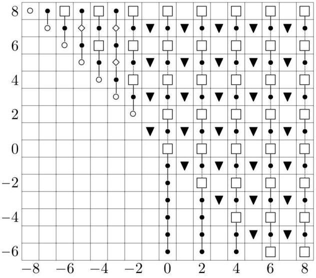
math for babies

math for grownups

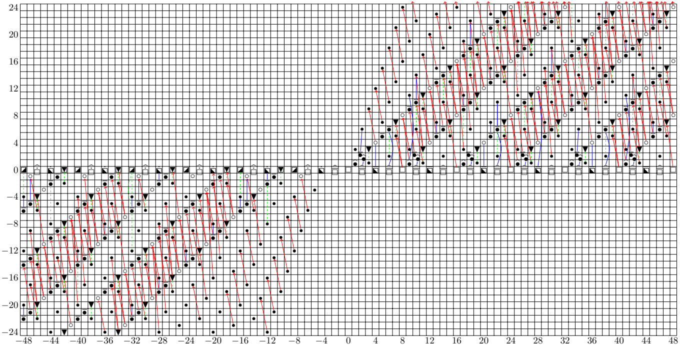


Studies show that shape-sorter toys help young children to learn tactile and motor skills, shape and color identification, and equivariant stable homotopy theory.

Derived memes for spectral schemes, March 20, 2019

The eightfold way: how to build the right model structure on orthogonal G-spectra

Mike Hill UCLA
Mike Hopkins Harvard University
Doug Ravenel University of Rochester



Electronic Computational Homotopy Theory Seminar

October 3, 2019

Functor categories

Orthogonal *G*-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category

Functor categories

Orthogonal *G*-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category,

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category.

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$,

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let *J* be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (*J*-shaped diagrams in $\mathcal{M}$) as follows:

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X,

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X, we denote its value on $j \in J$ by X_j,

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X, we denote its value on $j \in J$ by X_j, and the jth component of a map (natural transformation) $f : X \to Y$ by f_j.

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X, we denote its value on $j \in J$ by X_j, and the jth component of a map (natural transformation) $f : X \to Y$ by f_j.
- A map $f : X \to Y$ is defined to be a fibration or a weak equivalence if f_j is one for each j.

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X, we denote its value on $j \in J$ by X_j, and the jth component of a map (natural transformation) $f : X \to Y$ by f_j.
- A map $f : X \to Y$ is defined to be a fibration or a weak equivalence if f_j is one for each j.
- Cofibrations are defined in terms of lifting properties.

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X, we denote its value on $j \in J$ by X_j, and the jth component of a map (natural transformation) $f : X \to Y$ by f_j.
- A map $f : X \to Y$ is defined to be a fibration or a weak equivalence if f_j is one for each j.
- Cofibrations are defined in terms of lifting properties. Each f_j must be a cofibration for f to be one,

A naive model structure

Let $\mathcal{M}$ be a pointed topological symmetric monoidal model category and let J be a small category, the indexing category. We define the projective model structure on $[J, \mathcal{M}]$, the category of functors $J \to \mathcal{M}$ (J-shaped diagrams in $\mathcal{M}$) as follows:

- For such a functor X, we denote its value on $j \in J$ by X_j, and the jth component of a map (natural transformation) $f : X \to Y$ by f_j.
- A map $f : X \to Y$ is defined to be a fibration or a weak equivalence if f_j is one for each j.
- Cofibrations are defined in terms of lifting properties. Each f_j must be a cofibration for f to be one, but this is not sufficient.

More about $[J, \mathcal{M}]$

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$.

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$,

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

$$(X \wedge K)_j = X_j \wedge K.$$

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

$$(X \wedge K)_j = X_j \wedge K.$$

Similarly a map $g : K \to L$ in $\mathcal{M}$ induces a map

$$X \wedge K \xrightarrow{X \wedge g} X \wedge L \qquad \text{in } [J, \mathcal{M}].$$

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

$$(X \wedge K)_j = X_j \wedge K.$$

Similarly a map $g : K \to L$ in $\mathcal{M}$ induces a map

$$X \wedge K \xrightarrow{X \wedge g} X \wedge L \qquad \text{in } [J, \mathcal{M}].$$

For each $j \in J$ we have the Yoneda functor よ^j in $[J, \mathcal{M}]$ defined by

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

$$(X \wedge K)_j = X_j \wedge K.$$

Similarly a map $g : K \to L$ in $\mathcal{M}$ induces a map

$$X \wedge K \xrightarrow{X \wedge g} X \wedge L \qquad \text{in } [J, \mathcal{M}].$$

For each $j \in J$ we have the Yoneda functor よ^j in $[J, \mathcal{M}]$ defined by

$$\left(\text{よ}^j\right)_k = J(j, k).$$

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

$$(X \wedge K)_j = X_j \wedge K.$$

Similarly a map $g : K \to L$ in $\mathcal{M}$ induces a map

$$X \wedge K \xrightarrow{X \wedge g} X \wedge L \qquad \text{in } [J, \mathcal{M}].$$

For each $j \in J$ we have the Yoneda functor よ^j in $[J, \mathcal{M}]$ defined by

$$\left(\text{よ}^j\right)_k = J(j, k).$$

If J is an ordinary category, this is a set and therefore a coproduct of points (terminal objects) in the model category $\mathcal{M}$.

More about $[J, \mathcal{M}]$

$[J, \mathcal{M}]$ is tensored over $\mathcal{M}$. This means for for a functor X and object K in $\mathcal{M}$, we can define a new functor $X \wedge K$ by

$$(X \wedge K)_j = X_j \wedge K.$$

Similarly a map $g : K \to L$ in $\mathcal{M}$ induces a map

$$X \wedge K \xrightarrow{X \wedge g} X \wedge L \qquad \text{in } [J, \mathcal{M}].$$

For each $j \in J$ we have the Yoneda functor よ^j in $[J, \mathcal{M}]$ defined by

$$\left(\text{よ}^j\right)_k = J(j, k).$$

If J is an ordinary category, this is a set and therefore a coproduct of points (terminal objects) in the model category $\mathcal{M}$.

If J is enriched over $\mathcal{M}$, each morphism object $J(j, k)$ is an object in $\mathcal{M}$ rather than a set.

More about $[J, \mathcal{M}]$ (continued)

More about $[J, \mathcal{M}]$ (continued)

Suppose in addition that $\mathcal{M}$ is cofibrantly generated

More about $[J, \mathcal{M}]$ (continued)

Suppose in addition that $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$.

More about $[J, \mathcal{M}]$ (continued)

Suppose in addition that $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$. Then $[J, \mathcal{M}]$ is also cofibrantly generated.

More about $[J, \mathcal{M}]$ (continued)

Suppose in addition that $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$. Then $[J, \mathcal{M}]$ is also cofibrantly generated. Its generating sets are

$$F^J\mathcal{I} := \left\{ \text{よ}^j \wedge f \colon f \in \mathcal{I}, j \in J \right\}$$

and

$$F^J\mathcal{J} := \left\{ \text{よ}^j \wedge f \colon f \in \mathcal{J}, j \in J \right\}.$$

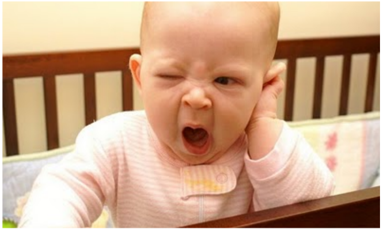
More about $[J, \mathcal{M}]$ (continued)

Suppose in addition that $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$. Then $[J, \mathcal{M}]$ is also cofibrantly generated. Its generating sets are

$$F^J\mathcal{I} := \left\{ よ^j \wedge f\colon f \in \mathcal{I}, j \in J \right\}$$

$$\text{and} \qquad F^J\mathcal{J} := \left\{ よ^j \wedge f\colon f \in \mathcal{J}, j \in J \right\}.$$

Are you bored yet?



More about $[J, \mathcal{M}]$ (continued)

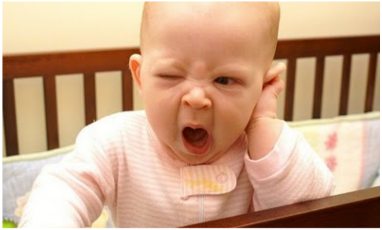
Suppose in addition that $\mathcal{M}$ is cofibrantly generated with generating sets $\mathcal{I}$ and $\mathcal{J}$. Then $[J, \mathcal{M}]$ is also cofibrantly generated. Its generating sets are

$$F^J\mathcal{I} := \left\{ \text{よ}^j \wedge f \colon f \in \mathcal{I}, j \in J \right\}$$

and

$$F^J\mathcal{J} := \left\{ \text{よ}^j \wedge f \colon f \in \mathcal{J}, j \in J \right\}.$$

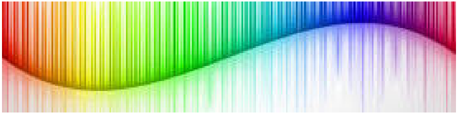
Are you bored yet?



WHY DO WE CARE ABOUT MODEL STRUCTURES ON FUNCTOR CATEGORIES?

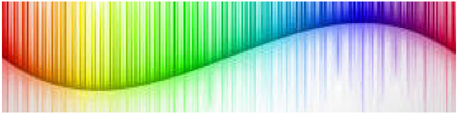
Orthogonal *G*-spectra as functors

Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

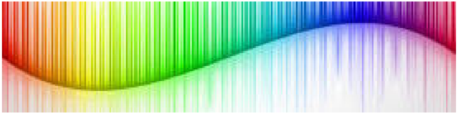
Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$,

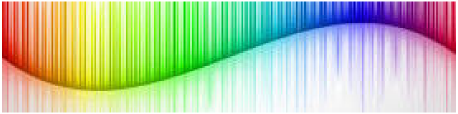
Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps.

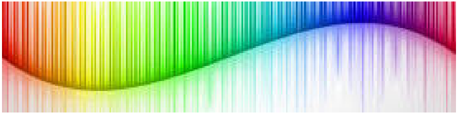
Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration

Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration if the same is true of the fixed point map $f^H : K^H \to L^H$

Functor categories

Orthogonal G-spectra as functors

Modifying the model structure

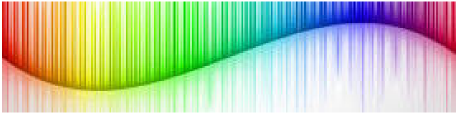
The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

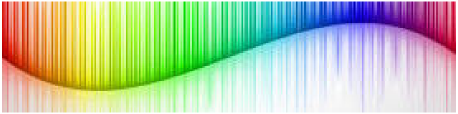
Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration if the same is true of the fixed point map $f^H : K^H \to L^H$ for each subgroup $H \subseteq G$.

Orthogonal G-spectra as functors

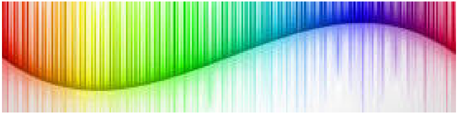


For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration if the same is true of the fixed point map $f^H : K^H \to L^H$ for each subgroup $H \subseteq G$.

Cofibrations are defined in terms of left lifting properties.

Orthogonal G-spectra as functors



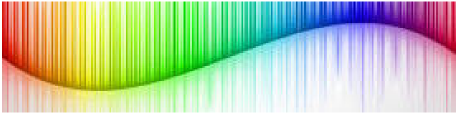
For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration if the same is true of the fixed point map $f^H : K^H \to L^H$ for each subgroup $H \subseteq G$.

Cofibrations are defined in terms of left lifting properties.

It is cofibrantly generated.

Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

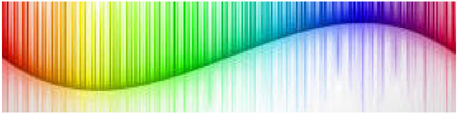
The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration if the same is true of the fixed point map $f^H : K^H \to L^H$ for each subgroup $H \subseteq G$.

Cofibrations are defined in terms of left lifting properties.

It is cofibrantly generated. Its generating sets are

$$\mathcal{I}^G = \left\{ G_+ \underset{H}{\wedge} (S^{n-1}_+ \hookrightarrow D^n_+) \colon H \subseteq G, n \geq 0 \right\}$$

Orthogonal G-spectra as functors



For a finite group G, the category $\mathcal{S}p^G$ of orthogonal G-spectra is such an enriched functor category $[J, \mathcal{M}]$.

The relevant model category is $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps. In it a map $f : K \to L$ is a weak equivalence or a fibration if the same is true of the fixed point map $f^H : K^H \to L^H$ for each subgroup $H \subseteq G$.

Cofibrations are defined in terms of left lifting properties.

It is cofibrantly generated. Its generating sets are

$$\mathcal{I}^G = \left\{ G_+ \underset{H}{\wedge} (S^{n-1}_+ \hookrightarrow D^n_+) \colon H \subseteq G, n \geq 0 \right\}$$

and

$$\mathcal{J}^G = \left\{ G_+ \underset{H}{\wedge} (I^n_+ \hookrightarrow I^{n+1}_+) \colon H \subseteq G, n \geq 0 \right\}.$$

Orthogonal G-spectra as functors (continued)

Orthogonal *G*-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$.

Orthogonal *G*-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$,

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is defined to be its Thom space.

Functor categories

Orthogonal G-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is defined to be its Thom space. It is a pointed G-space.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is defined to be its Thom space. It is a pointed G-space.

For representations U, V and W there is a composition morphism in $\mathcal{T}^G$,

$$j_{U,V,W} : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$$

Functor categories
Orthogonal G-spectra as functors
Modifying the model structure
The Crans-Kan transfer theorem
Equifibrant enlargement
Positivization
Stabilization

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is defined to be its Thom space. It is a pointed G-space.

For representations U, V and W there is a composition morphism in $\mathcal{T}^G$,

$$j_{U,V,W} : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$$

induced by composition of orthogonal embeddings $U \to V \to W$.

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is defined to be its Thom space. It is a pointed G-space.

For representations U, V and W there is a composition morphism in $\mathcal{T}^G$,

$$j_{U,V,W} : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$$

induced by composition of orthogonal embeddings $U \to V \to W$. It is equivariant,

Orthogonal G-spectra as functors (continued)

The relevant indexing category is the Mandell-May category $\mathscr{J}_G$, which is enriched over $\mathcal{T}^G$. Its objects are finite dimensional orthogonal representations V of G.

To define the morphism object (pointed G-space) $\mathscr{J}_G(V, W)$, let $O(V, W)$ denote the space of (nonequivariant) orthogonal embeddings of V into W. It is a Stiefel manifold which could be empty. The group G acts on it by conjugation.

Each such embedding $t : V \to W$ defines an orthogonal complement $t(V)^{\perp} \subseteq W$. Thus we get a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is defined to be its Thom space. It is a pointed G-space.

For representations U, V and W there is a composition morphism in $\mathcal{T}^G$,

$$j_{U,V,W} : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$$

induced by composition of orthogonal embeddings $U \to V \to W$. It is equivariant, even though the embeddings of vector spaces need not be.

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point,

Functor categories

Orthogonal G-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.
- When the dimension of V exceeds that of W, then the embedding space is empty, and $\mathscr{J}_G(V, W)$ is a point.

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.
- When the dimension of V exceeds that of W, then the embedding space is empty, and $\mathscr{J}_G(V, W)$ is a point.
- When V and W have the same dimension, the embedding space is the orthogonal group $O(V)$,

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.
- When the dimension of V exceeds that of W, then the embedding space is empty, and $\mathscr{J}_G(V, W)$ is a point.
- When V and W have the same dimension, the embedding space is the orthogonal group $O(V)$, with an action of G defined in terms of its actions on V and W.

Functor categories

Orthogonal G-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.
- When the dimension of V exceeds that of W, then the embedding space is empty, and $\mathscr{J}_G(V, W)$ is a point.
- When V and W have the same dimension, the embedding space is the orthogonal group $O(V)$, with an action of G defined in terms of its actions on V and W. The vector bundle is zero dimensional,

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.
- When the dimension of V exceeds that of W, then the embedding space is empty, and $\mathscr{J}_G(V, W)$ is a point.
- When V and W have the same dimension, the embedding space is the orthogonal group $O(V)$, with an action of G defined in terms of its actions on V and W. The vector bundle is zero dimensional, so its Thom space $\mathscr{J}_G(V, W)$ is $O(V)_+$,

Orthogonal G-spectra as functors (continued)

The morphism object $\mathscr{J}_G(V, W)$ is the Thom space of the orthogonal complement vector bundle over the space $O(V, W)$ of (nonequivariant) orthogonal embeddings of V into W.

Some examples:

- For $V = 0$, the embedding space $O(V, W)$ is a single point, and $\mathscr{J}_G(0, W) = S^W$, the one point compactification of W.
- When the dimension of V exceeds that of W, then the embedding space is empty, and $\mathscr{J}_G(V, W)$ is a point.
- When V and W have the same dimension, the embedding space is the orthogonal group $O(V)$, with an action of G defined in terms of its actions on V and W. The vector bundle is zero dimensional, so its Thom space $\mathscr{J}_G(V, W)$ is $O(V)_+$, the orthogonal group with a disjoint base point.

Orthogonal G-spectra as functors (continued)

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$.

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$.

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$. In particular, X_V has an action of the orthogonal group $O(V)$.

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$. In particular, X_V has an action of the orthogonal group $O(V)$.

The Yoneda functor よ^V becomes the Yoneda spectrum S^{-V}

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$. In particular, X_V has an action of the orthogonal group $O(V)$.

The Yoneda functor よ^V becomes the Yoneda spectrum S^{-V} defined by $(S^{-V})_W = \mathscr{J}_G(V, W)$.

Functor categories

Orthogonal G-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$. In particular, X_V has an action of the orthogonal group $O(V)$.

The Yoneda functor $よ^{V}$ becomes the Yoneda spectrum S^{-V} defined by $(S^{-V})_W = \mathscr{J}_G(V, W)$. Its structure maps are composition morphisms in $\mathscr{J}_G$.

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$. In particular, X_V has an action of the orthogonal group $O(V)$.

The Yoneda functor よ^V becomes the Yoneda spectrum S^{-V} defined by $(S^{-V})_W = \mathscr{J}_G(V, W)$. Its structure maps are composition morphisms in $\mathscr{J}_G$.

In particular, $(S^{-0})_W = \mathscr{J}_G(0, W) = S^W$

Orthogonal G-spectra as functors (continued)

An orthogonal G-spectrum X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$. This means it consists of

- a collection pointed G-spaces X_V, one for each representation V of G, and
- structure maps $\mathscr{J}_G(V, W) \wedge X_V \to X_W$. In particular, X_V has an action of the orthogonal group $O(V)$.

The Yoneda functor よ^V becomes the Yoneda spectrum S^{-V} defined by $(S^{-V})_W = \mathscr{J}_G(V, W)$. Its structure maps are composition morphisms in $\mathscr{J}_G$.

In particular, $(S^{-0})_W = \mathscr{J}_G(0, W) = S^W$ and S^{-0} is the sphere spectrum.

The projective model structure for orthogonal G-spectra

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use!

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.
2. It needs to play nicely with change of groups.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.
2. It needs to play nicely with change of groups. For $H \subseteq G$ there is a change of group adjunction

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.
2. It needs to play nicely with change of groups. For $H \subseteq G$ there is a change of group adjunction

   $$\begin{tikzcd} G_+ \underset{H}{\wedge} (-) : \mathcal{S}p^H \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G : i_H^G, \arrow[l, shift left] \end{tikzcd}$$

   where i_H^G is the restriction function.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.
2. It needs to play nicely with change of groups. For $H \subseteq G$ there is a change of group adjunction

   $$\begin{tikzcd} G_+ \underset{H}{\wedge} (-) : \mathcal{S}p^H \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G : i_H^G, \arrow[l, shift left] \end{tikzcd}$$

   where i_H^G is the restriction functor. It needs to be a Quillen adjunction.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.

2. It needs to play nicely with change of groups. For $H \subseteq G$ there is a change of group adjunction

$$\begin{tikzcd} G_+ \underset{H}{\wedge} (-) : \mathcal{S}p^H \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G : i_H^G, \arrow[l, shift left] \end{tikzcd}$$

   where i_H^G is the restriction functor. It needs to be a Quillen adjunction. This means the class of cofibrations in $\mathcal{S}p^G$ needs to be enlarged

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.
2. It needs to play nicely with change of groups. For $H \subseteq G$ there is a change of group adjunction

   $$\begin{tikzcd} G_+ \underset{H}{\wedge} (-) : \mathcal{S}p^H \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G : i_H^G, \arrow[l, shift left] \end{tikzcd}$$

   where i_H^G is the restriction functor. It needs to be a Quillen adjunction. This means the class of cofibrations in $\mathcal{S}p^G$ needs to be enlarged to include cofibrations induced up from H.

The projective model structure for orthogonal G-spectra

The category $\mathcal{S}p^G$ of orthogonal G-spectra is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. Hence it has a projective model structure as boringly described above. It is NOT the one we want to use! It needs to be modified in three different ways.

1. The levelwise weak equivalences of the projective model structure need to be replaced by stable equivalences. This is a form of Bousfield localization.
2. It needs to play nicely with change of groups. For $H \subseteq G$ there is a change of group adjunction

   $$\begin{tikzcd} G_+ \underset{H}{\wedge} (-) : \mathcal{S}p^H \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{S}p^G : i_H^G, \arrow[l, shift left] \end{tikzcd}$$

   where i_H^G is the restriction functor. It needs to be a Quillen adjunction. This means the class of cofibrations in $\mathcal{S}p^G$ needs to be enlarged to include cofibrations induced up from H. When we have this for each H, we say the model structure is equifibrant.

The projective model structure for orthogonal G-spectra (continued)

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized,

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized, a term to be defined later.

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized, a term to be defined later. This is needed to define a model structure on the category of commutative ring spectra.

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized, a term to be defined later. This is needed to define a model structure on the category of commutative ring spectra. It involves confining the class of cofibrations in a certain way.

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized, a term to be defined later. This is needed to define a model structure on the category of commutative ring spectra. It involves confining the class of cofibrations in a certain way. After we do this,

The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized, a term to be defined later. This is needed to define a model structure on the category of commutative ring spectra. It involves confining the class of cofibrations in a certain way. After we do this, the sphere spectrum S^{-0} will no longer be cofibrant.

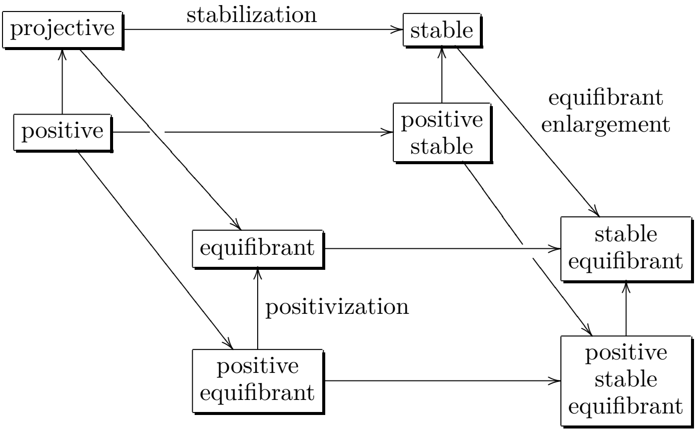
The projective model structure for orthogonal G-spectra (continued)

The projective model structure on $\mathcal{S}p^G$ needs to be modified in three different ways.

3. It needs to be positivized, a term to be defined later. This is needed to define a model structure on the category of commutative ring spectra. It involves confining the class of cofibrations in a certain way. After we do this, the sphere spectrum S^{-0} will no longer be cofibrant. This strange state of affairs is unavoidable.

Eight model structures for orthogonal *G*-spectra

Eight model structures for orthogonal *G*-spectra



Eight model structures for orthogonal G-spectra

$$\begin{tikzcd}
\boxed{\text{projective}} \arrow[rr, "\text{stabilization}"] \arrow[ddr] & & \boxed{\text{stable}} \arrow[ddr, "\text{equifibrant enlargement}"] & \\
\boxed{\text{positive}} \arrow[u] \arrow[rr] \arrow[ddr] & & \boxed{\text{positive stable}} \arrow[u] \arrow[ddr] & \\
& \boxed{\text{equifibrant}} \arrow[rr] & & \boxed{\text{stable equifibrant}} \\
& \boxed{\text{positive equifibrant}} \arrow[u, "\text{positivization}"] \arrow[rr] & & \boxed{\text{positive stable equifibrant}} \arrow[u]
\end{tikzcd}$$

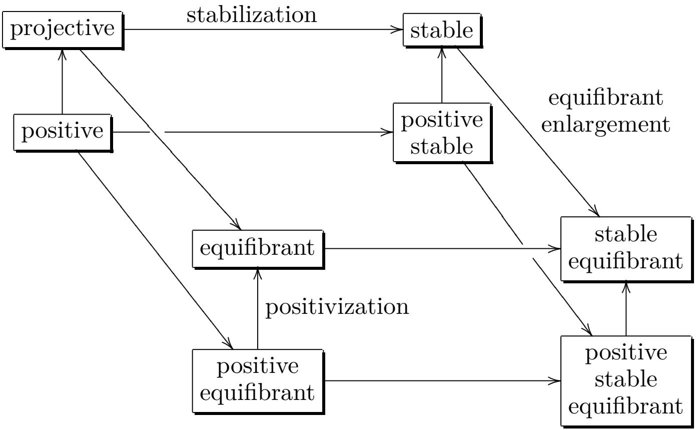
Each arrow denotes the identity functor as a left Quillen functor.

Eight model structures for orthogonal *G*-spectra

$$\begin{tikzcd}
\text{projective} \arrow[rr, "\text{stabilization}"] \arrow[ddr] & & \text{stable} \arrow[ddr, "\text{equifibrant enlargement}"] & \\
\text{positive} \arrow[u] \arrow[rr] \arrow[ddr] & & \text{positive stable} \arrow[u] \arrow[ddr] & \\
& \text{equifibrant} \arrow[rr] & & \text{stable equifibrant} \\
& \text{positive equifibrant} \arrow[u, "\text{positivization}"] \arrow[rr] & & \text{positive stable equifibrant} \arrow[u]
\end{tikzcd}$$

Each arrow denotes the identity functor as a left Quillen functor. The top four model structures were described by Mandell-May.

Eight model structures for orthogonal *G*-spectra



$$\begin{tikzcd}
\boxed{\text{projective}} \arrow[rr, "\text{stabilization}"] \arrow[ddr] & & \boxed{\text{stable}} \arrow[ddr, "\text{equifibrant enlargement}"] & \\
\boxed{\text{positive}} \arrow[u] \arrow[rr] \arrow[ddr] & & \boxed{\text{positive stable}} \arrow[u] \arrow[ddr] & \\
& \boxed{\text{equifibrant}} \arrow[rr] & & \boxed{\text{stable equifibrant}} \\
& \boxed{\text{positive equifibrant}} \arrow[u, "\text{positivization}"] \arrow[rr] & & \boxed{\text{positive stable equifibrant}} \arrow[u]
\end{tikzcd}$$

Each arrow denotes the identity functor as a left Quillen functor. The top four model structures were described by Mandell-May. Our model structure of choice is the positive stable equifibrant one on the lower right.

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category,

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let

$$\begin{tikzcd}
\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

be a pair of adjoint functors.

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let

$$\begin{tikzcd}
\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

be a pair of adjoint functors. For cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ be of $\mathcal{M}$,

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a pair of adjoint functors. For cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ be of $\mathcal{M}$, let $F\mathcal{I} = \{Fi\colon i \in \mathcal{I}\}$ and $F\mathcal{J} = \{Fj\colon j \in \mathcal{J}\}$.

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let

$$\begin{tikzcd}
\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

be a pair of adjoint functors. For cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ be of $\mathcal{M}$, let $F\mathcal{I} = \{Fi \colon i \in \mathcal{I}\}$ and $F\mathcal{J} = \{Fj \colon j \in \mathcal{J}\}$. Then the above is a **transfer adjunction**, and (F, U) is a **transfer pair**, if

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let

$$\begin{tikzcd}\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]\end{tikzcd}$$

be a pair of adjoint functors. For cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ be of $\mathcal{M}$, let $F\mathcal{I} = \{Fi\colon i \in \mathcal{I}\}$ and $F\mathcal{J} = \{Fj\colon j \in \mathcal{J}\}$. Then the above is a **transfer adjunction**, and (F, U) is a **transfer pair**, if

① both $F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument in $\mathcal{N}$ and

The Crans-Kan transfer theorem

Definition

Let $\mathcal{M}$ be a cofibrantly generated model category, let $\mathcal{N}$ be a bicomplete category and let

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a pair of adjoint functors. For cofibrant generating sets $\mathcal{I}$ and $\mathcal{J}$ be of $\mathcal{M}$, let $F\mathcal{I} = \{Fi\colon i \in \mathcal{I}\}$ and $F\mathcal{J} = \{Fj\colon j \in \mathcal{J}\}$. Then the above is a **transfer adjunction**, and (F, U) is a **transfer pair**, if

1. both $F\mathcal{I}$ and $F\mathcal{J}$ permit the small object argument in $\mathcal{N}$ and
2. U takes relative $F\mathcal{J}$-cell complexes in $\mathcal{N}$ to weak equivalences in $\mathcal{M}$.

The Crans-Kan transfer theorem (continued)

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a transfer adjunction as above.

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd}\mathcal{M} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left, "U"]\end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**),

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd}\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]\end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{F}\mathcal{I}$ and $\mathcal{F}\mathcal{J}$ are cofibrant generating sets,

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{F}\mathcal{I}$ and $\mathcal{F}\mathcal{J}$ are cofibrant generating sets, and the weak equivalences and fibrations are the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$.

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd}\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]\end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{F}\mathcal{I}$ and $\mathcal{F}\mathcal{J}$ are cofibrant generating sets, and the weak equivalences and fibrations are the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen pair.

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{F}\mathcal{I}$ and $\mathcal{F}\mathcal{J}$ are cofibrant generating sets, and the weak equivalences and fibrations are the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen pair.

This is our main tool for constructing new model structures.

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{F}\mathcal{I}$ and $\mathcal{F}\mathcal{J}$ are cofibrant generating sets, and the weak equivalences and fibrations are the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen pair.

This is our main tool for constructing new model structures. Note that $\mathcal{N}$ does not have a model structure to begin with.

The Crans-Kan transfer theorem (continued)

Crans-Kan Transfer Theorem

Let

$$\begin{tikzcd}\mathcal{M} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, shift left=2, "U"]\end{tikzcd}$$

be a transfer adjunction as above. Then there is a cofibrantly generated model structure on $\mathcal{N}$ (the **transferred model structure**), for which $\mathcal{FI}$ and $\mathcal{FJ}$ are cofibrant generating sets, and the weak equivalences and fibrations are the maps taken by U to weak equivalences and fibrations in $\mathcal{M}$. Furthermore, with respect to this model structure, (F, U) is a Quillen pair.

This is our main tool for constructing new model structures. Note that $\mathcal{N}$ does not have a model structure to begin with. It gets one though the adjunction.

Enlarging the class of cofibrations in a model category

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}\mathcal{M}' \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "U"]\end{tikzcd}$$

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences.

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences. It need not be a Quillen adjunction.

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences. It need not be a Quillen adjunction. Consider the following composite adjunction, which we will refer to as an enlarging adjunction.

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences. It need not be a Quillen adjunction. Consider the following composite adjunction, which we will refer to as an enlarging adjunction.

$$\begin{tikzcd}
(X, X') \arrow[r, mapsto] & (X, FX') \arrow[r, mapsto] & X \vee FX' \\
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left=2, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left=2, "\mathcal{M}\times U"] \arrow[r, shift left=2, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "\Delta"] \\
(Y, UY) & (Y, Y) \arrow[l, mapsto] & Y, \arrow[l, mapsto]
\end{tikzcd}$$

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences. It need not be a Quillen adjunction. Consider the following composite adjunction, which we will refer to as an enlarging adjunction.

$$\begin{tikzcd}
(X, X') \arrow[r, mapsto] & (X, FX') \arrow[r, mapsto] & X \vee FX' \\
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left=2, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left=2, "\mathcal{M}\times U"] \arrow[r, shift left=2, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "\Delta"] \\
(Y, UY) & (Y, Y) \arrow[l, mapsto] & Y, \arrow[l, mapsto]
\end{tikzcd}$$

It is a transfer adjunction, so it induces a new model structure on $\mathcal{M}$.

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences. It need not be a Quillen adjunction. Consider the following composite adjunction, which we will refer to as an enlarging adjunction.

$$\begin{tikzcd}
(X, X') \arrow[r, mapsto] & (X, FX') \arrow[r, mapsto] & X \vee FX' \\
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[r, shift left, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\Delta"] \\
(Y, UY) & (Y, Y) \arrow[l, mapsto] & Y, \arrow[l, mapsto]
\end{tikzcd}$$

It is a transfer adjunction, so it induces a new model structure on $\mathcal{M}$. It has the same weak equivalences but more cofibrations than the original one.

Enlarging the class of cofibrations in a model category

Suppose we have pointed model categories $\mathcal{M}$ and $\mathcal{M}'$ with an adjunction

$$\begin{tikzcd}
\mathcal{M}' \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

in which the right adjoint U preserves weak equivalences. It need not be a Quillen adjunction. Consider the following composite adjunction, which we will refer to as an enlarging adjunction.

$$\begin{tikzcd}
(X, X') \arrow[r, mapsto] & (X, FX') \arrow[r, mapsto] & X \vee FX' \\
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left=2, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left=2, "\mathcal{M}\times U"] \arrow[r, shift left=2, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "\Delta"] \\
(Y, UY) & (Y, Y) \arrow[l, mapsto] & Y, \arrow[l, mapsto]
\end{tikzcd}$$

It is a transfer adjunction, so it induces a new model structure on $\mathcal{M}$. It has the same weak equivalences but more cofibrations than the original one. They include the images under F of cofibrations in $\mathcal{M}'$.

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[r, shift left, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\Delta"]
\end{tikzcd}$$

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left=2, "\mathcal{M} \times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left=2, "\mathcal{M} \times U"] \arrow[r, shift left=2, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "\Delta"]
\end{tikzcd}$$

to induce a new model structure on $\mathcal{M}$.

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[r, shift left, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\Delta"]
\end{tikzcd}$$

to induce a new model structure on $\mathcal{M}$. The case of interest for us is

$$\mathcal{M} = \mathcal{S}p^G \qquad \text{and} \qquad \mathcal{M}' = \prod_{H \subset G} \mathcal{S}p^H.$$

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}' \arrow[r, shift left, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M}\times\mathcal{M} \arrow[l, shift left, "\mathcal{M}\times U"] \arrow[r, shift left, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left, "\Delta"]
\end{tikzcd}$$

to induce a new model structure on $\mathcal{M}$. The case of interest for us is

$$\mathcal{M} = \mathcal{S}p^G \qquad \text{and} \qquad \mathcal{M}' = \prod_{H\subset G} \mathcal{S}p^H .$$

The product here is over all proper subgroups H. The functor U is built out of restriction functors i_H^G,

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M} \times \mathcal{M}' \arrow[r, shift left=2, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M} \times \mathcal{M} \arrow[l, shift left=2, "\mathcal{M}\times U"] \arrow[r, shift left=2, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=2, "\Delta"]
\end{tikzcd}$$

to induce a new model structure on $\mathcal{M}$. The case of interest for us is

$$\mathcal{M} = \mathcal{S}p^G \qquad \text{and} \qquad \mathcal{M}' = \prod_{H \subset G} \mathcal{S}p^H.$$

The product here is over all proper subgroups H. The functor U is built out of restriction functors i_H^G, and F is built out of induction functors

$$X \mapsto G_+ \underset{H}{\wedge} X \qquad \text{for } X \in \mathcal{S}p^H.$$

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}' \arrow[r, shift left=1.5ex, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M}\times\mathcal{M} \arrow[l, shift left=1.5ex, "\mathcal{M}\times U"] \arrow[r, shift left=1.5ex, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=1.5ex, "\Delta"]
\end{tikzcd}$$

to induce a new model structure on $\mathcal{M}$. The case of interest for us is

$$\mathcal{M} = \mathcal{S}p^G \qquad \text{and} \qquad \mathcal{M}' = \prod_{H\subset G} \mathcal{S}p^H.$$

The product here is over all proper subgroups H. The functor U is built out of restriction functors i_H^G, and F is built out of induction functors

$$X \mapsto G_+ \underset{H}{\wedge} X \qquad \text{for } X \in \mathcal{S}p^H.$$

We call this process equifibrant enlargement.

Enlarging the class of cofibrations in a model category (continued)

We are using an adjunction of the form

$$\begin{tikzcd}
\mathcal{M}\times\mathcal{M}' \arrow[r, shift left=1.5, "\mathcal{M}\times F"] \arrow[r, phantom, "\perp"] & \mathcal{M}\times\mathcal{M} \arrow[l, shift left=1.5, "\mathcal{M}\times U"] \arrow[r, shift left=1.5, "\vee"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, shift left=1.5, "\Delta"]
\end{tikzcd}$$

to induce a new model structure on $\mathcal{M}$. The case of interest for us is

$$\mathcal{M} = \mathcal{S}p^G \qquad \text{and} \qquad \mathcal{M}' = \prod_{H\subset G} \mathcal{S}p^H.$$

The product here is over all proper subgroups H. The functor U is built out of restriction functors i_H^G, and F is built out of induction functors

$$X \mapsto G_+ \underset{H}{\wedge} X \qquad \text{for } X \in \mathcal{S}p^H.$$

We call this process equifibrant enlargement. The resulting model structure on $\mathcal{S}p^G$ plays nicely with the norm and with geometric fixed points.

Positivization: back to functor categories

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category,

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category.

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$.

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint.

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint. They are the left and right Kan extensions $\alpha_!$ and $\alpha_¡$.

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint. They are the left and right Kan extensions $\alpha_!$ and $\alpha_¡$. (This notation for the right Kan extension is new.)

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint. They are the left and right Kan extensions $\alpha_!$ and $\alpha_{\text{¡}}$. (This notation for the right Kan extension is new.) Consider the adjunction

$$\begin{tikzcd}
\mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left, "\alpha^*"]
\end{tikzcd}$$

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint. They are the left and right Kan extensions $\alpha_!$ and $\alpha_¡$. (This notation for the right Kan extension is new.) Consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J. \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction.

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint. They are the left and right Kan extensions $\alpha_!$ and $\alpha_{\text{¡}}$. (This notation for the right Kan extension is new.) Consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left=2, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left=2, "\alpha^*"] \end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. The Crans-Kan transfer theorem gives us a new model structure on $\mathcal{M}^J$

Positivization: back to functor categories

As in the start of this talk, let $\mathcal{M}$ be a pointed topological symmetric monoidal cofibrantly generated model category, and let J be a small category. Suppose further that J has a full subcategory K with inclusion functor $\alpha : K \to J$.

This induces a precomposition functor $\alpha^* : \mathcal{M}^J \to \mathcal{M}^K$. It has both a left and a right adjoint. They are the left and right Kan extensions $\alpha_!$ and $\alpha_{¡}$. (This notation for the right Kan extension is new.) Consider the adjunction

$$\begin{tikzcd}
\mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left, "\alpha^*"]
\end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. The Crans-Kan transfer theorem gives us a new model structure on $\mathcal{M}^J$ which differs from the projective one.

Back to functor categories (continued)

For a full subcategory K of J,

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd}
\mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J. \arrow[l, shift left, "\alpha^*"]
\end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction.

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J. \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. For a functor X in $\mathcal{M}^K$, in favorable cases we have

$$(\alpha_! X)_j = \begin{cases} X_j & \text{for } j \in \mathrm{Im}\alpha \\ * & \text{otherwise} \end{cases}$$

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. For a functor X in $\mathcal{M}^K$, in favorable cases we have

$$(\alpha_! X)_j = \begin{cases} X_j & \text{for } j \in \mathrm{Im}\alpha \\ * & \text{otherwise} \end{cases}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ which differs from the projective one.

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd}\mathcal{M}^K \arrow[r, shift left=2, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left=2, "\alpha^*"]\end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. For a functor X in $\mathcal{M}^K$, in favorable cases we have

$$(\alpha_! X)_j = \begin{cases} X_j & \text{for } j \in \mathrm{Im}\alpha \\ * & \text{otherwise} \end{cases}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ which differs from the projective one. In it a map $f : X \to Y$ is a weak equivalence or a fibration if f_j is one for each $j \in \mathrm{Im}\,\alpha$,

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left=2, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J. \arrow[l, shift left=2, "\alpha^*"] \end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. For a functor X in $\mathcal{M}^K$, in favorable cases we have

$$(\alpha_! X)_j = \begin{cases} X_j & \text{for } j \in \mathrm{Im}\alpha \\ * & \text{otherwise} \end{cases}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ which differs from the projective one. In it a map $f : X \to Y$ is a weak equivalence or a fibration if f_j is one for each $j \in \mathrm{Im}\,\alpha$, but not necessarily for other objects j of J.

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd}\mathcal{M}^K \arrow[r, shift left=2, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left=2, "\alpha^*"]\end{tikzcd}$$

In terms of the projective model structure on $\mathcal{M}^K$, this is a transfer adjunction. For a functor X in $\mathcal{M}^K$, in favorable cases we have

$$(\alpha_! X)_j = \begin{cases} X_j & \text{for } j \in \mathrm{Im}\alpha \\ * & \text{otherwise} \end{cases}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ which differs from the projective one. In it a map $f : X \to Y$ is a weak equivalence or a fibration if f_j is one for each $j \in \mathrm{Im}\,\alpha$, but not necessarily for other objects j of J. It has more weak equivalences and fibrations than the projective model structure.

Back to functor categories (continued)

For a full subcategory K of J,

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd}\mathcal{M}^K \arrow[r, shift left=2, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left=2, "\alpha^*"]\end{tikzcd}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ with more weak equivalences and fibrations, and therefore fewer cofibrations, than the projective one.

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left=2, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J . \arrow[l, shift left=2, "\alpha^*"] \end{tikzcd}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ with more weak equivalences and fibrations, and therefore fewer cofibrations, than the projective one. A map $f : X \to Y$ is an induced cofibration only when f_j is an isomorphism for each j not in the subcategory K.

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd}\mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J. \arrow[l, shift left, "\alpha^*"]\end{tikzcd}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ with more weak equivalences and fibrations, and therefore fewer cofibrations, than the projective one. A map $f : X \to Y$ is an induced cofibration only when f_j is an isomorphism for each j not in the subcategory K. This also means that there are fewer cofibrant objects.

Functor categories
Orthogonal G-spectra as functors
Modifying the model structure
The Crans-Kan transfer theorem
Equifibrant enlargement
Positivization
Stabilization

Back to functor categories (continued)

For a full subcategory K of J, consider the adjunction

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & \mathcal{M}^J. \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$

The Crans-Kan transfer theorem gives us an induced model structure on $\mathcal{M}^J$ with more weak equivalences and fibrations, and therefore fewer cofibrations, than the projective one. A map $f : X \to Y$ is an induced cofibration only when f_j is an isomorphism for each j not in the subcategory K. This also means that there are fewer cofibrant objects.

We call this new model structure on $[J, \mathcal{M}]$ a confinement of the projective one.

A relevant example of confinement

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G].$$

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G].$$

For this we need a full subcategory of $\mathscr{J}_G$.

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G].$$

For this we need a full subcategory of $\mathscr{J}_G$.

We say an orthogonal representation V of G is positive if its invariant subspace V^G is nontrivial.

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{I}_G, \mathcal{T}^G].$$

For this we need a full subcategory of $\mathscr{I}_G$.

We say an orthogonal representation V of G is positive if its invariant subspace V^G is nontrivial. The subcategory we want is $\mathscr{I}_G^+$, whose objects are positive representations.

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G].$$

For this we need a full subcategory of $\mathscr{J}_G$.

We say an orthogonal representation V of G is positive if its invariant subspace V^G is nontrivial. The subcategory we want is $\mathscr{J}_G^+$, whose objects are positive representations.

The positive model structure on $\mathcal{S}p^G$ is the one induced by the transfer adjunction

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G].$$

For this we need a full subcategory of $\mathscr{J}_G$.

We say an orthogonal representation V of G is positive if its invariant subspace V^G is nontrivial. The subcategory we want is $\mathscr{J}_G^+$, whose objects are positive representations.

The positive model structure on $\mathcal{S}p^G$ is the one induced by the transfer adjunction

$$\begin{tikzcd} \mathcal{S}p_+^G := [\mathscr{J}_G^+, \mathcal{T}^G] \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & {[\mathscr{J}_G, \mathcal{T}^G] = \mathcal{S}p^G.} \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$

A relevant example of confinement

We want to confine the projective model structure on the category of orthogonal G-spectra

$$\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G].$$

For this we need a full subcategory of $\mathscr{J}_G$.

We say an orthogonal representation V of G is positive if its invariant subspace V^G is nontrivial. The subcategory we want is $\mathscr{J}_G^+$, whose objects are positive representations.

The positive model structure on $\mathcal{S}p^G$ is the one induced by the transfer adjunction

$$\begin{tikzcd} \mathcal{S}p_+^G := [\mathscr{J}_G^+, \mathcal{T}^G] \arrow[r, shift left, "\alpha_!"] \arrow[r, phantom, "\perp"] & {[\mathscr{J}_G, \mathcal{T}^G] = \mathcal{S}p^G.} \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$

We call this type of confinement positivization.

Positivization: why do it?

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product,

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it,

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra.

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

We want to define a model structure on it.

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

We want to define a model structure on it. The issue here is not equivariant, so we assume for simplicity that the group is trivial.

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

We want to define a model structure on it. The issue here is not equivariant, so we assume for simplicity that the group is trivial. We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left=2, "U"] \end{tikzcd}$$

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

We want to define a model structure on it. The issue here is not equivariant, so we assume for simplicity that the group is trivial. We want to define a transfer adjunction

$$\begin{tikzcd}
\mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

where U is the forgetful functor,

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

We want to define a model structure on it. The issue here is not equivariant, so we assume for simplicity that the group is trivial. We want to define a transfer adjunction

$$\begin{tikzcd}
\mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

where U is the forgetful functor, and Sym is the free commutative algebra functor

$$X \mapsto \mathrm{Sym}\,(X) := \bigvee_{n \geq 0} \mathrm{Sym}^n X,$$

Positivization: why do it?

The category $\mathcal{S}p^G$ is closed symmetric monoidal under smash product, so we can speak of commutative ring objects in it, also known as E_∞-ring spectra. We denote the category of such spectra by $\mathrm{Comm}\,\mathcal{S}\mathrm{p}^{\mathrm{G}}$.

We want to define a model structure on it. The issue here is not equivariant, so we assume for simplicity that the group is trivial. We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left=2, "U"] \end{tikzcd}$$

where U is the forgetful functor, and Sym is the free commutative algebra functor

$$X \mapsto \mathrm{Sym}\,(X) := \bigvee_{n\geq 0} \mathrm{Sym}^n X,$$

where Sym^n is the nth symmetric product functor,

$$X \mapsto (X^{\wedge n})_{\Sigma_n}.$$

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}. \arrow[l, shift left, "U"] \end{tikzcd}$$

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}. \arrow[l, shift left, "U"] \end{tikzcd}$$

This means the functor Sym^n for each n must preserve weak equivalences between cofibrant objects in $\mathcal{S}p$.

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}. \arrow[l, shift left, "U"] \end{tikzcd}$$

This means the functor Sym^n for each n must preserve weak equivalences between cofibrant objects in $\mathcal{S}p$. We need this to work for the stable model structure, which we have not yet defined.

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}. \arrow[l, shift left, "U"] \end{tikzcd}$$

This means the functor Sym^n for each n must preserve weak equivalences between cofibrant objects in $\mathcal{S}p$. We need this to work for the stable model structure, which we have not yet defined. There is a map

$$s_1 : S^{-1} \wedge S^1 \to S^{-0},$$

which is a stable weak equivalence.

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}. \arrow[l, shift left=2, "U"] \end{tikzcd}$$

This means the functor Sym^n for each n must preserve weak equivalences between cofibrant objects in $\mathcal{S}p$. We need this to work for the stable model structure, which we have not yet defined. There is a map

$$s_1 : S^{-1} \wedge S^1 \to S^{-0},$$

which is a stable weak equivalence. Applying Sym^2 gives a map

$$\mathrm{Sym}^2 s_1 : \mathrm{Sym}^2(S^{-1} \wedge S^1) \to \mathrm{Sym}^2 S^{-0} = S^{-0}.$$

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}. \arrow[l, shift left, "U"] \end{tikzcd}$$

This means the functor Sym^n for each n must preserve weak equivalences between cofibrant objects in $\mathcal{S}p$. We need this to work for the stable model structure, which we have not yet defined. There is a map

$$s_1 : S^{-1} \wedge S^1 \to S^{-0},$$

which is a stable weak equivalence. Applying Sym^2 gives a map

$$\mathrm{Sym}^2 s_1 : \mathrm{Sym}^2(S^{-1} \wedge S^1) \to \mathrm{Sym}^2 S^{-0} = S^{-0}.$$

These two spectra are wildly different, so we have a problem.

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd}\mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left, "U"]\end{tikzcd}$$

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd}
\mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

but the functor Sym^2 fails to preserve the stable weak equivalence

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}.$$

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left=2, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left=2, "U"] \end{tikzcd}$$

but the functor Sym^2 fails to preserve the stable weak equivalence

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}.$$

This difficulty was first noticed by Jeff Smith in the 1990s.

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left, "U"] \end{tikzcd}$$

but the functor Sym^2 fails to preserve the stable weak equivalence

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}.$$

This difficulty was first noticed by Jeff Smith in the 1990s. We need Sym^2 to preserve weak equivalences between cofibrant objects.

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd}
\mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left, "U"]
\end{tikzcd}$$

but the functor Sym^2 fails to preserve the stable weak equivalence

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}.$$

This difficulty was first noticed by Jeff Smith in the 1990s. We need Sym^2 to preserve weak equivalences between cofibrant objects.

After positivizing the stable model structure on $\mathcal{S}p$, the sphere spectrum S^{-0} is no longer cofibrant,

Positivization: why do it? (continued)

We want to define a transfer adjunction

$$\begin{tikzcd} \mathcal{S}p \arrow[r, shift left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{Comm}\,\mathcal{S}\mathrm{p}, \arrow[l, shift left, "U"] \end{tikzcd}$$

but the functor Sym^2 fails to preserve the stable weak equivalence

$$s_1 : S^{-1} \wedge S^1 \to S^{-0}.$$

This difficulty was first noticed by Jeff Smith in the 1990s. We need Sym^2 to preserve weak equivalences between cofibrant objects.

After positivizing the stable model structure on $\mathcal{S}p$, the sphere spectrum S^{-0} is no longer cofibrant, and (Sym, U) above becomes a transfer pair as desired.

Bousfield localization

Bousfield localization may be the best construction in model category theory.

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

Functor categories

Orthogonal *G*-spectra as functors

Modifying the model structure

The Crans-Kan transfer theorem

Equifibrant enlargement

Positivization

Stabilization

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before,

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before, and hence fewer fibrations.

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before, and hence fewer fibrations. This often leads to a more interesting fibrant replacement function.

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before, and hence fewer fibrations. This often leads to a more interesting fibrant replacement functor.

The hard part of this is proving that each morphism can be factored as a trivial cofibration followed by a fibration.

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before, and hence fewer fibrations. This often leads to a more interesting fibrant replacement function.

The hard part of this is proving that each morphism can be factored as a trivial cofibration followed by a fibration. It often involves some delicate set theory.

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before, and hence fewer fibrations. This often leads to a more interesting fibrant replacement functor.

The hard part of this is proving that each morphism can be factored as a trivial cofibration followed by a fibration. It often involves some delicate set theory. It requires $\mathcal{M}$ to have certain properties,

Bousfield localization

Bousfield localization may be the best construction in model category theory. We start with a model category $\mathcal{M}$ and make a new model structure on its underlying category by

- keeping the same class of cofibrations and
- expanding the class of weak equivalences in some way.

This means there will be more trivial cofibrations than before, and hence fewer fibrations. This often leads to a more interesting fibrant replacement functor.

The hard part of this is proving that each morphism can be factored as a trivial cofibration followed by a fibration. It often involves some delicate set theory. It requires $\mathcal{M}$ to have certain properties, but there are no restrictions on how we expand the class of weak equivalences.

Some examples of Bousfield localization

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups,

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h. We can do the same in the category of spectra.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h. We can do the same in the category of spectra. The functors $L_{K(n)}$ and $L_{E(n)}$ are fundamental in chromatic homotopy theory.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h. We can do the same in the category of spectra. The functors $L_{K(n)}$ and $L_{E(n)}$ are fundamental in chromatic homotopy theory.
- Let $\mathcal{S}p = [\mathscr{J}, \mathcal{T}]$ be the category of spectra with its projective model structure.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h. We can do the same in the category of spectra. The functors $L_{K(n)}$ and $L_{E(n)}$ are fundamental in chromatic homotopy theory.
- Let $\mathcal{S}p = [\mathscr{J}, \mathcal{T}]$ be the category of spectra with its projective model structure. We can expand the class of weak equivalences to include all maps inducing isomorphisms in stable homotopy groups.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h. We can do the same in the category of spectra. The functors $L_{K(n)}$ and $L_{E(n)}$ are fundamental in chromatic homotopy theory.
- Let $\mathcal{S}p = [\mathscr{J}, \mathcal{T}]$ be the category of spectra with its projective model structure. We can expand the class of weak equivalences to include all maps inducing isomorphisms in stable homotopy groups. The resulting fibrant objects are precisely the Ω-spectra.

Some examples of Bousfield localization

- Let $\mathcal{T}$ be the category of pointed topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups, and all objects are fibrant. We can expand the class of weak equivalences by requiring the to induce inducing isomorphisms of homotopy groups only in dimensions $\leq n$. The resulting fibrant replacement functor is the nth Postnikov section.
- Let h_* be your favorite homology theory. We can expand the class of weak equivalences in $\mathcal{T}$ by including all h_*-isomorphisms. The resulting fibrant replacment functor is Bousfield's famous functor L_h. We can do the same in the category of spectra. The functors $L_{K(n)}$ and $L_{E(n)}$ are fundamental in chromatic homotopy theory.
- Let $\mathcal{S}p = [\mathscr{J}, \mathcal{T}]$ be the category of spectra with its projective model structure. We can expand the class of weak equivalences to include all maps inducing isomorphisms in stable homotopy groups. The resulting fibrant objects are precisely the Ω-spectra. We call this process stabilization.

More about stabilization

More about stabilization

In general there are two ways to describe Bousfield localization:

More about stabilization

In general there are two ways to describe Bousfield localization:

① Describe set or class of maps that are to become weak equivalences.

More about stabilization

In general there are two ways to describe Bousfield localization:

1. Describe set or class of maps that are to become weak equivalences. You need not specify all of them.

More about stabilization

In general there are two ways to describe Bousfield localization:

1. Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party,

More about stabilization

In general there are two ways to describe Bousfield localization:

① Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party, she will bring all of her friends.

More about stabilization

In general there are two ways to describe Bousfield localization:

① Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party, she will bring all of her friends.

② Describe the new fibrant replacement functor.

More about stabilization

In general there are two ways to describe Bousfield localization:

1. Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party, she will bring all of her friends.

2. Describe the new fibrant replacement functor. It is usually the case that a map $f : X \to Y$ is a weak equivalence in the new model structure

More about stabilization

In general there are two ways to describe Bousfield localization:

1. Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party, she will bring all of her friends.

2. Describe the new fibrant replacement functor. It is usually the case that a map $f : X \to Y$ is a weak equivalence in the new model structure iff the induced map between fibrant replacements is a weak equivalence in the original model structure.

More about stabilization

In general there are two ways to describe Bousfield localization:

1. Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party, she will bring all of her friends.

2. Describe the new fibrant replacement functor. It is usually the case that a map $f : X \to Y$ is a weak equivalence in the new model structure iff the induced map between fibrant replacements is a weak equivalence in the original model structure. For example, a map of spaces or spectra is a $K(n)_*$-equivalence

More about stabilization

In general there are two ways to describe Bousfield localization:

1. Describe set or class of maps that are to become weak equivalences. You need not specify all of them. If you invite one to the party, she will bring all of her friends.

2. Describe the new fibrant replacement functor. It is usually the case that a map $f : X \to Y$ is a weak equivalence in the new model structure iff the induced map between fibrant replacements is a weak equivalence in the original model structure. For example, a map of spaces or spectra is a $K(n)_*$-equivalence iff its $K(n)_*$ localization (fibrant replacement) is an ordinary weak equivalence.

More about stabilization (continued)

In general there are two ways to describe Bousfield localization.

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal *G*-spectra we can do both.

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal G-spectra we can do both.

① For each representation V, we define a stabilizing map $s_V : S^{-V} \wedge S^V \to S^{-0}$ as follows.

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal G-spectra we can do both.

1. For each representation V, we define a stabilizing map $s_V : S^{-V} \wedge S^V \to S^{-0}$ as follows. Its Wth component is the composition morphism

$$\begin{tikzcd}
(S^{-V} \wedge S^V)_W \arrow[d, phantom, "||"] & (S^{-0})_W \arrow[d, phantom, "||"] \\
\mathscr{J}_G(V, W) \wedge \mathscr{J}_G(0, V) \arrow[r, "j_{0,V,W}"] & \mathscr{J}_G(0, W).
\end{tikzcd}$$

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal G-spectra we can do both.

1. For each representation V, we define a stabilizing map $s_V : S^{-V} \wedge S^V \to S^{-0}$ as follows. Its Wth component is the composition morphism

$$\begin{tikzcd}
(S^{-V} \wedge S^V)_W \arrow[d, phantom, "||"] & (S^{-0})_W \arrow[d, phantom, "||"] \\
\mathscr{J}_G(V, W) \wedge \mathscr{J}_G(0, V) \arrow[r, "j_{0,V,W}"] & \mathscr{J}_G(0, W).
\end{tikzcd}$$

   For $V \neq 0$ this map is a stable equivalence but not a projective one.

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal G-spectra we can do both.

1. For each representation V, we define a stabilizing map $s_V : S^{-V} \wedge S^V \to S^{-0}$ as follows. Its Wth component is the composition morphism

$$\begin{tikzcd}
(S^{-V} \wedge S^V)_W \arrow[d, equal] & (S^{-0})_W \arrow[d, equal] \\
\mathscr{J}_G(V, W) \wedge \mathscr{J}_G(0, V) \arrow[r, "j_{0,V,W}"] & \mathscr{J}_G(0, W).
\end{tikzcd}$$

   For $V \neq 0$ this map is a stable equivalence but not a projective one.

2. To define the fibrant replacement RX of a spectrum X,

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal *G*-spectra we can do both.

1. For each representation V, we define a stabilizing map $s_V : S^{-V} \wedge S^V \to S^{-0}$ as follows. Its Wth component is the composition morphism

$$\begin{tikzcd}
(S^{-V} \wedge S^V)_W \arrow[d, phantom, "||"] & (S^{-0})_W \arrow[d, phantom, "||"] \\
\mathscr{J}_G(V, W) \wedge \mathscr{J}_G(0, V) \arrow[r, "j_{0,V,W}"] & \mathscr{J}_G(0, W).
\end{tikzcd}$$

   For $V \neq 0$ this map is a stable equivalence but not a projective one.

2. To define the fibrant replacement RX of a spectrum X, let ρ denote the regular representation of G.

More about stabilization (continued)

In general there are two ways to describe Bousfield localization. In the case of orthogonal *G*-spectra we can do both.

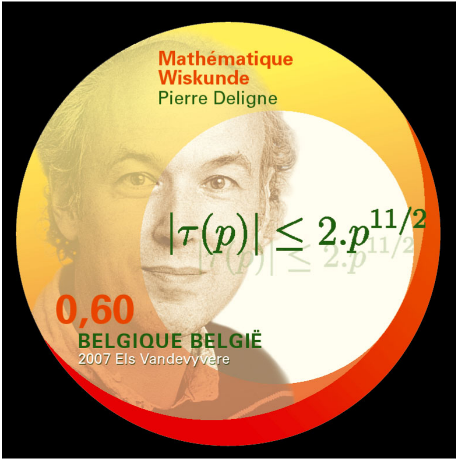
1. For each representation V, we define a stabilizing map $s_V : S^{-V} \wedge S^V \to S^{-0}$ as follows. Its Wth component is the composition morphism

$$\begin{tikzcd}
(S^{-V} \wedge S^V)_W \arrow[d, equal] & (S^{-0})_W \arrow[d, equal] \\
\mathscr{J}_G(V, W) \wedge \mathscr{J}_G(0, V) \arrow[r, "j_{0,V,W}"] & \mathscr{J}_G(0, W).
\end{tikzcd}$$

   For $V \neq 0$ this map is a stable equivalence but not a projective one.

2. To define the fibrant replacement RX of a spectrum X, let ρ denote the regular representation of G. Then

$$(RX)_V = \operatorname*{hocolim}_n \Omega^{n\rho} X_{V+n\rho}.$$



Happy 75th birthday to Pierre Deligne!