What is the telescope conjecture?

Lecture 2

Morava’s vision and the chromatic spectral sequence

Doug Ravenel

University of Rochester

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Recollections

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The group $G = G_{\mathbb{Z}}$ acts on $L \cong MU_*$ as follows. We can conjugate $F_L$ by $f$, defining

$$F^f_L(x, y) := f^{-1} F_L(f(x), f(y)).$$

This formal group law is induced by a ring automorphism $\theta_f : L \to L$. 
The Adams-Novikov $E_2$-term

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Invariant prime ideals

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which is related to formal group laws of height (at $p$) at least $h$. 

Landweber's theorem says they are the only ones which are also comodules over $BP^*$. There is a short exact sequence of comodules

$$0 \to \Sigma \mid v_{h-1} \mid BP^* / I_{h-1} v_h / BP^* / I_{h-1} / BP^* / 0,$$

where $I_0 = (0)$ and $v_0 = p$. The $(h)$th Greek letter sequence.
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Morava’s vision

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Let $V$ denote the “vector space” of ring homomorphisms $\theta : L \to \overline{\mathbb{F}}_p$. 

\begin{itemize}
  \item Each point in $V$ corresponds to a formal group law over $\mathbb{F}_p$.
  \item $V$ has an action of $G = G_{\mathbb{F}_p} \rtimes \mathbb{F}_p$ for which each orbit is an isomorphism class of formal group laws over $\mathbb{F}_p$. Hence there is one orbit for each height.
  \item For each $x \in V$, the isotropy or stabilizer group $G_x = \{ \gamma \in G : \gamma(x) = x \}$ is the automorphism group of the corresponding formal group law. When $x$ has height $h$, this group is isomorphic to the Morava stabilizer group $S_h$.
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Morava’s vision
The Morava stabilizer group
The change-of-rings isomorphism
The chromatic spectral sequence
Morava’s vision (continued)

Let $V$ denote the “vector space” of ring homomorphisms $\theta : L \to \overline{F}_p$.

- There are $G$-invariant finite codimensional linear subspaces
  
  $$V = V_1 \supset V_2 \supset V_3 \supset \cdots$$

  where $V_h = \{ \theta \in V : \theta(v_1) = \cdots = \theta(v_{h-1}) = 0 \}$. 

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- The height $h$ orbit is $V_h - V_{h+1}$. It is the set of $\overline{F}_p$-valued homomorphisms on $v_h^{-1}L/I_h$. 

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- The height $\infty$ orbit is the linear subspace
  
  \[ \bigcap_{h>0} V_h. \]
The Morava stabilizer group

Here we describe the endomorphism ring and automorphism group of a height $h$ formal group law over a field $K$ of characteristic $p$ containing $\mathbb{F}_{p^h}$. 
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We will see that it is the endomorphism ring of our formal group law.
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- $W := W(\mathbb{F}_{p^h})$ denotes the Witt ring for $\mathbb{F}_{p^h}$. It is the extension of the $p$-adic integers $\mathbb{Z}_p$ obtained by adjoining the $(p^h - 1)$th roots of unity. It is a complete local ring with residue field $\mathbb{F}_{p^h}$ and an extension of $\mathbb{Z}_p$ of degree $h$. It has an automorphism $\sigma$ that lifts the Frobenius automorphism ($p$th power map) in the residue field. We denote the image of $w \in W$ under $\sigma$ by $w^\sigma$. 
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- $\text{End}_h$ denotes the $\mathbb{Z}_p$-algebra obtained from $W$ by adjoining an indeterminate $S$ with $Sw = w^\sigma S$ for $w \in W$ and setting $S^h = p$. We will see that it is the endomorphism ring of our formal group law.
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To describe the action of $\text{End}_h$ on the mod $p$ reduction of the Honda formal group law $F_h$ of height $h$ over $W = W(F_{p^h})$, we note first that each element $e \in \text{End}_h$ can be written uniquely as

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where $e_i \in F_{p^h}$ for each $i$, meaning that each $e_i$ is either zero or a $(p^h - 1)$th root of unity. Recall that the logarithm of $F_h$ is

$$\log(x) = \sum_{k \geq 0} x^{p^k h} p^k = x + x^{p^h} + x^{p^{2h}} + \cdots$$

Now let $\omega \in W$ satisfy $\omega^{p^h} = \omega$. Then $\log(\omega x) = \omega \log(x)$, so $F_h$ has an endomorphism $x \mapsto \omega x$. The endomorphism for $\sum_{i \geq 0} e_i S_i \in \text{End}_h$ is $x \mapsto \sum_{i \geq 0} F_h e_i x^{p^i} \in F_{p^h}$. 
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- Each such expression with $e_0 \neq 0$ is invertible. The Morava stabilizer group $S_h$ is the group of units $\text{End}_h^x$. We also have the extended Morava stabilizer group $G_h = S_h \rtimes \text{Gal}(F_p^h : F_p)$. $\text{Div}_h : = \text{End}_h \otimes Z_p$ is a division algebra over the $p$-adic numbers $Q_p$ with Brauer invariant $1/h$, in which $\text{End}_h$ is a maximal order.
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Recollect the prime ideals $\text{End}_h$ and the Morava stabilizer group $S_h$. The change-of-rings isomorphism $\text{End}_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ contains every degree $h$ field extension of $\mathbb{Q}_p$. Its maximal order $\text{End}_h$ contains the ring of integers of every such field. This means that $S_h$ has an element of order $p^i$ iff $(p - 1)p^{i-1}$ divides $h$. The finite subgroups of $G_h$ have been classified by Bujard. The subgroup of order 8 in $S_4$ for $p = 2$ odd was used in the solution of Kervaire invariant problem with Hill and Hopkins. The subgroup of order $p^i$ in $S_{p^i - 1}$ for $p$ odd was used earlier in the solution of the odd primary Kervaire invariant problem. We know the mod $p$ cohomology of $S_1$ and $S_2$ for all primes, and of $S_3$ for $p \geq 5$. We also know $H^1$ and $H^2$ for all heights. $S_h$ has cohomological dimension $h/2$ when $p - 1$ does not divide $h$. $H^* S_4$ for $p > 5$ has been announced by Andrew Salch.
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This implies the following change-of-rings isomorphism due to Miller and myself:

$$\text{Ext} \left( v_h^{-1} L/ I_h, H^* \left( S_{h}; \mathbb{F}_p \right) \right)$$

This is not quite right; there are caveats having to do with grading. Details can be found in Chapter 6 of the green book, which describes methods for computing the cohomology group on the right.
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The chromatic spectral sequence

Periodic phenomena in the Adams-Novikov spectral sequence

By Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson
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The chromatic spectral sequence (continued)

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\[ E_2^{h,s} = \text{Ext}^s(M^h) \Rightarrow \text{Ext}^{s+h}(BP_*) \]

Roughly speaking, its \( h \)th column, \( \text{Ext}(M^h) \), displays \( \nu_h \)-periodic phenomena.
The chromatic spectral sequence (continued)

Roughly speaking, its $h$th column, $\Ext (M^h)$, displays $\nu_h$-periodic phenomena. This decomposition of the Adams-Novikov $E_2$-term into its various frequencies is our reason for the use of the word chromatic.
The chromatic spectral sequence (continued)

We will construct a long exact sequence of $BP_\ast BP$-comodules of the form

$$0 \rightarrow BP_\ast \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \ldots$$

called the chromatic resolution.
The chromatic spectral sequence (continued)

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We will do so by splicing together the chromatic short exact sequences.
The chromatic spectral sequence (continued)

We will construct a long exact sequence of $BP_*BP$-comodules of the form

$$0 \rightarrow BP_* \rightarrow M^0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \ldots$$

called the chromatic resolution.

We will do so by splicing together the chromatic short exact sequences

$$0 \rightarrow N^0 \coloneqq BP_* \rightarrow M^0 \rightarrow N^1 \rightarrow 0,$$

$$0 \rightarrow N^1 \rightarrow M^1 \rightarrow N^2 \rightarrow 0,$$

$$0 \rightarrow N^2 \rightarrow M^2 \rightarrow N^3 \rightarrow 0,$$

and so on.
The chromatic spectral sequence (continued)

\[ 0 \rightarrow N^0 := BP_* \rightarrow M^0 \rightarrow N^1 \rightarrow 0, \]
0 → N^0 := BP_* → M^0 → N^1 → 0,

We set M^0 := BP_0 ⊗ Q, so N^1 = BP_* ⊗ Q/\mathbb{Z}(p), which we write as

N^1 = BP_* / p^\infty := \colim_i BP_* / p^i.

Inverting v_1 in the comodule category requires some care.
The chromatic spectral sequence (continued)

\[
0 \rightarrow N^0 := BP_* \rightarrow M^0 \rightarrow N^1 \rightarrow 0,
\]

We set \( M^0 := BP_0 \otimes \mathbb{Q} \), so
\[
N^1 = BP_* \otimes \mathbb{Q}/\mathbb{Z}(p),
\]
which we write as
\[
N^1 = BP_*/p^\infty := \operatorname{colim}_i BP_*/p^i.
\]

Our first chromatic short exact sequence is

\[
0 \rightarrow N^0 \xrightarrow{\cong} M^0 \xrightarrow{\cong} N^1 \xrightarrow{\cong} 0,
\]

\[
BP_* \quad p^{-1}BP_* \quad BP_*/(p^\infty)
\]
The chromatic spectral sequence (continued)

0 → N^0 := BP_* → M^0 → N^1 → 0,

We set M^0 := BP_0 ⊗ Q, so N^1 = BP_* ⊗ Q/\mathbb{Z}(p), which we write as

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0 → N^0 → M^0 → N^1 → 0,

\begin{array}{c}
  BP_* \\
  \downarrow p^{-1}BP_* \\
  BP_*/(p^\infty)
\end{array}

We want the next one to be

0 → N^1 → M^1 → N^2 → 0,

\begin{array}{c}
  BP_*/(p^\infty) \\
  \downarrow v_1^{-1}BP_*/(p^\infty) \\
  BP_*/(p^\infty, v_1^\infty)
\end{array}
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\[ 0 \longrightarrow N^0 := BP_* \longrightarrow M^0 \longrightarrow N^1 \longrightarrow 0, \]

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\[ N^1 = BP_*/p^\infty := \text{colim}_i BP_*/p^i. \]

Our first chromatic short exact sequence is

\[ 0 \longrightarrow N^0 \overset{\text{id}}{\longrightarrow} M^0 \overset{p^{-1}BP_*}{\longrightarrow} N^1 \overset{\text{id}}{\longrightarrow} 0, \]

\[ BP_* \quad p^{-1}BP_* \quad BP_*/(p^\infty) \]

We want the next one to be

\[ 0 \longrightarrow N^1 \overset{\text{id}}{\longrightarrow} M^1 \overset{\text{id}}{\longrightarrow} N^2 \overset{\text{id}}{\longrightarrow} 0, \]

\[ BP_*/(p^\infty) \quad v_1^{-1}BP_*/(p^\infty) \quad BP_*/(p^\infty, v_1^\infty), \]

but inverting \( v_1 \) in the comodule category requires some care.
The chromatic spectral sequence (continued)

We want a short exact sequence of comodules

\[
0 \rightarrow N^1 \rightarrow M^1 \rightarrow N^2 \rightarrow 0,
\]

\[
BP^* / (p^\infty) \rightarrow v_1^{-1} BP^* / (p^\infty) \rightarrow BP^* / (p^\infty, v_1^\infty),
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Consider the \( BP_* \)-module \( v_1^{-1}BP_* \). Since \( \eta_R(v_1) = v_1 + pt_1 \),
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Consider the \( BP_* \)-module \( v_1^{-1}BP_* \). Since \( \eta_R(v_1) = v_1 + pt_1 \), formally we have

\[ \eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i. \]
The chromatic spectral sequence (continued)

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When \( k < 0 \), this sum is infinite and therefore does not lie in \( v_1^{-1}BP_*BP \).
We want a short exact sequence of comodules

\[
0 \rightarrow N^1 \rightarrow M^1 \rightarrow N^2 \rightarrow 0, \\
\xrightarrow{\text{BP}_*/(p^\infty)} \quad \xrightarrow{v_1^{-1} \text{BP}_*/(p^\infty)} \quad \xrightarrow{\text{BP}_*/(p^\infty, v_1^\infty)},
\]

but inverting \( v_1 \) in the comodule category requires some care.

Consider the \( \text{BP}_* \)-module \( v_1^{-1} \text{BP}_* \). Since \( \eta_R(v_1) = v_1 + pt_1 \), formally we have

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\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.
\]

When \( k < 0 \), this sum is infinite and therefore does not lie in \( v_1^{-1} \text{BP}_* \text{BP} \). This means that \( v_1^{-1} \text{BP}_* \) is not a comodule.
We want a short exact sequence of comodules

\[ 0 \rightarrow N^1 \rightarrow M^1 \rightarrow N^2 \rightarrow 0, \]

\[ BP_*/(p^\infty) \quad \nu_1^{-1} BP_*/(p^\infty) \quad BP_*/(p^\infty, v_1^\infty), \]

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Consider the \( BP_* \)-module \( \nu_1^{-1} BP_* \). Since \( \eta_R(v_1) = v_1 + pt_1 \), formally we have

\[ \eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i. \]

When \( k < 0 \), this sum is infinite and therefore does not lie in \( \nu_1^{-1} BP_* BP \). This means that \( \nu_1^{-1} BP_* \) is not a comodule. We claim that \( \nu_1^{-1} BP_* / p^\infty \) is one nevertheless.
The following sum is infinite for $k < 0$. 

$$\eta_R(v^k_1) = (v^1_1 + pt^1_1)^k = \sum_{i \geq 0} (k^i_1)p^i_1v^k_1 - i_1t^i_1.$$

Each element in $BP^* / p^\infty$ can be written as a fraction of the form $x^{p^j}$ where $j > 0$ and $x \in BP^*$ is not divisible by $p$. This element is killed by $p^j$.

It follows that $\eta_R(x^{p^j}) = \sum_{0 \leq i < j} (k^i_1)v^k_1 - i_1t^i_1\eta_R(x)$

This sum is finite for all $k$, unlike the previous one, so $v^1_1BP^* / p^\infty$ is a comodule as claimed.
The chromatic spectral sequence (continued)

The following sum is infinite for $k < 0$.

$$
\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.
$$
The chromatic spectral sequence (continued)

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The chromatic spectral sequence (continued)

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$$

Each element in $BP_* / p^\infty$ can be written as a fraction of the form

$$
\frac{x}{p^j}
$$

where $j > 0$ and $x \in BP_*$ is not divisible by $p$. 

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This element is killed by \( p^j \).
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Each element in $BP_* / p^\infty$ can be written as a fraction of the form

$$\frac{x}{p^j}$$

where $j > 0$ and $x \in BP_*$ is not divisible by $p$.

This element is killed by $p^j$. It follows that

$$\eta_R \left( \frac{v_1^k x}{p^j} \right) = \sum_{0 \leq i < j} \binom{k}{i} \frac{v_1^{k-i} t_1^i \eta_R(x)}{p^{j-i}}.$$
The chromatic spectral sequence (continued)

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\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.
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$$
\eta_R \left( \frac{v_1^k x}{p^j} \right) = \sum_{0 \leq i < j} \binom{k}{i} v_1^{k-i} t_1^i \eta_R(x) \frac{1}{p^{j-i}}.
$$

This sum is finite for all $k$, unlike the previous one, so $v_1^{-1} BP_* / p^\infty$ is a comodule as claimed.
Thus we have our second **chromatic short exact sequence**

\[ 0 \longrightarrow N^1 \longrightarrow M^1 \longrightarrow N^2 \longrightarrow 0 \]

\[ BP_*/(p^\infty) \quad \nu_1^{-1} BP_*/(p^\infty) \quad BP_*/(p^\infty, v_1^\infty). \]
Thus we have our second chromatic short exact sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & N^1 & \rightarrow & M^1 & \rightarrow & N^2 & \rightarrow & 0 \\
\| & & \| & & \| & & \\
BP_*/(p^\infty) & \rightarrow & \nu_1^{-1}BP_*/(p^\infty) & \rightarrow & BP_*/(p^\infty, v_1^\infty).
\end{array}
\]

In a similar manner we can work by induction on \( h \) and construct
Thus we have our second chromatic short exact sequence

\[
0 \to N^1 \to M^1 \to N^2 \to 0
\]

\[
BP_*/(p^\infty) \xrightarrow{v_1^{-1}} BP_*/(p^\infty) \to BP_*/(p^\infty, v_1^\infty).
\]

In a similar manner we can work by induction on \( h \) and construct

\[
0 \to N^h \to M^h \to N^{h+1} \to 0
\]

\[
BP_*/(p^\infty, \ldots, v_{h-1}^\infty) \xrightarrow{v_h^{-1}N^h} BP_*/(p^\infty, \ldots, v_h^\infty).
\]
The chromatic spectral sequence (continued)

Thus we have our second **chromatic short exact sequence**

\[
0 \longrightarrow N^1 \longrightarrow M^1 \longrightarrow N^2 \longrightarrow 0
\]

\[
BP^* / (p^\infty) \quad v_1^{-1} BP^* / (p^\infty) \quad BP^* / (p^\infty, v_1^\infty).
\]

In a similar manner we can work by induction on \( h \) and construct

\[
0 \longrightarrow N^h \longrightarrow M^h \longrightarrow N^{h+1} \longrightarrow 0
\]

\[
BP^* / (p^\infty, \ldots, v_{h-1}^\infty) \quad v_h^{-1} N^h \quad BP^* / (p^\infty, \ldots, v_h^\infty).
\]

Splicing these together for all \( h \) gives the desired long exact sequence,

\[
0 \longrightarrow BP^* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow M^4 \longrightarrow \cdots.
\]
Recall that the change-of-ring-isomorphism gives us a handle on $\text{Ext} \left( v_h^{-1} BP_* / I_h \right)$.
Recall that the change-of-ring-isomorphism gives us a handle on $\Ext(v_h^{-1}BP_*/I_h)$. For $h = 1$, consider the short exact sequence
The chromatic spectral sequence (continued)

Recall that the change-of-ring-isomorphism gives us a handle on $\text{Ext}(v^{-1}_h BP_*/I_h)$. For $h = 1$, consider the short exact sequence

$$
0 \longrightarrow M_1^0 \xrightarrow{\cong} M_1^1 \xrightarrow{p} M_1^1 \longrightarrow 0.
$$

$$
\begin{align*}
0 & \longrightarrow v_1^{-1} BP_*/(p) \xrightarrow{\cong} v_1^{-1} BP_*/(p^\infty) \xrightarrow{\cong} v_1^{-1} BP_*/(p^\infty)
\end{align*}
$$
Recall that the change-of-ring-isomorphism gives us a handle on $\text{Ext} \left( v_1^{-1} BP_{\ast} / I_h \right)$. For $h = 1$, consider the short exact sequence

\[ 0 \rightarrow M_1^0 \rightarrow M_1^1 \rightarrow M_1^1 \xrightarrow{p} M_1^1 \rightarrow 0. \]

\[ v_1^{-1} BP_{\ast} / (p) \rightarrow v_1^{-1} BP_{\ast} / (p^\infty) \rightarrow v_1^{-1} BP_{\ast} / (p^\infty) \]

This leads to a Bockstein spectral sequence of the form
Recall that the change-of-ring-isomorphism gives us a handle on $\text{Ext} \left( v^{-1}_h BP_* / I_h \right)$. For $h = 1$, consider the short exact sequence

$$0 \rightarrow M_1^0 \rightarrow M_1 \rightarrow M_1^1 \rightarrow 0.$$ 

This leads to a Bockstein spectral sequence of the form

$$\text{Ext} \left( M_1^0 \right) \otimes P(a_0) \rightarrow \text{Ext} \left( M_1^1 \right).$$

$$x \otimes a_0^i \rightarrow \frac{x}{p^{i+1}}.$$
The chromatic spectral sequence (continued)

For $h = 2$ we have two short exact sequences
For $h = 2$ we have two short exact sequences

$$
0 \rightarrow M_1^1 \rightarrow M^2 \xrightarrow{p} M^2 \rightarrow 0
$$

\[ v_2^{-1} BP_* / (p, v_1^\infty) \]

\[ v_2^{-1} BP_* / (p^\infty, v_1^\infty) \]
For $h = 2$ we have two short exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M_1^1 & \rightarrow & M^2 & \rightarrow & M^2 & \rightarrow & 0 \\
& \parallel & v_2^{-1}BP_*/(p, v_1^\infty) & \parallel & v_2^{-1}BP_*/(p^\infty, v_1^\infty) \\
\end{array}
\]

and

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M_0^2 & \rightarrow & \Sigma |v_1| M_1^1 & \rightarrow & M_1^1 & \rightarrow & 0. \\
& \parallel & v_2^{-1}BP_*/(p, v_1) & \parallel & v_1 & \parallel & x & \rightarrow & x \\
\end{array}
\]
The chromatic spectral sequence (continued)

For $h = 2$ we have two short exact sequences

$$
\begin{array}{ccccccc}
0 & \rightarrow & M_1^1 & \rightarrow & M^2 & \rightarrow & M^2 \\
\| & & \| & & p & & \\
& & v_2^{-1}BP_*/(p, v_1^\infty) & & v_2^{-1}BP_*/(p^\infty, v_1^\infty) & & 0
\end{array}
$$

and

$$
\begin{array}{ccccccc}
0 & \rightarrow & M_0^2 & \rightarrow & \Sigma |v_1| M_1^1 & \rightarrow & M_1^1 \\
\| & & \| & & v_1 & & \\
v_2^{-1}BP_*/(p, v_1) & & X/\Sigma |v_1| M_1^1 & & X/pv_1 & & 0
\end{array}
$$

Each one leads to a Bockstein spectral sequence, making the desired $\text{Ext}(M^2)$ two steps removed from the known $\text{Ext}(v_2^{-1}BP_*/(p, v_1))$. 
More generally we have a short exact sequence of comodules

\[ 0 \rightarrow \Sigma |v_i| M_{i+1}^{h-i-1} \rightarrow \Sigma |v_i| M_i^{h-i} \rightarrow M_i^{h-i} \rightarrow 0 \]
The chromatic spectral sequence (continued)

More generally we have a short exact sequence of comodules

\[ 0 \to \sum |v_i| M^h_{i+1} \to \sum |v_i| M^h_i \xrightarrow{v_i} M^h_i \to 0 \]

for \( 0 \leq i < h \), where \( M^h_0 = M^h \) and \( v_0 = 0 \).
More generally we have a short exact sequence of comodules

$$0 \longrightarrow \sum |v_i| M_{i+1}^{h-i-1} \longrightarrow \sum |v_i| M_i^{h-i} \longrightarrow M_i^{h-i} \longrightarrow 0$$

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More generally we have a short exact sequence of comodules

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for \( 0 \leq i < h \), where \( M_0^h = M^h \) and \( v_0 = 0 \). This leads to a Bockstein spectral sequence

\[ \text{Ext} \left( M_{i+1}^{h-i-1} \right) \otimes P(a_i) \rightsquigarrow \text{Ext} \left( M_i^{h-i} \right) \]

\[ \frac{x}{pv_1 \cdots v_{i-1} v_i v_{i+1} \cdots v_{h-1} \otimes a_j^i \rightsquigarrow \frac{x}{pv_1 \cdots v_{i-1} v_i^{j+1} \cdots v_{h-1}}}. \]
More generally we have a short exact sequence of comodules

\[ 0 \longrightarrow \sum |v| M^{h-i-1}_{i+1} \longrightarrow \sum |v| M^{h-i}_{i} \xrightarrow{v_i} M^{h-i}_{i} \longrightarrow 0 \]

for \(0 \leq i < h\), where \(M^h_0 = M^h\) and \(v_0 = 0\). This leads to a Bockstein spectral sequence

\[ \operatorname{Ext}(M^{h-i-1}_{i+1}) \otimes P(a_i) \xrightarrow{\times} \operatorname{Ext}(M^{h-i}_i) \]

This makes \(\operatorname{Ext}(M^h)\) \(h\) steps removed from the cohomology of \(S^h\). 

\(x\text{ pv}_1 \cdots v_{i-1} v_i v_{i+1}^{j+1} \cdots v_{h-1}^{j-1} \otimes a^j_i \xrightarrow{\times} p v_1 \cdots v_{i-1} v_i^{j+1} \cdots v_{h-1}^{j-1} \)
Computations with these Bockstein spectral sequence can be quite delicate.
Computations with these Bockstein spectral sequence can be quite delicate. Nearly all of them published since 1977 have been due to Katsumi Shimomura and various coauthors.
THANK YOU!