1 Review

In the previous three lectures we described

- The algebraic machinery behind complex cobordism theory, in particular the theory of formal
group laws, their classification and endomorphism rings in characteristic $p$ in Lecture 1.

- The chromatic resolution in its algebraic form leading to the chromatic spectral sequence and
the chromatic filtration of the Adams-Novikov $E_2$-term in Lecture 2.

- The geometric form of the chromatic resolution defined using Bousfield localization with re-
spect to the theories $E(h)$ in Lecture 3.

We have left out a motivating development in the stable homotopy groups of spheres: the dis-
covery in the early 70s of periodic families known as Greek letter elements. We will describe these
now.

2 Greek letter elements

Recall the $h$th Greek letter sequence,

\[ 0 \longrightarrow \Sigma^{v_{h-1}} \mathcal{B}_p \longrightarrow \mathcal{B}_p/I_{h-1} \longrightarrow \mathcal{B}_p/I_h \longrightarrow 0. \]

where $I_h = (p, v_1, \ldots, v_{h-1}), v_0 = p$ and $I_0 = (0)$. It leads to a long exact sequence of Ext groups
in which we denote the connecting homomorphism by $\delta_h$. We know

\[ \text{Ext}^0(\mathcal{B}_p) \cong \mathbb{Z}/(p) \quad \text{and} \quad \text{Ext}^0(\mathcal{B}_p/I_h) \cong \mathbb{Z}/p[v_h] \]

for each $h > 0$. For each $t > 0$, we define

\[ \alpha_{t} := \delta_1(v'_1) \in \text{Ext}^{1/|v'_1|}(\mathcal{B}_p). \]

For $p$ odd this represents an element or order $p$ in $\pi_{|v'_1| - 1} S$. For $t = 1$, this dimension is $2p - 3$,
and $\alpha_1$ is the first positive dimensional element in the $p$-component of the stable homotopy groups
of spheres.
These $\alpha_i$s comprise a $v_1$-periodic family.  
To repeat, the $\alpha$ sequence,

$$
0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_*/(p) \longrightarrow 0.
$$

enables us to define

$$
\alpha_i := \delta_1(v_1^i) \in \text{Ext}^1_{BP_*}(BP_*/(p)).
$$

This algebraic construction has a geometric antecedent.

Let $V(0)$ the cofiber of the degree $p$ map of the sphere spectrum. Adams showed that for $p$ odd, there is a map

$$
\Sigma^{2p-2}V(0) \xrightarrow{\alpha} V(0)
$$

inducing multiplication by $v_1$.

Then the homotopy element $\alpha_i$ is the composite

$$
S^{|v_1|} \xrightarrow{i} \Sigma^{|v_1|}V(0) \xrightarrow{\alpha^i} V(0) \xrightarrow{j} S^1,
$$

where $i$ is the inclusion of the bottom cell and $j$ is the pinch map onto the top cell. Again the $\alpha_i$s comprise a $v_1$-periodic family.

We can construct a $v_2$-periodic family as follows. Let $V(1)$ be the cofiber of the Adams map

$$
\Sigma^{2p-2}V(0) \xrightarrow{\alpha} V(0).
$$

inducing multiplication by $v_1$. It is a CW-spectrum of the form

$$
V(1) = S^0 \cup_p e^1 \cup \alpha_1 e^{2p-1} \cup_p e^{2p}.
$$

Independently Larry Smith and Hirosi Toda showed that for $p \geq 5$, there is a map

$$
\Sigma^{2p^2-2}V(1) \xrightarrow{\beta} V(1)
$$

inducing multiplication by $v_2$ in $BP_*(-)$.

Then the element

$$
\beta_i := \delta_1 \delta_2 v_2^i \in \text{Ext}^1_{BP_*}(BP_*/(p)).
$$

is represented by the composite

$$
S^{2|v_2|} \xrightarrow{i} \Sigma^{2|v_2|}V(1) \xrightarrow{\beta^i} V(1) \xrightarrow{j} S^{2p}.
$$

Algebraically we can do a similar thing at all heights and at all primes. We can define

$$
\eta_{(h)} := \delta_1 \delta_2 \ldots \delta_h(v_1^i) \in \text{Ext}^1_{BP_*}(BP_*/(p)).
$$

where $\eta_{(h)}$ denotes the $h$th letter of the Greek alphabet and $w_h = |v_1| + \cdots + |v_{h-1}|$.

However, we can go only one step further geometrically, defining elements $\gamma$ for $p \geq 7$. Nobody knows how to construct a map

$$
\Sigma^{2p^2-2}V(3) \xrightarrow{\delta} V(3)
$$

inducing multiplication by $v_4$ in $BP_*(-)$ at any prime.

## 3 Type $h$ finite complexes

For a $p$-local finite spectrum $X$, we know that $K(h)_*X = 0$ implies $K(h-1)_*X = 0$, and that $K(h)_*X \neq 0$ for $h \gg 0$ unless $X$ is contractible. We say that $X$ has type $h$ if $h$ is the smallest integer with $K(h)_*X \neq 0$. Hence Toda’s $V(h-1)$ has type $h$. If $K(h)_*X = 0$ for all $h$, then $X$ is contractible.

The following was conjectured in [Rav84] and proved by Ethan Devinatz, Mike Hopkins and Jeff Smith in [DHS88].

**Class Invariance Theorem.** The Bousfield equivalence class of a $p$-local finite spectrum is determined by its type.
In particular any $p$-local finite spectrum $X$ with nontrivial rational homology is Bousfield equivalent to $S(p)$.

A few years later in [HS98], Hopkins and Smith proved the following.

**Periodicity Theorem.** Let $X$ be $p$-local finite spectrum of type $h$. Then there is a map $v : \Sigma^d X \to X$ for some $d > 0$ that induces an isomorphism in $K(h)_*(-)$ and a nilpotent map in every other Morava $K$-theory. We call it a $v_h$ self-map.

This map is asymptotically unique in the following sense. Given a second such map $v' : \Sigma^{d'} X \to X$, there exist integers $e$ and $e'$ with $ed = e'd'$ and $v^e = (v')^{e'}$.

If follows that the cofiber $C_v$ has type $h + 1$. Hence we can produce finite spectra of all higher types by iterating this process. The Class Invariance theorem implies that the Bousfield class of the telescope $v^{-1}X$ is independent of the choices of both $X$ and $v$. We denote it by $\langle T(h) \rangle$.

4 The telescope conjecture

**Periodicity Theorem.** Let $X$ be $p$-local finite spectrum of type $h$. Then there is a map $v : \Sigma^d X \to X$ for some $d > 0$ that induces an isomorphism in $K(h)_*(-)$. We call it a $v_h$ self-map.

The map $X \to v^{-1}X$ is a $K(h)_*$-equivalence, so we have a map

$$\lambda : v^{-1}X \to L_{K(h)}X = L_hX,$$

where the equality holds because the lower Morava $K$-theories vanish on $X$. The following appeared in [Rav84].

**Telescope Conjecture.** The map

$$\lambda : v^{-1}X \to L_{K(h)}X$$

is an equivalence.

This is trivially true for $h = 0$, and for $h = 1$ it was proved around 1980 by Mahowald for $p = 2$ and by Miller for $p$ odd. In 1989 I began to think it was false for $h \geq 2$. This is now a theorem of Robert Burklund, Jeremy Hahn, Ishan Levy and Tomer Schlank.

Jeremy, Tomer, myself, Ishan and Robert at Oxford University, June 9, 2023.

Photo by Matteo Barucco.

**Telescope Conjecture.** The map

$$\lambda : v^{-1}X \to L_{K(h)}X$$

is an equivalence.

This conjecture equated the geometrically interesting object $v^{-1}X$, the $v_h$-periodic telescope associated with the type $h$ finite complex $X$, with the more computationally accessible spectrum $L_{K(h)}X$.

For example, we know how to compute $\pi_* L_{K(h)} V(1)$ for $p \geq 5$, where $V(1)$ is Toda’s 4-cell complex. It consists of exactly 12 $v_2$-periodic families.

For example, we know how to compute $\pi_* L_{K(h)} V(1)$ for $p \geq 5$, where $V(1)$ is Toda’s 4-cell complex. It consists of exactly 12 $v_2$-periodic families.
We do not know $\pi_*v_{2}^{-1}V(1)$, which is likely to be much larger. There are possibly infinitely many such families not detected by the localized Adams-Novikov spectral sequence, which is known to converge to $\pi_*L_{K(2)}V(1)$, but not to $\pi_*v_{2}^{-1}V(1)$.

Meanwhile the ordinary Adams-Novikov spectral sequence does converge to $\pi_*V(1)$ but only sees $12v_2$-periodic families there. How can this be? One could have a $v_2$-periodic family (or many of them) that are spread out over infinitely many Adams-Novikov filtrations.

References

