ECHT Minicourse

What is the telescope conjecture?

Lecture 4

v_h-periodic families and telescopes

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1 Review

In the previous three lectures we described

- The algebraic machinery behind complex cobordism theory, in particular the theory of formal group laws, their classification and endomorphism rings in characteristic p in Lecture 1.
- The chromatic resolution in its algebraic form leading to the chromatic spectral sequence and the chromatic filtration of the Adams-Novikov E_2-term in Lecture 2.
- The geometric form of the chromatic resolution defined using Bousfield localization with respect to the theories $E(h)$ in Lecture 3.

We have left out a motivating development in the stable homotopy groups of spheres: the discovery in the early 70s of periodic families known as Greek letter elements. We will describe these now.

2 Greek letter elements

Recall the hth Greek letter sequence,

$$0 \longrightarrow \Sigma^{|v_{h-1}|}BP_*/I_{h-1} \xrightarrow{v_{h-1}} BP_*/I_{h-1} \longrightarrow BP_*/I_h \longrightarrow 0.$$

where $I_h = (p, v_1, \ldots, v_{h-1})$, $v_0 = p$ and $I_0 = (0)$. It leads to a long exact sequence of Ext groups in which we denote the connecting homomorphism by δ_h. We know

$$\mathrm{Ext}^0(BP_*) \cong \mathbb{Z}_{(p)} \quad \text{and} \quad \mathrm{Ext}^0(BP_*/I_h) \cong \mathbb{Z}/p[v_h]$$

for each $h > 0$. For each $t > 0$, we define

$$\alpha_t := \delta_1(v_1^t) \in \mathrm{Ext}^{1,t|v_1|}(BP_*).$$

For p odd this represents an element or order p in $\pi_{t|v_1|-1}\mathbb{S}$. For $t = 1$, this dimension is $2p-3$, and α_1 is the first positive dimensional element in the p-component of the stable homotopy groups of spheres.

These α_ts comprise a v_1-periodic family.
To repeat, the α sequence,

$$0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_*/(p) \longrightarrow 0.$$

enables us to define
$$\alpha_t := \delta_1(v_1^t) \in \mathrm{Ext}^{1,t|v_1|}(BP_*)$$
This algebraic construction has a geometric antecedent.

Let $V(0)$ the cofiber of the degree p map of the sphere spectrum. Adams showed that for p odd, there is a map

$$\Sigma^{2p-2}V(0) \xrightarrow{\alpha} V(0)$$

inducing multiplication by v_1.

Then the homotopy element α_t is the composite

$$S^{t|v_1|} \xrightarrow{i} \Sigma^{t|v_1|}V(0) \xrightarrow{\alpha^t} V(0) \xrightarrow{j} S^1,$$

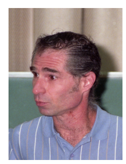
where i is the inclusion of the bottom cell and j is the pinch map onto the top cell. Again the α_ts comprise a v_1-periodic family.

We can construct a v_2-periodic family as follows. Let $V(1)$ be the cofiber of the Adams map

$$\Sigma^{2p-2}V(0) \xrightarrow{\alpha} V(0).$$

inducing multiplication by v_1. It is a CW-spectrum of the form
$$V(1) = S^0 \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}.$$

Independently Larry Smith and Hirosi Toda showed that for $p \geq 5$, there is a map



$$\Sigma^{|2p^2-2|}V(1) \xrightarrow{\beta} V(1)$$

inducing multiplication by v_2 in $BP_*(-)$.

Then the element
$$\beta_t := \delta_1\delta_2 v_2^t \in \mathrm{Ext}^{2,t|v_2|-|v_1|}(BP_*)$$
is represented by the composite

$$S^{t|v_2|} \xrightarrow{i} \Sigma^{t|v_2|}V(1) \xrightarrow{\beta^t} V(1) \xrightarrow{j} S^{2p}.$$

Algebraically we can do a similar thing at all heights and at all primes. We can define

$$\eta_t^{(h)} := \delta_1\delta_2 \dots \delta_h(v_h^t) \in \mathrm{Ext}^{h,t|v_h|-w_h}(BP_*)$$

where $\eta^{(h)}$ denotes the hth letter of the Greek alphabet and $w_h = |v_1| + \cdots + |v_{h-1}|$.

However, we can go only one step further geometrically, defining elements γ_t for $p \geq 7$. Nobody knows how to construct a map
$$\Sigma^{|2p^4-2|}V(3) \xrightarrow{\delta} V(3)$$
inducing multiplication by v_4 in $BP_*(-)$ at any prime.

3 Type h finite complexes

For a p-local finite spectrum X, we know that $K(h)_*X = 0$ implies $K(h-1)_*X = 0$, and that $K(h)_*X \neq 0$ for $h \gg 0$ unless X is contractible. We say that X has type h if h is the smallest integer with $K(h)_*X \neq 0$. Hence Toda's $V(h-1)$ has type h. If $K(h)_*X = 0$ for all h, then X is contractible.

The following was conjectured in [Rav84] and proved by Ethan Devinatz, Mike Hopkins and Jeff Smith in [DHS88].

Class Invariance Theorem. *The Bousfield equivalence class of a p-local finite spectrum is determined by its type.*

In particular any p-local finite spectrum X with nontrivial rational homology is Bousfield equivalent to $\mathbb{S}_{(p)}$.

A few years later in [HS98], Hopkins and Smith proved the following.

Periodicity Theorem. *Let X by p-local finite spectrum of type h. Then there is a map $v : \Sigma^d X \to X$ for some $d > 0$ that induces an isomorphism in $K(h)_*(-)$ and a nilpotent map in every other Morava K-theory. We call it a v_h self-map.*

This map is asymptotically unique in the following sense. Given a second such map $v' : \Sigma^{d'} X \to X$, there exist integers e and e' with $ed = e'd'$ and $v^e = (v')^{e'}$.

If follows that the cofiber C_v has type $h+1$. Hence we can produce finite spectra of all higher types by iterating this process. The Class Invariance theorem implies that the Bousfield class of the telescope $v^{-1}X$ is independent of the choices of both X and v. We denote it by $\langle T(h) \rangle$.

4 The telescope conjecture

Periodicity Theorem. *Let X by p-local finite spectrum of type h. Then there is a map $v : \Sigma^d X \to X$ for some $d > 0$ that induces an isomorphism in $K(h)_*(-)$. We call it a v_h self-map.*

The map $X \to v^{-1}X$ is a $K(h)_*$- equivalence, so we have a map

$$\lambda : v^{-1}X \to L_{K(h)}X = L_h X,$$

where the equality holds because the lower Morava K-theories vanish on X. The following appeared in [Rav84].

Telescope Conjecture. *The map*

$$\lambda : v^{-1}X \to L_{K(h)}X$$

is an equivalence.

This is trivially true for $h = 0$, and for $h = 1$ it was proved around 1980 by Mahowald for $p = 2$ and by Miller for p odd. In 1989 I began to think it was false for $h \geq 2$. This is now a theorem of Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank.

Jeremy, Tomer, myself, Ishan and Robert at Oxford University, June 9, 2023.
Photo by Matteo Barucco.

Telescope Conjecture. *The map*

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is an equivalence.

This conjecture equated the geometrically interesting object $v^{-1}X$, the v_h-periodic telescope associated with the type h finite complex X, with the more computationally accessible spectrum $L_{K(h)}X$.

For example, we know how to compute $\pi_* L_{K(2)} V(1)$ for $p \geq 5$, where $V(1)$ is Toda's 4-cell complex. It consists of exactly 12 v_2-periodic families.

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We do not know $\pi_* v_2^{-1} V(1)$, which is likely to be much larger. There are possibly infinitely many such families not detected by the localized Adams-Novikov spectral sequence, which is known to converge to $\pi_* L_{K(2)} V(1)$, but not to $\pi_* v_2^{-1} V(1)$.

Meanwhile the ordinary Adams-Novikov spectral sequence does converge to $\pi_* V(1)$ but only sees 12 v_2-periodic families there. How can this be? One could have a v_2-periodic family (or many of them) that are spread out over infinitely many Adams-Novikov filtrations.

References

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

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[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.