ECHT Minicourse

What is the telescope conjecture? Lecture 2 Morava's vision and the chromatic spectral sequence

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1 Recollections

Recollections

The Lazard ring $L = \mathbb{Z}[x_1, x_2, \ldots]$ is the graded ring (with $|x_i| = 2i$) over which the universal formal group law F_L is defined. This means that any formal group law F over any ring R is induced from F_L by a ring homomorphism $\theta : L \to R$.

Quillen showed that for the formal group law F_{MU} that one sees in $MU^*\mathbb{C}P^\infty$, the map θ is an isomorphism. There is a Hopf algebroid, i.e., an affine groupoid scheme, (MU_*, MU_*MU). It represents the functor that assigns to each ring R the groupoid of formal group laws over R and strict isomorphisms between them.

We have $MU_*MU = MU_*[b_1, b_2, \ldots]$ with $|b_i| = 2i$. There is an affine group scheme, i.e., a Hopf algebra, represented by the ring $B = \mathbb{Z}[b_1, b_2, \ldots]$. The corresponding functor assigns to each R the group (under functional composition) G_R of formally invertible power series of the form

$$f(x) = x + \sum_{i>0} b_i x^{i+1} \in Rx.$$

Recollections (continued)

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$$f(x) = x + \sum_{i>0} b_i x^{i+1} \in Rx.$$

The group $G = G_{\mathbb{Z}}$ acts on $L \cong MU_*$ as follows. We can conjugate F_L by f, defining

$$F_L^f(x, y) := f^{-1}F_L(f(x), f(y)).$$

This formal group law is induced by a ring automorphism $\theta_f : L \to L$.

2 The Adams-Novikov E_2-term

The Adams-Novikov E_2-term

We have $MU_*MU = MU_*[b_1, b_2, \dots]$ with $|b_i| = 2i$. For any spectrum X, MU_*X is a comodule over MU_*MU. For any such comodule M, we can define

$$\mathrm{Ext}\,(M) := \mathrm{Ext}_{MU_*MU}\,(MU_*, M)\,.$$

When $M = MU_*X$, this is the E_2-term of the Adams-Novikov spectral sequence converging (in favorable circumstances) to π_*X.

In the p-local setting, it is more convenient to look at

$$BP_*X = BP_* \otimes_{MU_*} MU_*X,$$

which is a comodule over BP_*BP. For any such comodule M, we can define

$$\mathrm{Ext}\,(M) := \mathrm{Ext}_{BP_*BP}\,(BP_*, M)\,.$$

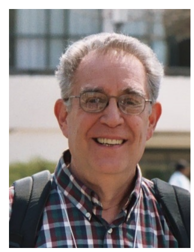
3 Invariant prime ideals

Invariant prime ideals

Fix a prime number p throughout.
For each $h > 0$, we have a prime ideal

$$I_h = (p, v_1, \dots, v_{h-1}) \subseteq L,$$

which is related to formal group laws of height (at p) at least h. In 1973 Peter Landweber showed that they are the only prime ideals in MU_* that are invariant under the action of G.



We will use the same notation for the analogous prime ideals in BP_*. Landweber's theorem says they are the only ones which are also comodules over BP_*BP. There is a short exact sequence of comodules

$$0 \longrightarrow \Sigma^{|v_{h-1}|} BP_*/I_{h-1} \xrightarrow{v_{h-1}} BP_*/I_{h-1} \longrightarrow BP_*/I_h \longrightarrow 0,$$

where $I_0 = (0)$ and $v_0 = p$, the (hth) Greek letter sequence.

4 Morava's vision

Morava's vision

I learned the following from Jack Morava in 1973 and have never forgotten it. It was the subject of an unpublished AMS Bulletin announcement that he recently dug up. You can find it on my archive.

Let V denote the "vector space" of ring homomorphisms $\theta : L \to \overline{\mathbb{F}}_p$.

- Each point in V corresponds to a formal group law over $\overline{\mathbb{F}}_p$.

- V has an action of $\mathbb{G} = G_{\overline{\mathbb{F}}_p} \rtimes \overline{\mathbb{F}}_p^\times$ for which each orbit is an isomorphism class of formal group laws over $\overline{\mathbb{F}}_p$. Hence there is one orbit for each height.
- For each $x \in V$, the isotropy or stabilizer group $\mathbb{G}_x = \{\gamma \in \mathbb{G} : \gamma(x) = x\}$ is the automorphism group of the corresponding formal group law. When x has height h, this group is isomorphic to the Morava stabilizer group $\mathbb{S}_h$.

Morava's vision (continued)

Let V denote the "vector space" of ring homomorphisms $\theta : L \to \overline{\mathbb{F}}_p$.

- There are G-invariant finite codimensional linear subspaces
$$V = V_1 \supset V_2 \supset V_3 \supset \cdots$$
where $V_h = \{\theta \in V : \theta(v_1) = \cdots = \theta(v_{h-1}) = 0\}$. We will call this the Morava filtration of V.
- The height h orbit is $V_h - V_{h+1}$. It is the set of $\overline{\mathbb{F}}_p$-valued homomorphisms on $v_h^{-1}L/I_h$. We use this fact later.
- The height ∞ orbit is the linear subspace
$$\bigcap_{h>0} V_h.$$

5 The Morava stabilizer group

The Morava stabilizer group

Here we describe the endomorphism ring and automorphism group of a height h formal group law over a field K of characteristic p containing $\mathbb{F}_{p^h}$.

We need some notation.

- $W := W(\mathbb{F}_{p^h})$ denotes the Witt ring for $\mathbb{F}_{p^h}$. It is the extension of the p-adic integers $\mathbb{Z}_p$ obtained by adjoining the $(p^h - 1)$th roots of unity. It is a complete local ring with residue field $\mathbb{F}_{p^h}$ and an extension of $\mathbb{Z}_p$ of degree h. It has an automorphism σ that lifts the Frobenius automorphism (pth power map) in the residue field. We denote the image of $w \in W$ under σ by w^σ.
- End_h denotes the $\mathbb{Z}_p$-algebra obtained from W by adjoining an indeterminate S with $Sw = w^\sigma S$ for $w \in W$ and setting $S^h = p$. We will see that it is the endomorphism ring of our formal group law.

The Morava stabilizer group (continued)

To describe the action of End_h on the mod p reduction of the Honda formal group law F_h of height h over $W = W(\mathbb{F}_{p^h})$, we note first that each element $e \in \mathrm{End}_h$ can be written uniquely as
$$\sum_{i\geq 0} e_i S^i \qquad \text{where } e_i^{p^h} = e_i \text{ for each } i,$$
meaning that each e_i is either zero or a $(p^h - 1)$th root of unity.

Recall that the logarithm of F_h is
$$\log(x) = \sum_{k\geq 0} \frac{x^{p^{kh}}}{p^k} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots .$$
Now let $\omega \in W$ satisfy $\omega^{p^h} = \omega$. Then $\log(\omega x) = \omega \log(x)$, so F_h has an endomorphism $x \mapsto \omega x$.

The endomorphism for
$$\sum_{i\geq 0} e_i S^i \in \mathrm{End}_h \qquad \text{is} \qquad x \mapsto \sum_{i\geq 0}{}^{F_h} e_i x^{p^i} \in \mathbb{F}_{p^h} x.$$

The Morava stabilizer group (continued)

Again, each element $e \in \mathrm{End}_h$ can be written uniquely as

$$\sum_{i \geq 0} e_i S^i \qquad \text{where } e_i^{p^h} = e_i \text{ for each } i.$$

Here are some additional properties of End_h:

- Each such expression with $e_0 \neq 0$ is invertible. The Morava stabilizer group $\mathbb{S}_h$ is the group of units $\mathrm{End}_h^\times$. We also have the extended Morava stabilizer group
$$\mathbb{G}_h = \mathbb{S}_h \rtimes \mathrm{Gal}(\mathbb{F}_{p^h} : \mathbb{F}_p).$$
- $\mathrm{Div}_h := \mathrm{End}_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a division algebra over the p-adic numbers $\mathbb{Q}_p$ with Brauer invariant $1/h$, in which End_h is a maximal order.

The Morava stabilizer group (continued)

The division algebra $\mathrm{Div}_h := \mathrm{End}_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ contains every degree h field extension of $\mathbb{Q}_p$. Its maximal order End_h contains the ring of integers of every such field. This means that $\mathbb{S}_h$ has an element of order p^i iff $(p-1)p^{i-1}$ divides h.

The finite subgroups of $\mathbb{G}_h$ have been classified by Bujard.

The subgroup of order 8 in $\mathbb{S}_4$ for $p = 2$ odd was used in the solution of Kervaire invariant problem with Hill and Hopkins.

The subgroup of order p in $\mathbb{S}_{p-1}$ for p odd was used earlier in the solution of the odd primary Kervaire invariant problem.

We know the mod p cohomology of $\mathbb{S}_1$ and $\mathbb{S}_2$ for all primes, and of $\mathbb{S}_3$ for $p \geq 5$. We also know H^1 and H^2 for all heights. $\mathbb{S}_h$ has cohomological dimension h^2 when $p-1$ does not divide h. $H^*\mathbb{S}_4$ for $p > 5$ has been announced by Andrew Salch.

6 The change-of-rings isomorphism

The change-of-rings isomorphism

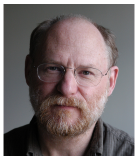
Recall that Morava's height h orbit is the set of $\overline{\mathbb{F}}_p$-valued ring homomorphisms on $v_h^{-1}L/I_h$. This implies the following change-of-rings isomorphism due to Miller and myself:

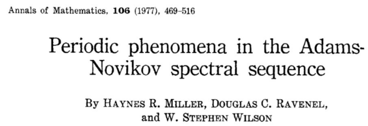
$$\mathrm{Ext}\left(v_h^{-1}BP_*/I_h\right) \cong H^*(\mathbb{S}_h; \mathbb{F}_p)$$

This is not quite right; there are caveats having to do with grading. Details can be found in Chapter 6 of the green book, which describes methods for computing the cohomology group on the right.

7 The chromatic spectral sequence

The chromatic spectral sequence

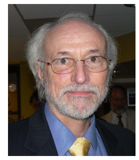




Annals of Mathematics, **106** (1977), 469-516

Periodic phenomena in the Adams-Novikov spectral sequence

By HAYNES R. MILLER, DOUGLAS C. RAVENEL, and W. STEPHEN WILSON



We now describe a way to see Morava's vision in the structure of the Adams-Novikov E_2-term. We will construct a long exact sequence of BP_*BP-comodules of the form

$$0 \longrightarrow BP_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow \dots$$

called the chromatic resolution. Then standard homological algebra gives a spectral sequence of the form

$$E_2^{h,s} = \mathrm{Ext}^s\left(M^h\right) \implies \mathrm{Ext}^{s+h}(BP_*)$$

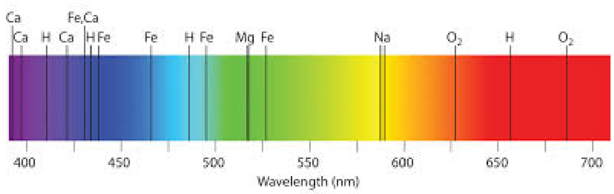
called the chromatic spectral sequence.

The chromatic spectral sequence (continued)

The chromatic spectral sequence

$$E_2^{h,s} = \mathrm{Ext}^s\left(M^h\right) \implies \mathrm{Ext}^{s+h}(BP_*)$$

Roughly speaking, its hth column, $\mathrm{Ext}\left(M^h\right)$, displays v_h-periodic phenomena. This decomposition of the Adams-Novikov E_2-term into its various frequencies is our reason for the use of the word chromatic.



The chromatic spectral sequence (continued)

We will construct a long exact sequence of BP_*BP-comodules of the form

$$0 \longrightarrow BP_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow \dots$$

called the chromatic resolution.

We will do so by splicing together the chromatic short exact sequences

$$\begin{aligned} 0 \longrightarrow N^0 := BP_* &\longrightarrow M^0 \longrightarrow N^1 \longrightarrow 0,\\ 0 \longrightarrow N^1 &\longrightarrow M^1 \longrightarrow N^2 \longrightarrow 0,\\ 0 \longrightarrow N^2 &\longrightarrow M^2 \longrightarrow N^3 \longrightarrow 0, \end{aligned}$$

and so on.

The chromatic spectral sequence (continued)

$$0 \longrightarrow N^0 := BP_* \longrightarrow M^0 \longrightarrow N^1 \longrightarrow 0,$$

We set $M^0 := BP_0 \otimes \mathbb{Q}$, so $N^1 = BP_* \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$, which we write as

$$N^1 = BP_*/p^\infty := \operatorname*{colim}_i BP_*/p^i.$$

Our first chromatic short exact sequence is

$$\begin{array}{ccccccc} 0 \longrightarrow & N^0 & \longrightarrow & M^0 & \longrightarrow & N^1 & \longrightarrow 0, \\ & \| & & \| & & \| & \\ & BP_* & & p^{-1}BP_* & & BP_*/(p^\infty) & \end{array}$$

We want the next one to be

$$\begin{array}{ccccccc} 0 \longrightarrow & N^1 & \longrightarrow & M^1 & \longrightarrow & N^2 & \longrightarrow 0, \\ & \| & & \| & & \| & \\ & BP_*/(p^\infty) & & v_1^{-1}BP_*/(p^\infty) & & BP_*/(p^\infty, v_1^\infty), & \end{array}$$

but inverting v_1 in the comodule category requires some care.

The chromatic spectral sequence (continued)

We want a short exact sequence of comodules

$$\begin{array}{ccccccc} 0 \longrightarrow & N^1 & \longrightarrow & M^1 & \longrightarrow & N^2 & \longrightarrow 0, \\ & \| & & \| & & \| & \\ & BP_*/(p^\infty) & & v_1^{-1}BP_*/(p^\infty) & & BP_*/(p^\infty, v_1^\infty), & \end{array}$$

but inverting v_1 in the comodule category requires some care.

Consider the BP_*-module $v_1^{-1}BP_*$. Since $\eta_R(v_1) = v_1 + pt_1$, formally we have

$$\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.$$

When $k < 0$, this sum is infinite and therefore does not lie in $v_1^{-1}BP_*BP$. This means that $v_1^{-1}BP_*$ is not a comodule. We claim that $v_1^{-1}BP_*/p^\infty$ is one nevertheless.

The chromatic spectral sequence (continued)

The following sum is infinite for $k < 0$.

$$\eta_R(v_1^k) = (v_1 + pt_1)^k = \sum_{i \geq 0} \binom{k}{i} p^i v_1^{k-i} t_1^i.$$

Each element in BP_*/p^∞ can be written as a fraction of the form

$$\frac{x}{p^j} \qquad \text{where } j > 0 \text{ and } x \in BP_* \text{ is not divisible by } p.$$

This element is killed by p^j. It follows that

$$\eta_R\left(\frac{v_1^k x}{p^j}\right) = \sum_{0 \leq i < j} \binom{k}{i} \frac{v_1^{k-i} t_1^i \eta_R(x)}{p^{j-i}}.$$

This sum is finite for all k, unlike the previous one, so $v_1^{-1}BP_*/p^\infty$ is a comodule as claimed.

The chromatic spectral sequence (continued)

Thus we have our second chromatic short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & N^1 \arrow[r] \arrow[d, equal] & M^1 \arrow[r] \arrow[d, equal] & N^2 \arrow[r] \arrow[d, equal] & 0 \\
& BP_*/(p^\infty) & v_1^{-1}BP_*/(p^\infty) & BP_*/(p^\infty, v_1^\infty). &
\end{tikzcd}$$

In a similar manner we can work by induction on h and construct

$$\begin{tikzcd}
0 \arrow[r] & N^h \arrow[r] \arrow[d, equal] & M^h \arrow[r] \arrow[d, equal] & N^{h+1} \arrow[r] \arrow[d, equal] & 0 \\
& BP_*/(p^\infty, \ldots, v_{h-1}^\infty) & v_h^{-1}N^h & BP_*/(p^\infty, \ldots, v_h^\infty). &
\end{tikzcd}$$

Splicing these together for all h gives the desired long exact sequence,

$$0 \longrightarrow BP_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow M^4 \longrightarrow \cdots.$$

The chromatic spectral sequence (continued)

Recall that the change-of-ring-isomorphism gives us a handle on $\operatorname{Ext}\left(v_h^{-1}BP_*/I_h\right)$. For $h = 1$, consider the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M_1^0 \arrow[r] \arrow[d, equal] & M^1 \arrow[r, "p"] \arrow[d, equal] & M^1 \arrow[r] \arrow[d, equal] & 0. \\
& v_1^{-1}BP_*/(p) & v_1^{-1}BP_*/(p^\infty) & v_1^{-1}BP_*/(p^\infty) &
\end{tikzcd}$$

This leads to a Bockstein spectral sequence of the form

$$\operatorname{Ext}\left(M_1^0\right) \otimes P(a_0) \Longrightarrow \operatorname{Ext}\left(M^1\right)$$

$$x \otimes a_0^j \rightsquigarrow \frac{x}{p^{j+1}}$$

The chromatic spectral sequence (continued)

For $h = 2$ we have two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_1^1 \arrow[r] \arrow[d, equal] & M^2 \arrow[r, "p"] & M^2 \arrow[r] \arrow[d, equal] & 0 \\
& v_2^{-1}BP_*/(p, v_1^\infty) & & v_2^{-1}BP_*/(p^\infty, v_1^\infty) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & M_0^2 \arrow[r] \arrow[d, equal] & \Sigma^{|v_1|}M_1^1 \arrow[r, "v_1"] & M_1^1 \arrow[r] & 0. \\
& v_2^{-1}BP_*/(p, v_1) & \frac{x}{pv_1^{i+1}} \arrow[r, mapsto] & \frac{x}{pv_1^i} &
\end{tikzcd}$$

$$x \longmapsto \frac{x}{pv_1}$$

Each one leads to a Bockstein spectral sequence, making the desired $\operatorname{Ext}\left(M^2\right)$ two steps removed from the known quantity $\operatorname{Ext}\left(v_2^{-1}BP_*/(p, v_1)\right)$.

The chromatic spectral sequence (continued)

More generally we have a short exact sequence of comodules

$$0 \longrightarrow \Sigma^{|v_i|}M_{i+1}^{h-i-1} \longrightarrow \Sigma^{|v_i|}M_i^{h-i} \xrightarrow{v_i} M_i^{h-i} \longrightarrow 0$$

for $0 \le i < h$, where $M_0^h = M^h$ and $v_0 = 0$. This leads to a Bockstein spectral sequence

$$\operatorname{Ext}\left(M_{i+1}^{h-i-1}\right) \otimes P(a_i) \Longrightarrow \operatorname{Ext}\left(M_i^{h-i}\right)$$

$$\frac{x}{pv_1 \cdots v_{i-1} v_i v_{i+1}^{j_{i+1}} \cdots v_{h-1}^{j_{h-1}}} \otimes a_i^j \rightsquigarrow \frac{x}{pv_1 \cdots v_{i-1} v_i^{j+1} \cdots v_{h-1}^{j_{h-1}}}.$$

This makes $\operatorname{Ext}\left(M^h\right)$ h steps removed from the cohomology of $\mathbb{S}_h$.

The chromatic spectral sequence (continued)

Computations with these Bockstein spectral sequence can be quite delicate. Nearly all of them published since 1977 have been due to Katsumi Shimomura and various coauthors.