ECHT Minicourse

What is the telescope conjecture? Lecture 1 An algebraic prelude to chromatic homotopy theory

Doug Ravenel
University of Rochester

December 5, 2023

1 The space BU

We begin with the space BU, the classifying space for the stable unitary group U. It also classifies complex vector bundles, and Whitney sum of such bundles induces a map

$$BU \times BU \to BU$$

that induces a multiplication

$$H_*BU \otimes H_*BU \to H_*BU$$

making H_*BU a graded ring. As such, it has the form

$$H_*BU \cong \mathbb{Z}[b_1, b_2, \ldots] \qquad \text{with } |b_i| = 2i.$$

Each b_i is the image of the standard generator $\beta_i \in H_{2i}\mathbb{C}P^\infty$ under the map $\mathbb{C}P^\infty = BU(1) \to BU$.

The best reference for this material is the 1974 book *Characteristic classes* by Milnor and Stasheff, [MS74].

For each finite n we have the inclusion map $BU(n) \to BU$. The image in homology is spanned by the monomials in the b_is of degree $\leq n$. The space $BU(n)$ is the Grassmannian of complex n-planes in $\mathbb{C}^\infty$. It classifies n-dimensional complex vector bundles. The total space of the universal n-plane bundle $\gamma_n^{\mathbb{C}}$ over it is

$$E(\gamma_n^{\mathbb{C}}) = \{(x,v) \in BU(n) \times \mathbb{C}^\infty : v \in [x]\},$$

where $[x]$ denotes the n-plane corresponding to x. By collapsing all points with $|v| \geq 1$ to a single point we get the Thom space $MU(n)$. One has a Thom isomorphism

$$H_k BU(n) \cong H_{k+2n} MU(n).$$

Since $E(\gamma_{n+1}^{\mathbb{C}})$ restricts under the map $BU(n) \to BU(n+1)$ to the bundle $\varepsilon_1^{\mathbb{C}} \oplus \gamma_n^{\mathbb{C}}$, where $\varepsilon_1^{\mathbb{C}}$ is the trivial line bundle, we get a map

$$\Sigma^2 MU(n) \to MU(n+1).$$

This means the spaces $MU(n)$ can be assembled into the spectrum MU with

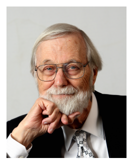
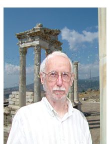
$$MU_{2n} = MU(n) \qquad \text{and} \qquad MU_{2n+1} = \Sigma MU(n).$$

MU has an E_∞-ring structure and is one of the nicest spectra you could ever hope to meet!

It gives us a very good tool for computing the homotopy groups of spheres, the Adams-Novikov spectral sequence, the subject of the green book [Rav86]. Hence we need to study its internal properties.

2 Properties of MU

Here are some wonderful things we know about MU:

- $H_*MU \cong \mathbb{Z}[b_1, b_2, \dots]$ by the Thom isomorphism.
- $MU_* := \pi_*MU \cong \mathbb{Z}[x_1, x_2, \dots]$, where $|x_i| = 2i$. This was proved independently by Milnor and Novikov around 1960.  
-   Localizing at a prime p gives a splitting
$$MU_{(p)} \simeq \bigvee \Sigma^? BP$$
with $\quad \pi_*BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$,
where $|v_h| = 2(p^h - 1)$.

  BP is the Brown-Peterson spectrum, first constructed in 1966.

More wonderful things we know about MU:

- $MU_*(MU)$, the MU-homology of MU itself, is $MU_*[b_1, b_2, \dots]$.
- The pair $(MU_*, MU_*(MU))$ forms a Hopf algebroid that we will say more about later. It was first studied by Novikov and Landweber in the late 60s.
- MU_* is also the complex cobordism ring. For each closed n-dimensional complex analytic manifold V there is an element $[V] \in \pi_{2n}MU$ represented by it.
- Recall that $MU_* \cong \mathbb{Z}[x_1, x_2, \dots]$. While the x_is do not have convenient descriptions, we know that
$$MU_* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots].$$

Still more wonderful things we know about MU:

- $MU^*\mathbb{C}P^\infty$, the MU cohomology of $\mathbb{C}P^\infty$, is the power series ring on one variable
$$MU^*x, \qquad \text{where } |x| = 2,$$
and MU^* is the negatively graded version of MU_*.
- The above is the limit over m of the MU^*-algebras
$$MU^*\mathbb{C}P^m \cong MU^*[x]/(x^{m+1}).$$
- Similarly
$$MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong MU^*x \otimes 1, 1 \otimes x.$$

The space $\mathbb{C}P^\infty$ classifies complex line bundles, and the tensor product of such is classified by a map

$$\mathbb{C}P^\infty \longleftarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty.$$

In cohomology this induces

$$\begin{array}{ccc} MU^*\mathbb{C}P^\infty & \longrightarrow & MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ \| & & \| \\ MU^*x & & MU^*x\otimes 1, 1\otimes x \end{array}$$

$$x \longmapsto F(x\otimes 1, 1\otimes x) := \sum_{i,j} a_{i,j} x^i \otimes x^j$$

where $a_{i,j} \in MU^{2(1-i-j)}$, and the sum is over all $i, j \geq 0$ with $i + j \geq 1$. Hence the sum is a homogeneous expression of dimension 2.

$$\begin{array}{ccc} MU^*\mathbb{C}P^\infty & \longrightarrow & MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ x & \longmapsto & F(x\otimes 1, 1\otimes x) := \sum_{i,j} a_{i,j} x^i \otimes x^j \end{array}$$

This power series is a linchpin of the theory. It is easily seen to have the following three properties:

1. Identity: $F(0,x) = F(x,0) = x$. This means $a_{1,0} = a_{0,1} = 1$ and $a_{i,0} = 0$ for $i > 1$.
2. Commutativity: $F(y,x) = F(x,y)$. This means $a_{j,i} = a_{i,j}$.
3. Associativity: $F(F(x,y),z) = F(x,F(y,z))$. This implies complicated relations among the $a_{i,j}$.

A power series $F(x,y) \in Rx,y$ satisfying these three conditions is called a formal group law over R. We know a lot about formal group laws.

3 Formal group laws

Here are some examples of formal group laws.

- The additive formal group law: $F(x,y) = x + y$.
- The multiplicative formal group law: $F(x,y) = x + y + xy$.
  Note that $(1+x)(1+y) = 1 + F(x,y)$.
- The tangent formal group law: $F(x,y) = (x+y)/(1-xy)$. Recall the trig identity $\tan(\alpha+\beta) = F(\tan\alpha, \tan\beta)$.
- Euler's elliptic integral addition formula:

$$F(x,y) = \frac{x\sqrt{1-y^4} + y\sqrt{1-x^4}}{1+x^2y^2} \in \mathbb{Z}[1/2]x,y.$$

Formal group laws were studied by Michel Lazard in 1955. He considered the ring $L = \mathbb{Z}[a_{i,j}]/(\sim)$, with relations implied by the three defining properties of the power series $F(x,y)$. This means that any formal group law F over any ring R is induced from G via a ring homomorphism $\theta : L \to R$. Hence L is the ground ring for the universal formal group law.

To describe L, we give it a grading with $|a_{i,j}| = 2(1-i-j)$. He then showed that

$$L \cong \mathbb{Z}[x_1, x_2, \dots] \qquad \text{with } |x_i| = -2i.$$

Quillen showed that the map $\theta : L \to MU^*$ (inducing the formal group law for complex cobordism) is an isomorphism! This means that homotopy theorists are married to one dimensional formal group laws.

3.1 Lazard's classification in characteristic p

For a formal group law F and a natural number n, we define power series $[n]_F(x)$, the n-series for F, recursively by

$$[0]_F(x) = 0 \qquad \text{and} \qquad [n]_F(x) = F(x, [n-1]_F(x)).$$

for example,

$$[n]_F(x) = \begin{cases} nx & \text{when } F \text{ is additive} \\ \displaystyle\sum_{1 \le i \le n} \binom{n}{i} x^i & \text{when } F \text{ is multiplicative .} \end{cases}$$

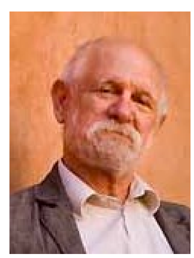
Of particular interest is the p-series $[p]_F(x)$ over R/p for each prime p. When R/p is a field, $[p]_F(x)$ is either 0 or has the form

$$ax^{p^h} + \cdots \qquad \text{for some } a \neq 0.$$

The exponent h is called the height of F at p. When $[p]_F(x) = 0$, the height is defined to be ∞. When R has characteristic zero, we can speak of its heights at the primes that are not invertible in it.

As for our previous examples,

- the additive formal group law has infinite height at all primes,
- the multiplicative formal group law has height 1 at all primes,
- the tangent formal group law has infinite height at $p = 2$ and height 1 at all odd primes, and
- Euler's formal group law over $\mathbb{Z}[1/2]$ has height 1 or 2 depending on whether p is congruent to 1 or 3 mod 4. Its height at $p = 2$ is not defined since 2 is invertible.



Lazard proved that two formal group laws over $\overline{\mathbb{F}}_p$ are isomorphic if and only if they have the same height. Jack Morava realized this has profound implications for homotopy theory. We will say much more about this later.

In addition

- Lazard described the automorphism group in each case. It is a compact p-adic Lie group now known as the Morava stabilizer group $\mathbb{S}_h$.
- We now know there are elements $v_h \in L$ with $|v_h| = 2(p^h - 1)$ such that the height of a formal group law induced by $\theta : L \to R$ is the smallest h with $\theta(v_h)0 \bmod p$.
- Consider the ideal $I_h = (v_0, v_1, \dots, v_{h-1}) \subset L$, where $v_0 = p$. Then the ascending chain of ideals
$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset L$$
leads to the chromatic filtration of the stable homotopy category. We will say more about this later.

Formal group laws with $[p](x) = x^{p^h}$ were constructed for all h and p in 1970 by Taira Honda. Hence all heights occur.

36cm

Billboard by Yuri Sulyma

3.2 The logarithm of a formal group law

Given two formal group laws F and G over a ring R, a map $f : F \to G$ is a power series $f(x)$ such that

$$f(F(x,y)) = G(f(x), f(y)).$$

It is an isomorphism if $f'(0)$ is unit in R and a strict isomorphism if $f'(0) = 1$. A logarithm for F is a strict isomorphism to the additive formal group law.

Logarithm Theorem. *Let F be a formal group law over R, and let*

$$f(x)=\int_0^x \frac{dt}{F_2(t,0)} \qquad \in \qquad (R\otimes\mathbb{Q})x,$$

where $F_2(x,y)=\partial F/\partial y$. Then f is a logarithm for F, i.e., $F(x,y)=f^{-1}(f(x)+f(y))$, and F is isomorphic over $R\otimes\mathbb{Q}$ to the additive formal group law.

Logarithm Theorem. *Let F be a formal group law over R, and let*

$$f(x)=\int_0^x \frac{dt}{F_2(t,0)} \qquad \in \qquad (R\otimes\mathbb{Q})x,$$

where $F_2(x,y)=\partial F/\partial y$. Then f is a logarithm for F, i.e., $F(x,y)=f^{-1}(f(x)+f(y))$, and F is isomorphic over $R\otimes\mathbb{Q}$ to the additive formal group law.

This is Theorem A2.1.6 of the green book and its proof is a calculus exercise. Applying it to the formal group law for complex cobordism gives Mischenko's theorem of 1967,

$$\log_{MU}(x)=\sum_{n\geq 0} m_n x^{n+1} := \sum_{n\geq 0}\frac{[\mathbb{C}P^n]x^{n+1}}{n+1}.$$

Again Mischenko's theorem is

$$\log_{MU}(x)=\sum_{n\geq 0} m_n x^{n+1} := \sum_{n\geq 0}\frac{[\mathbb{C}P^n]x^{n+1}}{n+1}.$$

The analogous formula for BP-theory is

$$\log_{BP}(x)=\sum_{k\geq 0}\ell_k x^{p^k} := \sum_{k\geq 0}\frac{[\mathbb{C}P^{p^k-1}]x^{p^k}}{p^k}.$$

Recall that

$$BP_*=\pi_*BP\cong\mathbb{Z}_{(p)}[v_1,v_2,\dots] \qquad \text{with } |v_h|=2(p^h-1).$$

The v_ks and the ℓ_ks are related by the following recursive formula due to Hazewinkel 1976

$$p\ell_k=\sum_{0\leq i<k}\ell_i v_{k-i}^{p^i}.$$

Again Hazewinkel's formula is

$$p\ell_k=\sum_{0\leq i<k}\ell_i v_{k-i}^{p^i}=v_k+\sum_{0<i<k}\ell_i v_{k-i}^{p^i}.$$

This yields

$$\begin{aligned}
\ell_1 &= \frac{v_1}{p},\\
\ell_2 &= \frac{v_2}{p}+\frac{v_1^{p+1}}{p^2},\\
\ell_3 &= \frac{v_3}{p}+\frac{v_1v_2^p+v_2v_1^{p^2}}{p^2}+\frac{v_1^{1+p+p^2}}{p^3},
\end{aligned}$$

and so on. Recall that the height of a formal group law F over a ring R in characteristic p is the smallest h such that v_h has nontrivial image under the homomorphism $L\to R$ inducing F. Here are some examples of logarithms.

- For the additive formal group law, $F(x,y)=x+y$, it is x.

- For the multiplicative formal group law, $F(x,y) = x+y+xy$, it is

$$\ln(1+x) = \sum_{i\geq 0} \frac{(-1)^i x^{i+1}}{i+1} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots.$$

- For the tangent formal group law, $F(x,y) = (x+y)/(1-xy)$, it is

$$\arctan x = \sum_{i\geq 0} \frac{(-1)^i x^{2i+1}}{2i+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots.$$

- For Euler's elliptic integral addition formula, it is

$$\sum_{i\geq 0} \binom{2i}{i} \frac{x^{4i+1}}{4^i(4i+1)} = x + \frac{x^5}{10} + \frac{x^9}{24} + \frac{5x^{13}}{208} + \cdots.$$

- For Honda's height h formal group law F_h, it is

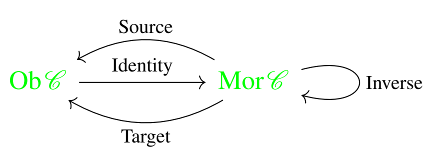
$$\sum_{k\geq 0} \frac{x^{p^{kh}}}{p^k} = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \cdots.$$

3.3 The Landweber-Novikov groupoid

We want to set up and study the Adams spectral sequences based on MU- and BP-theories. This requires a working knowledge of the structures of $MU_*(MU)$ and $BP_*(BP)$. These are the analogs of dual Steenrod algebra in ordinary mod p homology.

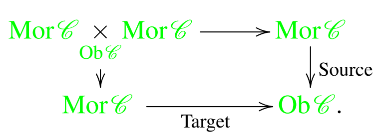
Rather than getting into the nuts and bolts of these objects, which are discussed thoroughly in the green book, we will present a conceptual picture of them.

Recall that a groupoid is a small category $\mathscr{C}$ in which each morphism is invertible. Thus we have sets of objects $\mathrm{Ob}\,\mathscr{C}$ and morphisms $\mathrm{Mor}\,\mathscr{C}$ and four maps between them shown below.

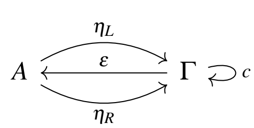


These satisfy some obvious identities.

We also have composition of morphisms, which is a map to $\mathrm{Mor}\,\mathscr{C}$ from a certain subset of its product with itself, that of composable pairs of morphisms, namely the pullback of the diagram



A groupoid scheme over a commutative ring K is a functor that assigns a groupoid to each commutative K-algebra R. It is affine if it representable. An affine groupoid scheme is also called a Hopf algebroid. This means there are K-algebras A and Γ such that the object and morphism sets for a K-algebra R are $\mathrm{Alg}_{\mathrm{K}}(A,R)$ and $\mathrm{Alg}_{\mathrm{K}}(\Gamma,R)$ respectively. There are corresponding maps between A and Γ shown below.



Here composition corresponds to a coproduct map $\Delta : \Gamma \to \Gamma \otimes_A \Gamma$, where the tensor product is defined using the right and left A-module structures on Γ given by the right and left units η_R and η_L corresponding to the target and source maps in $\mathscr{C}$. The maps c and ε are the conjugation and counit.

The case of interest to us is the affine groupoid scheme that assigns to each commutative ring R the category of formal group laws over it and (possibly strict) isomorphisms between them. The ring representing the object set is the Lazard ring L, which is isomorphic to MU_*. An isomorphism can be any power series of the form

$$f(x) = \sum_{i \geq 0} b_i x^{i+1},$$

where b_0 is invertible and, in the strict isomorphism case, $b_0 = 1$. This means the morphism set is represented by the ring

$$L[b_0^{\pm 1}, b_1, b_2, \ldots] \quad \text{or} \quad L[b_1, b_2, \ldots] \cong MU_*(MU).$$

In the late 1960s Landweber and Novikov found explicit descriptions of the structure maps.

For explicit computations, it is more convenient to use BP-theory, even though the spectrum BP does not have as much multiplicative structure as MU.

We have

$$BP_*BP \cong BP_*[t_1, t_{2,\ldots}] \qquad \text{where } |t_i| = 2(p^i - 1).$$

The following formulas are due to Quillen. The right unit and coproduct maps, after tensoring with $\mathbb{Q}$, are

$$\eta_R(\ell_i) = \sum_{0 \leq i \leq h} \ell_i \otimes t_{h-i}^{p^i}$$

and

$$\sum_{0 \leq i \leq h} \ell_i \Delta(t_{h-i})^{p^i} = \sum_{\substack{0 \leq i \leq h \\ 0 \leq j \leq h-i}} \ell_i t_j^{p^i} \otimes t_{h-i-j}^{p^{i+j}}.$$

References

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. Second edition available online at author's home page.