Model categories and spectra

Mike Hill UCLA
Mike Hopkins Harvard University
Doug Ravenel University of Rochester

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1.1

1 Introduction

Introduction

This expository talk is a self contained variant of the one I gave in Shenzhen. Its purpose is to introduce the use of Quillen model categories in stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \dots\}$ with structure maps $\varepsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps.

There are two different notions of weak equivalence in the category of spectra $\mathscr{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

1.2

Introduction (continued)

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- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.
(ii) Define a functor $\Lambda : \mathscr{S}p \to \mathscr{S}p$ where $(\Lambda X)_n$ is the homotopy colimit (meaning the mapping telescope) of

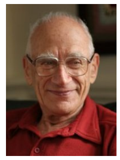
$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

and then require Λf to be a strict equivalence.

These two definitions are known to be equivalent. 1.3

Iintroduction (continued)

Dan Quillen
1940-2011



Dan Kan
1928-2013

Pete
Bousfield

In order to understand this better we need to discuss

- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation
- Bousfield localization

We will see that the passage from strict equivalence to stable equivalence is a form of Bousfield localization. 1.4

2 Quillen model categories

Quillen model categories

Definition. **A Quillen model category** $\mathcal{M}$ *is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 **Bicompleteness axiom.** *$\mathcal{M}$ has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that $\mathcal{M}$ has initial and terminal objects, denoted by $\varnothing$ and $*$.*

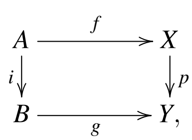
MC2 **2-out-of-3 axiom.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{M}$. Then if any two of f, g and gf are weak equivalences, so is the third.*

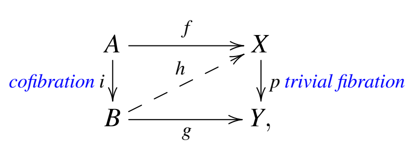
MC3 **Retract axiom.** *A retract of a weak equivalence, fibration or cofibration is again a weak equivalence, fibration or cofibration.*

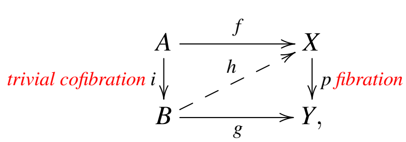
We say that a fibration or cofibration is trivial (or acyclic) if it is also a weak equivalence. 1.5

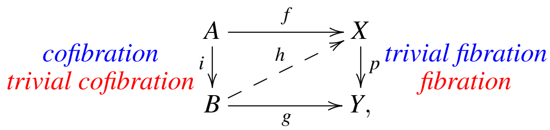
Quillen model categories (continued)

Definition. **MC4** **Lifting axiom.** *Given a commutative diagram*



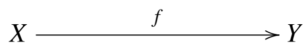


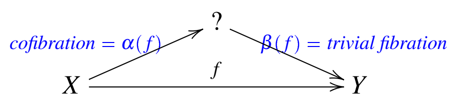


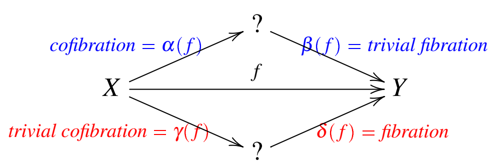


a morphism h (called a lifting) exists for i and p as indicated.

MC5 Factorization axiom. *Any morphism $f : X \to Y$ can be functorially factored in two ways as*



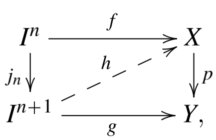




This last axiom is the hardest one to verify in practice. 1.6

Two classical examples

Let $\mathscr{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations, that is is maps $p : X \to Y$ with the right lifting property



for each $n \geqslant 0$, where I^n is the unit n-cube.

Cofibrations are maps (such as $i_n : S^{n-1} \to D^n$ for $n \geqslant 0$) having the left lifting property with respect to all trivial Serre fibrations.

Similar definitions can be made for $\mathscr{T}$, the category of pointed topological spaces and basepoint preserving maps. 1.7

Some definitions

Recall that we denote the initial and terminal objects of $\mathscr{M}$ by $\varnothing$ and $*$. When they are the same, we say that $\mathscr{M}$ is pointed.

Definition. *An object X is cofibrant if the unique map $\varnothing \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.*

All objects in $\mathscr{T}$ and $\mathscr{T}op$ are fibrant. The cofibrant objects are the CW-complexes.

By **MC5**, for any object X, the unique maps $\varnothing \to X$ and $X \to *$ have factorizations

$$\varnothing \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X,
and RX is a fibrant object weakly equivalent to X. 1.8

Some definitions (continued)

By **MC5**, for any object X, the unique maps $\varnothing \to X$ and $X \to *$ have factorizations

$$\varnothing \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X,
and RX is a fibrant object weakly equivalent to X.

These maps to and from X are called cofibrant and fibrant approximations. The objects QX and RX are called cofibrant and fibrant replacements of X. 1.9

3 Cofibrant generation

Cofibrant generation

In $\mathscr{T}op$, let

$$\mathscr{I} = \{i_n : S^{n-1} \to D^n, n \geqslant 0\} \text{ and } \mathscr{J} = \{j_n : I^n \to I^{n+1}, n \geqslant 0\}.$$

It is known that every (trivial) cofibration in $\mathscr{T}op$ can be derived from the ones in ($\mathscr{J}$) $\mathscr{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathscr{I}$) $\mathscr{J}$. This condition is easier to verify than the previous one.

In $\mathscr{T}$, the category of pointed topological spaces, one can define similar sets $\mathscr{I}_+$ and $\mathscr{J}_+$, by adding disjoint basepoints to the above. 1.10

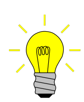
Cofibrant generation (continued)

Definition. *A cofibrantly generated model category $\mathscr{M}$ is one with morphism sets $\mathscr{I}$ and $\mathscr{J}$ having similar properties to the ones in $\mathscr{T}op$. $\mathscr{I}$ ($\mathscr{J}$) is a generating set of (trivial) cofibrations.*

In practice, specifying the generating sets $\mathscr{I}$ and $\mathscr{J}$, and defining weak equivalences is the most convenient way to describe a model category. 1.11

4 Bousfield localization

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea. 

Suppose we have a model category $\mathscr{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations, satisfy the second Factorization Axiom MC5. The proof involves some delicate set theory. 1.12

An elementary examples of Bousfield localization

Let $\mathscr{T}op$ be the category of topological spaces with its usual model structure.

Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leqslant n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n. 1.13

5 The strict model structure on the category of spectra

The strict model structure on the category of spectra

Recall that a spectrum X is a sequence of pointed spaces $\{X_0, X_1, X_2, \dots\}$ with structure maps $\varepsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps. We will denote the category of spectra by $\mathscr{S}p$.

Definition. *The m***th Yoneda spectrum** S^{-m} *is given by*

$$(S^{-m})_n = \begin{cases} * & \text{for } n < m \\ S^{n-m} & \text{otherwise,} \end{cases}$$

with the evident structure maps.

In particular, S^{-0} is the sphere spectrum. 1.14

The strict model structure on the category of spectra (continued)

Definition. The strict or projective model structure on $\mathscr{S}p$. *A map of spectra $f : X \to Y$ is a weak equivalence or a fibration if f_n is one for each $n \geqslant 0$. A map is a cofibration if it has the left lifting property with respect to all trivial fibrations.*

This model structure is known to be cofibrantly generated. Recall that $\mathscr{T}$, the category of pointed topological spaces, has generating sets $\mathscr{I}_+$ and $\mathscr{J}_+$. The ones for $\mathscr{S}p$ are

$$\widetilde{\mathscr{I}} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{I}_+ \qquad \text{and} \qquad \widetilde{\mathscr{J}} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{J}_+.$$

Note that here we are smashing a spectrum X with a map of pointed spaces $g : A \to B$. The nth component of $X \wedge g$ is the map $X_n \wedge A \to X_n \wedge B$. The categorical term for being able to smash a spectrum with a spaces is to say that $\mathscr{S}p$ is tensored over $\mathscr{T}$. 1.15

6 The stable model structure

The stable model structure

Definition. The strict or projective model structure on $\mathscr{S}p$. *A map of spectra $f : X \to Y$ is a weak equivalence or a fibration if f_n is one for each $n \geqslant 0$. A map is a cofibration if it has the left lifting property with respect to all trivial fibrations.*

Experience has taught us that to do stable homotopy theory, we need a looser notion of weak equivalence, one which involves stable homotopy groups. To define them, recall our functor $\Lambda : \mathscr{S}p \to \mathscr{S}p$ where $(\Lambda X)_n$ is the homotopy colimit (meaning the mapping telescope) of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

Each space $(\Lambda X)_n$ is an infinite loop space, and the adjoint structure map

$$\eta_n^{\Lambda X} : (\Lambda X)_n \to \Omega(\Lambda X)_{n+1}$$

is a weak equivalence for all n, so ΛX is an Ω-spectrum. 1.16

The stable model structure (continued)

Again, $(\Lambda X)_n$ is the homotopy colimit (meaning the mapping telescope) of

$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \dots$$

We can use it to define the stable homotopy groups of X by

$$\pi_k X := \pi_{n+k}(\Lambda X)_n,$$

which is independent of n. We say a map $f : X \to Y$ is a stable equivalence if $\pi_* f$ is an isomorphism. This is equivalent to Λf being a strict equivalence.

Thus we have expanded the class of weak equivalences, so we can use Bousfield localization to construct the stable model structure. It turns out that the fibrant objects are precisely the Ω-spectra and that our functor Λ is fibrant replacement! 1.17

7 Cofibrant generation for spectra

Cofibrant generation

We will now describe cofibrant generating sets for the stable model structure on the category of spectra $\mathscr{S}p$. Recall that the strict model structure has generating sets

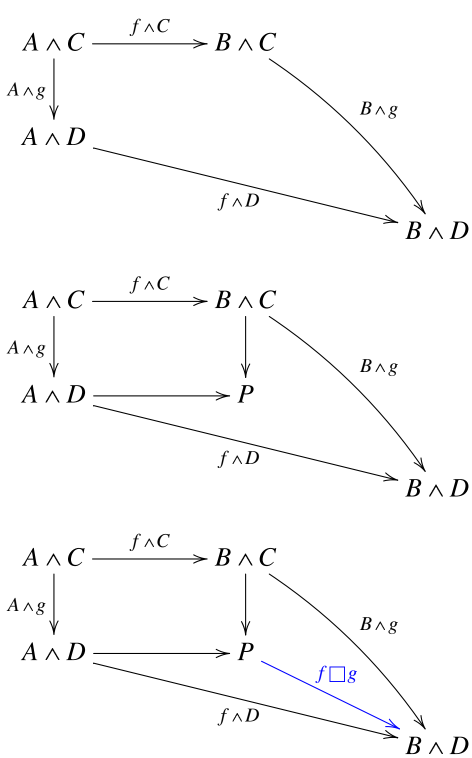
$$\mathscr{I}^{strict} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{I}_+ \qquad \text{and} \qquad \mathscr{J}^{strict} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{J}_+.$$

The stable model structure has the same cofibrations, but more trivial cofibrations. This means we need to enlarge $\mathscr{J}^{strict}$.

In order to do so, we need another construction, the pushout product or corner map. 1.18

Cofibrant generation (continued)

Suppose $f : A \to B$ and $g : C \to D$ are maps of pointed spaces. Consider the diagram



Here P is the pushout of the two maps from $A \wedge C$. Since the outer diagram commutes, there is a unique map from it to $B \wedge D$ which we denote by $f \square g$. This is the pushout product or corner map of f and g.

This construction also makes sense if $f : A \to B$ is a map of spectra, with $g : C \to D$ still being a map of spaces. 1.19

Cofibrant generation (continued)

Recall that we need to enlarge the generating set of trivial cofibrations,

$$\mathscr{J}^{strict} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{J}_+.$$

We will do so by defining a set $\mathscr{S}$ of stable equivalences of spectra and adjoining the set $\mathscr{S} \square \mathscr{I}_+$ to $\mathscr{J}^{strict}$.

Recall the Yoneda spectrum S^{-k} given by

$$(S^{-k})_n = \begin{cases} * & \text{for } n < k \\ S^{n-k} & \text{otherwise.} \end{cases}$$

It follows that $S^{-k} \wedge S^k$ given by

$$(S^{-k} \wedge S^k)_n = \begin{cases} * & \text{for } n < k \\ S^{n} & \text{otherwise.} \end{cases}$$

This is the same as the sphere spectrum S^{-0} for $n \geqslant k$. Hence there is a stable equivalence $s_k : S^{-k} \wedge S^k \to S^{-0}$ whose nth component is the identity map for $n \geqslant k$. 1.20

Cofibrant generation (continued)

Recall that the strict model structure on the category of spectra $\mathscr{S}p$ is cofibrantly generated by the sets

$$\mathscr{I}^{strict} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{I}_+ \qquad \text{and} \qquad \mathscr{J}^{strict} = \bigcup_{m \geqslant 0} S^{-m} \wedge \mathscr{J}_+.$$

The stable model structure has the same cofibrations, but more trivial cofibrations. This means we need to enlarge $\mathscr{J}^{strict}$.

The stable model structure is cofibrantly generated by the sets

$$\begin{aligned} & \mathscr{I}^{stable} = \mathscr{I}^{strict} \\ \text{and} \qquad & \mathscr{J}^{stable} = \mathscr{J}^{strict} \cup \bigcup_{k \geqslant 0} s_k \square \mathscr{I}_+, \end{aligned}$$

where $s_k : S^{-k} \wedge S^k \to S^{-0}$ is the stable equivalence defined above. 1.21

1.22