed space under the compact open topology, the base point being the constant map. Here composition leads to a map $\mathscr{T}(X,Y) \wedge \mathscr{T}(W,X) \to \mathscr{T}(W,Y)$. (From now on, all topological spaces will be assumed to be compactly generated weak Hausdorff.)

We say that both of these categories are enriched over themselves. 1.7

Some categorical notions: Enrichment, I (continued)

Let G be a finite group. There are two categories whose objects are pointed G-spaces, where the base point is always fixed by G, because there are two types of morphisms to consider.

(i) Let $\mathscr{T}^G$ denote the category of pointed G-spaces and equivariant continuous pointed maps. Then $\mathscr{T}^G(X,Y)$ is a pointed topological space, so $\mathscr{T}^G$ is enriched over $\mathscr{T}$.

(ii) Let $\mathscr{T}_G$ denote the category of pointed G-spaces and all (not necessarily equivariant) continuous pointed maps. Then $\mathscr{T}_G(X,Y)$ is a pointed G-space. For $f : X \to Y$ and $\gamma \in G$, we define $\gamma(f) = \gamma f \gamma^{-1}$, the lower composite map in the noncommutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\gamma^{-1}"'] & Y \\ X \arrow[r, "f"] & Y. \arrow[u, "\gamma"'] \end{tikzcd}$$

$\mathscr{T}_G$ is enriched $\mathscr{T}^G$ and hence over itself. $\mathscr{T}_G(X,Y)^G = \mathscr{T}^G(X,Y)$. 1.8

2.2 Symmetric monoidal categories

Symmetric monoidal categories

A symmetric monoidal category is a category $\mathscr{V}$ equipped with a map $\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ with natural associativity isomorphisms $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, natural symmetry isomorphisms $X \otimes Y \to Y \otimes X$ and a unit object 1 with unit isomorphisms $\iota_X : 1 \otimes X \to X$. We will denote this structure by $(\mathscr{V}, \otimes, 1)$, surpressing the required isomorphisms from the notation.

The monoidal structure is closed if the functor $A \otimes (\cdot)$ has a right adjoint $(\cdot)^A$, the internal Hom with $\mathscr{V}(1, X^A) = \mathscr{V}(A, X)$. 1.9

Symmetric monoidal categories (continued)

Here are some familiar examples:

(i) $(\mathscr{S}ets, \times, *)$, the category of sets under Cartesian product, where the unit is a set $*$ with one element.

(ii) $(\mathscr{A}b, \otimes, \mathbf{Z})$, the category of abelian groups under tensor product, with the integers $\mathbf{Z}$ as unit.

(ii) $(\mathscr{A}b, \oplus, 0)$, the category of abelian groups under direct sum, with the trivial group as unit.

(iv) $(\mathscr{T}op, \times, *)$, the category of topological spaces (without base point) under Cartesian product with the one point space $*$ as unit.

(v) $(\mathscr{T}_G, \wedge, S^0)$, the category of pointed G-spaces and nonequivariant maps under smash product with the 0-sphere S^0 as unit.

(vi) $(\mathscr{T}^G, \wedge, S^0)$, the category of pointed G-spaces and equivariant maps under smash product with S^0 as unit.

1.10

2.3 Enrichment II

Enrichment II

The following definitions were first published by Eilenberg-Kelly in 1966.

Sammy Eilenberg
1913-1998

Max Kelly
1930-2007

Let $\mathscr{V} = (\mathscr{V}_0, \otimes, 1)$ be a symmetric monoidal category. A $\mathscr{V}$-category $\mathscr{C}$ (or a category enriched over $\mathscr{V}$) has a collection of objects $ob(\mathscr{C})$ and for each pair of objects X, Y an object $\mathscr{C}(X,Y)$ in $\mathscr{V}_0$, instead of a morphism set.

For each object X in $\mathscr{C}$ we have a morphism $1 \to \mathscr{C}(X,X)$ in $\mathscr{V}_0$ instead of an identity morphism. For each triple of objects X,Y,Z in $\mathscr{C}$, we have composition morphism $\mathscr{C}(Y,Z) \otimes \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$ in $\mathscr{V}_0$. 1.11

Enrichment II (continued)

A $\mathscr{V}$-category $\mathscr{C}$ is underlain by an ordinary category $\mathscr{C}_0$ having the same objects as $\mathscr{C}$ and morphism sets $\mathscr{C}_0(X,Y) = \mathscr{V}_0(1, \mathscr{C}(X,Y))$.

A functor $F : \mathscr{C} \to \mathscr{D}$ between $\mathscr{V}$-categories consists of a function $F : ob(\mathscr{C}) \to ob(\mathscr{D})$, and for each pair of objects X and Y in $\mathscr{C}$, a morphism $\mathscr{C}(X,Y) \to \mathscr{D}(FX,FY)$ in $\mathscr{V}_0$ satisfying suitable naturality conditions.

A symmetric monoidal category is closed iff it is enriched over itself.

When $\mathscr{V} = (\mathscr{T}, \wedge, S^0)$, we say, $\mathscr{C}$ is a topological category.

When $\mathscr{V} = (\mathscr{T}^G, \wedge, S^0)$, we say, $\mathscr{C}$ is a topological G-category. It is also enriched over $\mathscr{T}_G$, since $\mathscr{T}_G$ has the same objects as $\mathscr{T}^G$, and more morphisms. 1.12

3 The main definition

The definition of a G-spectrum

We will define spectra as functors to $\mathscr{T}_G$ from a certain indexing category $\mathscr{J}_G$. Both are topological G-categories.

Definition. *The indexing category $\mathscr{J}_G$ is the topological G-category whose objects are finite dimensional real orthogonal representations V of G. Let $O(V,W)$ denote the Stiefel manifold of (possibly nonequivariant) orthogonal embeddings $V \to W$. For each such embedding we have an orthogonal complement $W - V$, giving us a vector bundle over $O(V,W)$. The morphism object $\mathscr{J}_G(V,W)$ is its Thom space, which is a pointed G-space.*

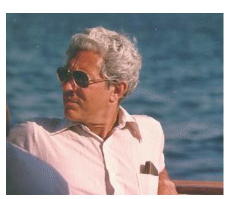
Informally, $\mathscr{J}_G(V,W)$ is a wedge of spheres S^{W-V} (where $W - V$ denotes the orthogonal complement of V embedded in W) parametrized by the orthogonal embeddings $V \to W$.

Main Definition. *An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathscr{T}_G$. We will denote its value on V by E_V.* 1.13

The definition of a G-spectrum (continued)

Main Definition. *An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathscr{T}_G$. We will denote its value on V by E_V.*

Mike Mandell



Peter May

This definition is due to Mandell–May and can be found in their book, *Equivariant orthogonal spectra and S-modules*, 2002.

There are similar definitions by other authors, such as that of symmetric spectra by Jeff Smith *et al* in 2000, in which $\mathscr{J}_G$ is replaced by other symmetric monoidal categories. 1.14

The definition of a G-spectrum (continued)

Main Definition. *An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathscr{T}_G$. We will denote its value on V by E_V.*

This definition requires some unpacking!

First we examine the indexing spaces $\mathscr{J}_G(V,W)$.

- When $dim(V) > dim(W)$, the embedding space $O(V,W)$ is empty, so $\mathscr{J}_G(V,W) = *$.
- When $dim(V) = dim(W)$, the vector bundle is 0-dimensional, so $\mathscr{J}_G(V,W) = O(V,W)_+$, the orthogonal group (equipped with a G-action) with a disjoint base point.
- When $dim(V) = 0$, the embedding space is a point, so $\mathscr{J}_G(0,W) = S^W$, the one point compactification of W.
- When $dim(V) = 1$, the embedding space is the unit sphere $S(W)$, and $\mathscr{J}_G(V,W)$ is its tangent Thom space.

1.15

The definition of a G-spectrum (continued)

Main Definition. *An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathscr{T}_G$. We will denote its value on V by E_V.*

There are equivariant structure maps $\mathscr{J}_G(V,W) \wedge \mathscr{J}_G(U,V) \to \mathscr{J}_G(U,W)$ (composition in $\mathscr{J}_G$)
$\oplus : \mathscr{J}_G(V,W) \wedge \mathscr{J}_G(V',W') \to \mathscr{J}_G(V \oplus V', W \oplus W')$
and $\varepsilon_{V,W} : \mathscr{J}_G(V,W) \wedge E_V \to E_W$.

In particular, $\mathscr{J}_G(U,V)$ and E_V each have a base point preserving left action of the orthogonal group $O(V) = O(V,V)$, and $\mathscr{J}_G(V,W)$ has a right $O(V)$-action.

The structure map $\varepsilon_{V,W}$ factors through the orbit space $\mathscr{J}_G(V,W) \underset{O(V)}{\wedge} E_V$. When $dim(V) = dim(W)$, this space equivariantly homeomorphic to E_W. This means that a G-spectrum E is determined by its values on vector spaces V with trivial G-action. We will come back to this later.

1.16

3.1 Comparison with the original definition

Comparison with the original definition

Main Definition. *An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathscr{T}_G$. We will denote its value on V by E_V.*

For trivial G we have a functor $\mathscr{J} \to \mathscr{T}$, where $\mathscr{J}$ is the topological category of finite dimensional orthogonal vector spaces with morphism spaces as before.

Such vector spaces are determined by their dimensions, so we study the structure map $\varepsilon_{n,n+1} : \mathscr{J}(n,n+1) \wedge E_n \to E_{n+1}$, which factors through $\mathscr{J}(n,n+1) \underset{O(n)}{\wedge} E_n$. We want to compare this with Whitehead's structure map $\varepsilon_n : S^1 \wedge E_n \to E_{n+1}$.

The latter is based on a previously chosen orthogonal embedding $\mathbf{R}^n \to \mathbf{R}^{n+1}$. Mandell-May's $\varepsilon_{n,n+1}$ amounts to a family of maps $S^1 \wedge E_n \to E_{n+1}$ parameterized by all such embeddings. This coordinate free approach is technically convenient.

1.17

4 Simple examples

4.1 Spaces and spectra

Smash products with spaces and the sphere spectrum

Given a G-spectrum E and a pointed G-space X, we can define a spectrum $E \wedge X$ by $(E \wedge X)_V = E_V \wedge X$. We will define the smash product of two spectra shortly. We can also define a spectrum $F_G(X,E)$ by $F_G(X,E)_V = \mathscr{T}_G(X, E_V)$. For $X = S^W$, these spectra also denoted by $\Sigma^W E$ and $\Omega^W E$.

We can also define limits and colimits object wise,

$$(\lim_{\rightarrow} E^\alpha)_V = \lim_{\rightarrow}(E_V^\alpha) \quad \text{and} \quad (\lim_{\leftarrow} E^\alpha)_V = \lim_{\leftarrow}(E_V^\alpha).$$

We will denote the sphere spectrum by S^{-0} to avoid confusion with the space S^0. It is defined by $(S^{-0})_V = S^V$ with structure map induced by composition in $\mathscr{J}_G$

$$\mathscr{J}_G(V,W) \wedge S^V = \mathscr{J}_G(V,W) \wedge \mathscr{J}_G(0,V) \to \mathscr{J}_G(0,W) = S^W.$$

For a pointed G-space X, the suspension spectrum $\Sigma^\infty X$ is $S^{-0} \wedge X$. 1.18

4.2 The spectrum S^{-V}

The spectrum S^{-V}

We define the spectrum S^{-V} by $(S^{-V})_W = \mathscr{J}_G(V,W)$. We have structure maps $j_{V,W} : S^{-W} \wedge \mathscr{J}_G(V,W) \to S^{-V}$ induced by composition in $\mathscr{J}_G$.

Let $\mathscr{S}_G$ denote the category of orthogonal G-spectra. Since its objects are functors $\mathscr{J}_G \to \mathscr{T}_G$, its morphisms are natural transformations between such functors. It is a topological G-category.

One can use the enriched Yoneda lemma to show that $\mathscr{S}_G(S^{-V}, E) = E_V$. In particular,

$$\mathscr{S}_G(S^{-0}, E) = E_0 = \mathscr{T}_G(S^0, E_0) = \mathscr{T}_G(S^0, \Omega^\infty E),$$

where the 0th space functor Ω^∞ sends a spectrum E to the space E_0. For a pointed G-space X we have

$$\mathscr{S}_G(\Sigma^\infty X, E) = \mathscr{S}_G(S^{-0} \wedge X, E) = \mathscr{T}_G(X, \Omega^\infty E),$$

so the functors $\Sigma^\infty : \mathscr{T}_G \to \mathscr{S}_G$ and $\Omega^\infty : \mathscr{S}_G \to \mathscr{T}_G$ are adjoint. 1.19

4.3 Naive G-spectra

Naive G-spectra

An ordinary orthogonal spectrum is a functor $\mathscr{J} \to \mathscr{T}$. Since $\mathscr{J}$ is a full subcategory of $\mathscr{J}_G$, an orthogonal G-spectrum induces a functor $\mathscr{J} \to \mathscr{T}_G$. This amounts to an ordinary spectrum equipped with a G-action, and is called a naive G-spectrum. We denote the corresponding category by $\mathscr{S}_G^{naive}$. A functor on $\mathscr{J}_G$ is sometimes called a genuine G-spectrum.

As noted above, a functor on $\mathscr{J}_G$ is determined by its value on $\mathscr{J}$. It can be shown that the categories of naive and genuine G-spectra are equivalent. However their homotopy theories are different. The category $\mathscr{S}_G$ has more weak equivalences than $\mathscr{S}_G^{naive}$. We will give an explicit example of this below if time permits.

Nevertheless, the categorical equivalence is useful for certain definitions. 1.20

4.4 Change of group

Fixed point spectra and change of group

The fixed point spectrum E^G of G-spectrum E is the ordinary spectrum (functor on $\mathscr{J}$) E^G defined by $(E^G)_n = (E_n)^G$.

For a subgroup $H \subseteq G$, there are forgetful functors $\mathscr{T}_G \to \mathscr{T}_H$ and $\mathscr{J}_G \to \mathscr{J}_H$. The latter is not surjective on objects since not every representation of H is the restriction of a representation on G. Hence these forgetful functors do not lead directly to one from the category of G-spectra $\mathscr{S}_G$ to the category of H-spectra $\mathscr{S}_H$.

However we do get a forgetful functor $\mathscr{S}_G^{naive} \to \mathscr{S}_H^{naive}$ since both are functor categories on $\mathscr{J}$. Then we can use the categorical equivalance of naive and genuine G (or H)-spectra to get the desired forgetful functor $i_H^G : \mathscr{S}_G \to \mathscr{S}_H$. 1.21

Change of group (continued)

The forgetful functor $i_H^G : \mathscr{S}_G \to \mathscr{S}_H$ has a left adjoint (induction) sending an H-spectrum E to the G-spectrum $G_+ \underset{H}{\wedge} E$, defined objectwise by

$$(G_+ \underset{H}{\wedge} E)_V = G_+ \underset{H}{\wedge} (E_{Res_H^G V}).$$

This may be written as a wedge indexed by the G-set G/H,

$$G_+ \underset{H}{\wedge} E = \bigvee_{i \in G/H} E_i \qquad \text{where } E_i = (H_i)_+ \underset{H}{\wedge} E$$

with $H_i \subseteq G$ the coset indexed by i. 1.22

Change of group (continued)

There is a similar construction with the smash product,

$$N_H^G E := \bigwedge_{i \in G/H} E_i \qquad \text{with } E_i \text{ as above,}$$

the norm of the H-spectrum E.

In proving the Kervaire invariant theorem we used this for $H = C_2$, $G = C_8$ and $E = MU_{\mathbf{R}}$. 1.23

5 The smash product

The tautological presentation and smash product

Any spectrum E is the reflexive coequalizer (i.e., the colimit) of the diagram

$$\bigvee_{V,W} S^{-W} \wedge \mathscr{J}_G(V,W) \wedge E_V \underset{S^{-W} \wedge \varepsilon_{V,W}}{\overset{j_{V,W} \wedge E_V}{\rightrightarrows}} \bigvee_V S^{-V} \wedge E_V$$

This is the tautological presentation of E. We abbreviate it by

$$\lim_{\overrightarrow{V}} S^{-V} \wedge E_V.$$

1.24

The tautological presentation and smash product (continued)

$$E = \varinjlim_{V} S^{-V} \wedge E_V.$$

Similarly we define the smash product of two spectra E and F by

$$E \wedge F = \varinjlim_{V,V'} S^{-V\oplus V'} \wedge E_V \wedge F_{V'},$$

the reflexive coequalizer of

$$\begin{array}{c} \bigvee_{V,V',W,W'} S^{-W\oplus W'} \wedge \mathscr{J}_G(V,W) \wedge \mathscr{J}_G(V',W') \wedge E_V \wedge F_{V'} \\ \downarrow\downarrow \\ \bigvee_{V,V'} S^{-V\oplus V'} \wedge E_V \wedge F_{V'} = \bigvee_{W,W'} S^{-W\oplus W'} \wedge E_W \wedge F_{W'}, \end{array}$$

which makes use of the map $\oplus : \mathscr{J}_G(V,W) \wedge \mathscr{J}_G(V',W') \to \mathscr{J}_G(V \oplus V', W \oplus W')$. 1.25

The tautological presentation and smash product (continued)

We want to say that the smash product as defined above makes $\mathscr{S}_G$ into a closed symmetric monoidal category with unit S^{-0}. This would mean that it is strictly associative and commutative, thereby solving decades of technical problems in stable homotopy theory!

It turns out that this is purely formal. We are looking at the category of functors from the (skeletally) small symmetric monoidal category $(\mathscr{J}_G, \oplus, 0)$ to the cocomplete closed symmetric monoidal category $(\mathscr{T}_G, \wedge, S^0)$. Both are topological G-categories and hence enriched over the target category $\mathscr{T}_G$. 1.26

The tautological presentation and smash product (continued)

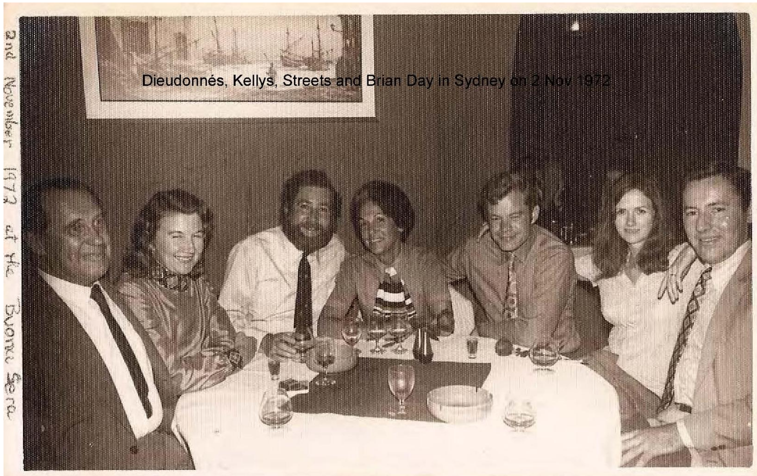
In 1970 the Australian category theorist Brian Day (1945-2012), a student of Max Kelly, studied this very problem. He defined a symmetric monoidal structure on the category of functors ($\mathscr{S}_G$ in our case) between two symmetric monoidal categories as above. It is called the Day convolution. It can be described as a left Kan extension.

Jeff Smith

Its relevance to spectra was first noticed by Jeff Smith in the 1990s.

(The symmetric monoidal structure on the category of spectra first discovered by Elmendorf, Kriz, Mandell and May (1997) is not of this type.) 1.27

The tautological presentation and smash product (continued)



Jean Dieudonné, Imogene Kelly, Max Kelly, Odette Dieudonné, Brian Day, Margery Street and Ross Street at a restaurant in Sydney in 1972

1.28

6 Homotopy theory

Homotopy theory of G-spectra

To do homotopy theory in $\mathscr{S}_G$, we need to define a weak equivalence of G-spectra. First we need to know how to recognize an equivariant homotopy equivalence of G-spaces.

Glen Bredon
1932-2000

A theorem of Bredon (1967) states that a map of G-CW-complexes $f : X \to Y$ is an equivariant homotopy equivalence (meaning an equivalence for which the homotopies are equivariant) iff the induced maps $X^H \to Y^H$ of fixed point sets are ordinary homotopy equivalences for all subgroups $H \subseteq G$. Fixed point maps tell all!

1.29

Homotopy theory of G-spectra (continued)

For a pointed G-space X, let $\pi_*^H X = \pi_* X^H$. Bredon's theorem leads us to define a weak equivalence of G-spaces to be an equivariant map $f : X \to Y$ inducing an isomorphism $\pi_*^H X \to \pi_*^H X$ for all H.

What about weak equivalences of spectra? Experience has shown that for a map $f : E \to E'$ of spectra, we do not want to require each map $E_V \to E'_V$ to be a weak equivalence. That would be far too rigid.

In the nonequivariant case we define $\pi_k E$ to be $\lim_{\to} \pi_{n+k} E_n$ and define a weak equivalence $f : E \to E'$ to be a map inducing an isomorphism in these homotopy groups. 1.30

Homotopy theory of G-spectra (continued)

In the nonequivariant case we define $\pi_k E$ to be $\lim_{\rightarrow} \pi_{n+k} E_n$, where the limit is over all $n \geq -k$, and define a weak equivalence $f : E \to E'$ to be a map inducing an isomorphism in these homotopy groups.

In the equivariant case we will replace the colimit above by one indexed by a family of orthogonal inclusions

$$V_0 \to V_1 \to V_2 \to V_3 \to \cdots$$

which is exhaustive, meaning that each V is contained in some V_n.

We define $\pi_k^H E$ to be $\lim_{\rightarrow} \pi_{k+V_n}^H E_{V_n}$, and define a weak equivalence of G-spectra to be a map $f : E \to E'$ inducing an isomorphism in π_k^H for all subgroups $H \subseteq G$ and all integers k. 1.31

Homotopy theory (continued)

This definition of weak equivalence leaves a lot of wiggle room. For example, in a G-spectrum E one could alter the G-spaces E_V arbitrarily for small V without changing the weak homotopy type of E.

CAUTION! Many functors one would like to use are not homotopical, meaning they do not convert weak equivalances to weak equivalences. They are not homotopically meaningful. For example, the functor $\mathscr{S}_G(S^{-V}, \cdot)$, which sends E to E_V, is not homotopical. It turns out that fixed points and symmetric products also fail to be homotopical.

This can lead to a lot of technical problems!

1.32

6.1 Quillen model structures

Quillen model category structures

A way out of this difficulty is to define a Quillen model category structure on $\mathscr{S}_G$ and related categories. This leads to two special collections of G-spectra, the fibrant and cofibrant ones. Each G-spectrum then comes equipped with a canonical weak equivalence to (from) a fibrant (cofibrant) one, called its fibrant (cofibrant) replacement.

Dan Quillen
1940-2011

Then it may happen that the functors one wants to use do preserve weak equivalences among either fibrant or cofibrant objects, depending on the nature of the functor. 1.33

Quillen model category structures (continued)

In the usual model structure on $\mathscr{T}$ (pointed spaces), the cofibrant objects are the CW-complexes, and all spaces are fibrant.

In any reasonable model category structure on $\mathscr{S}$ or $\mathscr{S}_G$, the fibrant objects are the Ω-spectra. One replaces each space E_W by the homotopy colimit (or mapping telescope) of

$$\Omega^{V_0} E_{W\oplus V_0} \to \Omega^{V_1} E_{W\oplus V_1} \to \Omega^{V_2} E_{W\oplus V_2} \to \cdots$$

for an exhaustive sequence $\{V_n\}$ as before.

Pete Bousfield

This observation (in the nonequivariant case) is due to Bousfield-Friedlander in a 1978 paper.

Eric Friedlander

1.34

6.2 A new model structure on $\mathscr{S}_G$

The positive complete model category structure on $\mathscr{S}_G$

One way to define a model category structure, once we know what the weak equivalences are, is to specify a set of generating cofibrations. For the classical model category structure on $\mathscr{T}$, it is

$$\left\{S^{n-1} \to D^n : n \geq 0\right\} \qquad \text{(inclusion of the boundary)}.$$

For the positive complete model category structure on $\mathscr{S}_G$ it is

$$\mathscr{A}_{cof} = \left\{G_+ \underset{H}{\wedge} S^{-W} \wedge (S_+^{n-1} \to D_+^n) : n \geq 0, H \subseteq G\right\}.$$

where W ranges over all representations of all subgroups H of G with $W^H \neq 0$. 1.35

The positive complete model category structure on $\mathscr{S}_G$ (continued)

In the positive complete model category structure on $\mathscr{S}_G$ the set of generating cofibrations is

$$\mathscr{A}_{cof} = \left\{G_+ \underset{H}{\wedge} S^{-W} \wedge (S_+^{n-1} \to D_+^n) : n \geq 0, H \subseteq G\right\}.$$

where W ranges over all representations of all subgroups H of G with $W^H \neq 0$.

The last requirement is the positivity condition of Jeff Smith. It is needed because the kth symmetric product functor does not convert the weak equivalence $S^{-1} \wedge S^1 \to S^{-0}$ into a weak equivalence. This issue came up around 2000 in the theory of symmetric spectra. We need a homotopically meaningful symmetric product functor to handle commutative ring spectra.

The positivity condition means the sphere spectrum S^{-0} is not cofibrant! Its cofibrant replacement is $S^{-1} \wedge S^1$. 1.36

The positive complete model category structure on $\mathscr{S}_G$ (continued)

In the positive complete model category structure on $\mathscr{S}_G$ the set of generating cofibrations is

$$\mathscr{A}_{cof} = \left\{ G_+ \underset{H}{\wedge} S^{-W} \wedge (S^{n-1}_+ \to D^n_+) \colon n \geq 0, H \subseteq G \right\}.$$

where W ranges over all representations of all subgroups H of G with $W^H \neq 0$.

The word "complete" refers to the use of representations of subgroups H as well as G itself. Completeness is needed to insure that certain fixed point functors preserve acyclic cofibrations. It also guarantees that wedges and smash products indexed by G-sets (such as the norm) of cofibrant objects are again cofibrant. 1.37

6.3 A counterexample

A counterexample: why we need genuine G-spectra

EXAMPLE. Let $G = C_2$ and let σ be the sign representation. We will show that there is a map $E := S^{-\sigma} \wedge S^{\sigma} \to S^{-0} =: F$ which is a weak equivalence in $\mathscr{S}_G$ but **NOT** in $\mathscr{S}_G^{naive}$.

The map $f : E \to F$ is defined by $f_V = \varepsilon_{\sigma,V} : \mathscr{J}_G(\sigma, V) \wedge S^{\sigma} \to S^V$, the structure map for S^{-0}.

For $G = C_2$, each V has the form $m\sigma \oplus n$ for integers $m, n \geq 0$. We have $\mathscr{J}_G(a\sigma \oplus b, c\sigma \oplus d)^G = O(a,c)_+ \wedge \mathscr{J}(b,d)$. In particular it is a point if $a > c$ or $b > d$.

Working in $\mathscr{S}_G^{naive}$, we have $E_n = \mathscr{J}_G(\sigma, n) \wedge S^{\sigma}$, so $E_n^G = *$ for all n, and $\pi_*^G E = 0$. On the other hand, $F_n = S^n$ with trivial G-action, so $\pi_*^G F$ is nontrivial. This means that E and F are homotopically distinct as naive G-spectra. 1.38

A counterexample: why we need genuine G-spectra (continued)

EXAMPLE. Let $G = C_2$ and let σ be the sign representation. We will show that there is a map $E := S^{-\sigma} \wedge S^{\sigma} \to S^{-0} =: F$ which is a weak equivalence in $\mathscr{S}_G$ but **NOT** in $\mathscr{S}_G^{naive}$.

In $\mathscr{S}_G$, we have $E_{m\sigma\oplus n} = \mathscr{J}_G(\sigma, m\sigma \oplus n) \wedge S^{\sigma}$, so

$$\begin{aligned} E^G_{m\sigma\oplus n} &= O(1,m)_+ \wedge S^n = S^n \vee S^{m+n-1} && \text{for } m > 0 \\ F^G_{m\sigma\oplus n} &= (S^{m\sigma\oplus n})^G = S^n, \end{aligned}$$

and the map f induces an isomorphism in π_*^G.

The map underlying $f_{m\sigma\oplus n} = \varepsilon_{\sigma, m\sigma\oplus n}$ has the form $\mathscr{J}(1, m+n) \wedge S^1 \to S^{m+n}$. Recall that $\mathscr{J}(1, m+n)$ is the tangent Thom space for S^{m+n-1}, so its suspension is the Thom space for the trivial $\mathbf{R}^{m+n}$-bundle over S^{m+n-1}, which is equivalent to $S^{m+n} \vee S^{2(m+n)-1}$. It follows that f also induces an isomorphism in π_* and is therefore a weak equivalence. 1.39

Happy Birthday Don!

1.40