ECHT Minicourse

What is the telescope conjecture?

Lecture 4

$\nu_h$-periodic families and telescopes

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December 14, 2023
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Greek letter elements
Type $h$ finite complexes
The telescope conjecture
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- The discovery in the early 70s of periodic families known as Greek letter elements.

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We have left out a motivating development in the stable homotopy groups of spheres: the discovery in the early 70s of periodic families known as Greek letter elements. We will describe these now.
Greek letter elements

Recall the $h$th Greek letter sequence,

$$0 \rightarrow \sum ^{|v_{h-1}|} BP_{\ast} / I_{h-1} \overset{v_{h-1}}{\rightarrow} BP_{\ast} / I_{h-1} \rightarrow BP_{\ast} / I_{h} \rightarrow 0.$$
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where $I_{h} = (p, v_{1}, \ldots, v_{h-1})$, $v_{0} = p$ and $I_{0} = (0)$. 
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where $I_h = (p, v_1, \ldots, v_{h-1})$, $v_0 = p$ and $I_0 = (0)$. It leads to a long exact sequence of Ext groups in which we denote the connecting homomorphism by $\delta_h$. 

For $p$ odd this represents an element of order $p$ in $\pi_{t|v_1| - 1} S$. 

For $t = 1$, this dimension is $2p - 3$, and $\alpha_1$ is the first positive dimensional element in the $p$-component of the stable homotopy groups of spheres. These $\alpha_t$s comprise a $v_1$-periodic family.
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$$\text{Ext}^0(BP_*) \cong \mathbb{Z}(p) \quad \text{and} \quad \text{Ext}^0(BP_* / I_h) \cong \mathbb{Z}/p[v_h]$$

for each $h > 0$. 
Recall the $h$th Greek letter sequence,

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These $\alpha_t$s comprise a $v_1$-periodic family.
Greek letter elements (continued)

To repeat, the \( \alpha \) sequence,

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Then the homotopy element $\alpha_t$ is the composite

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where $i$ is the inclusion of the bottom cell and $j$ is the pinch map onto the top cell. Again the $\alpha_t$'s comprise a $v_1$-periodic family.
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However, we can go only one step further geometrically, defining elements $\gamma_t$ for $p \geq 7$. Nobody knows how to construct a map

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inducing multiplication by $v_4$ in $BP_*(-)$ at any prime.
Finite complexes of type $h$

For a $p$-local finite spectrum $X$, we know that $K(h)_* X = 0$ implies $K(h - 1)_* X = 0$,
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Hence Toda’s $V_{h-1}$ has type $h$.

If $K(h)_* X = 0$ for all $h$, then $X$ is contractible.

The following was conjectured in [Rav84] and proved by Ethan Devinatz, Mike Hopkins and Jeff Smith in [DHS88].

Class Invariance Theorem

The Bousfield equivalence class of a $p$-local finite spectrum is determined by its type. In particular any $p$-local finite spectrum $X$ with nontrivial rational homology is Bousfield equivalent to $S(p)$. 

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For a $p$-local finite spectrum $X$, we know that $K(h)_* X = 0$ implies $K(h-1)_* X = 0$, and that $K(h)_* X \neq 0$ for $h \gg 0$ unless $X$ is contractible. We say that $X$ has type $h$ if $h$ is the smallest integer with $K(h)_* X \neq 0$. Hence Toda’s $V(h-1)$ has type $h$. If $K(h)_* X = 0$ for all $h$, then $X$ is contractible.

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In particular any $p$-local finite spectrum $X$ with nontrivial rational homology is Bousfield equivalent to $S(p)$. 

Finite complexes of type $h$ (continued)

A few years later in [HS98], Hopkins and Smith proved the following.

Periodicity Theorem

Let $X$ be a $p$-local finite spectrum of type $h$. Then there is a map $\nu : \Sigma^d X \to X$ for some $d > 0$ that induces an isomorphism in $K^*_h(-)$ and a nilpotent map in every other Morava K-theory. We call it a $\nu^h$-self-map.

This map is asymptotically unique in the following sense. Given a second such map $\nu' : \Sigma^{d'} X \to X$, there exist integers $e$ and $e'$ with $ed = e'd'$ and $\nu^e = (\nu')^{e'}$.

It follows that the cofiber $C_{\nu}$ has type $h+1$.

Hence we can produce finite spectra of all higher types by iterating this process. The Class Invariance theorem implies that the Bousfield class of the telescope $\nu^{-1} X$ is independent of the choices of both $X$ and $\nu$. We denote it by $\langle T(h) \rangle$. 
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What is the telescope conjecture?

Review
Greek letter elements
Type h finite complexes
The telescope conjecture
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The telescope conjecture
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Jeremy, Tomer, myself, Ishan and Robert at Oxford University, June 9, 2023.

Photo by Matteo Barucco.
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THANK YOU!
References

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith.
Nilpotence and stable homotopy theory. I.

Nilpotence and stable homotopy theory. II.

[Rav84] Douglas C. Ravenel.
Localization with respect to certain periodic homology theories.