Model categories and stable homotopy theory
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1.1

1 Introduction

Introduction

The purpose of this talk is to describe a theorem about a cofibrantly generated Quillen model structure on certain categories of spectra. It came up in the process of writing a book about equivariant stable homotopy theory.

A spectrum X was originally defined to be a sequence of pointed spaces or simplicial sets $\{X_0, X_1, X_2, \ldots\}$ with structure maps $\varepsilon_n^X : \Sigma X_n \to X_{n+1}$. A map of spectra $f : X \to Y$ is a collection of pointed maps $f_n : X_n \to Y_n$ compatible with the structure maps.

There are two different notions of weak equivalence in the category of spectra $\mathscr{S}p$:

- $f : X \to Y$ is a strict equivalence if each map f_n is a weak equivalence.
- $f : X \to Y$ is a stable equivalence if ...

1.2

Introduction (continued)

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- $f : X \to Y$ is a stable equivalence if ...

There are at least two different ways to finish the definition of stable equivalence:

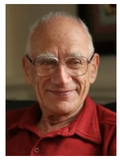
(i) Define stable homotopy groups of spectra and require $\pi_* f$ to be an isomorphism.
(ii) Define a functor $\Lambda : \mathscr{S}p \to \mathscr{S}p$ where $(\Lambda X)_n$ is the homotopy colimit (meaning the mapping telescope) of
$$X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \ldots$$
and then require Λf to be a strict equivalence.

Classically these two definitions are equivalent, but in certain variants of the definition of spectra themselves, they are different. They differ in the category $\mathscr{S}p^{\Sigma}$ of symmetric spectra of Hovey-Shipley-Smith.

1.3

Iintroduction (continued)


Dan Quillen 1940-2011


Dan Kan 1928-2013


Pete Bousfield


Max Kelly 1930-2007

In order to understand this better we need to discuss

- Quillen model categories
- Fibrant and cofibrant replacement
- Cofibrant generation
- Bousfield localization
- Enriched category theory

We will see that the passage from strict equivalence to stable equivalence is a form of Bousfield localization. We will give an explicit description of the cofibrant generating sets for the stable category. 1.4

2 Quillen model categories

Quillen model categories

Definition. A Quillen model category *$\mathscr{M}$ is a category equipped with three classes of morphisms: weak equivalences, fibrations and cofibrations, each of which includes all isomorphisms, satisfying the following five axioms:*

MC1 Bicompleteness axiom. *$\mathscr{M}$ has all small limits and colimits. These include products, coproducts, pullbacks and pushouts. This implies that $\mathscr{M}$ has initial and terminal objects.*

MC2 2-out-of-3 axiom. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathscr{M}$. Then if any two of f, g and gf are weak equivalences, so is the third.*

MC3 Retract axiom. *A retract of a weak equivalence, fibration or cofibration is again a weak equivalence, fibration or cofibration.*

We say that a fibration or cofibration is trivial (or acyclic) if it is also a weak equivalence. 1.5

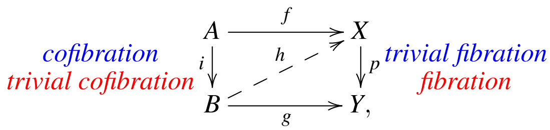
Quillen model categories (continued)

Definition. MC4 Lifting axiom. *Given a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

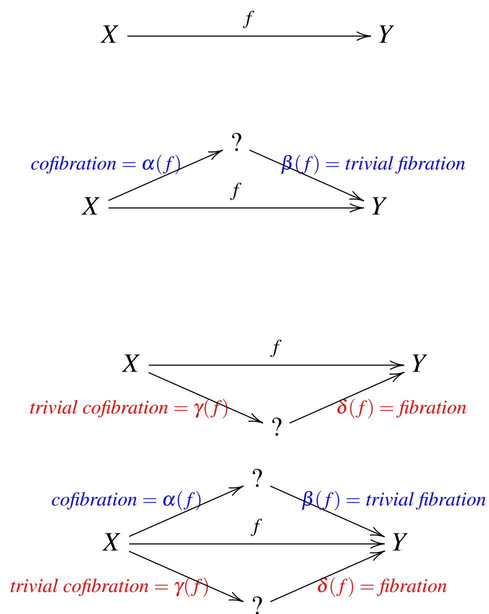
$$\text{cofibration}\ \begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}\ \text{trivial fibration}$$

$$\text{trivial cofibration}\ \begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}\ \text{fibration}$$



a morphism h exists for i and p as indicated.

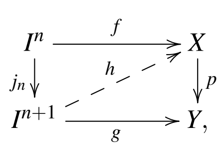
MC5 Factorization axiom. *Any morphism $f : X \to Y$ can be functorially factored in two ways as*



This is the hardest axiom to verify in practice. 1.6

A classical example

Let $\mathscr{T}op$ denote the category of (compactly generated weak Hausdorff) topological spaces. Weak equivalences are maps inducing isomorphisms of homotopy groups. Fibrations are Serre fibrations, that is is maps $p : X \to Y$ with the right lifting property



for each $n \geq 0$.

Cofibrations are maps (such as $i_n : S^{n-1} \to D^n$ for $n \geq 0$) having the left lifting property with respect to all trivial Serre fibrations. 1.7

Some definitions

We will denote the initial and terminal objects of $\mathscr{M}$ by $\emptyset$ and $*$. When they are the same, we say that $\mathscr{M}$ is pointed.

Definition. *An object X is cofibrant if the unique map $\emptyset \to X$ is a cofibration. It X is fibrant if the unique map $X \to *$ is a fibration.*

All objects in $\mathscr{T}$ and $\mathscr{T}op$ are fibrant. The cofibrant objects are the CW-complexes.

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X,
and RX is a fibrant object weakly equivalent to X. 1.8

Some definitions (continued)

By **MC5**, for any object X, the unique maps $\emptyset \to X$ and $X \to *$ have factorizations

$$\emptyset \to QX \to X \qquad \text{and} \qquad X \to RX \to *$$

where QX is a cofibrant object weakly equivalent to X,
and RX is a fibrant object weakly equivalent to X.

These maps to and from X are called cofibrant and fibrant approximations. The objects QX and RX are called cofibrant and fibrant replacements of X. 1.9

3 Cofibrant generation

Cofibrant generation

Example. *In $\mathscr{T}op$, let*

$$\mathscr{I} = \{i_n : S^{n-1} \to D^n, n \geq 0\} \text{ and } \mathscr{J} = \{j_n : I^n \to I^{n+1}, n \geq 0\}.$$

It is known that every (trivial) cofibration in $\mathscr{T}op$ can be derived from the ones in ($\mathscr{J}$) $\mathscr{I}$ by iterating certain elementary constructions. A map is a (trivial) fibration iff it has the right lifting property with respect to each map in ($\mathscr{I}$) $\mathscr{J}$.

Definition. *A cofibrantly generated model category $\mathscr{M}$ is one with morphism sets $\mathscr{I}$ and $\mathscr{J}$ having properties as above. $\mathscr{I}$ ($\mathscr{J}$) is a generating set of (trivial) cofibrations.*

In practice, defining weak equivalences and specifying generating sets $\mathscr{I}$ and $\mathscr{J}$ is the most convenient way to describe a model category. 1.10

Cofibrant generation (continued)

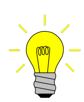
Definition. *A cofibrantly generated model category $\mathscr{M}$ is one with morphism sets $\mathscr{I}$ and $\mathscr{J}$ having similar properties to the ones in $\mathscr{T}op$. $\mathscr{I}$ ($\mathscr{J}$) is a generating set of (trivial) cofibrations.*

In practice, specifying the generating sets $\mathscr{I}$ and $\mathscr{J}$, and defining weak equivalences is the most convenient way to describe a model category.

The Kan Recognition Theorem gives four necessary and sufficient conditions for morphism sets $\mathscr{I}$ and $\mathscr{J}$ to be generating sets as above, assuming that weak equivalences have already been defined. They are too technical for this talk. 1.11

4 Bousfield localization

Bousfield localization

Around 1975 Pete Bousfield had a brilliant idea. 

Suppose we have a model category $\mathscr{M}$, and we wish to change the model structure (without altering the underlying category) as follows.

- Enlarge the class of weak equivalences in some way.
- Keep the same class of cofibrations as before.
- Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

Since there are more weak equivalences, there are more trivial cofibrations. Hence there are fewer fibrations and fewer fibrant objects. This could make the fibrant replacement functor much more interesting.

The hardest part of this is showing that the new classes of weak equivalences and fibrations, along with the original class of cofibrations, satisfy the Factorization Axiom MC5. The proof involves some delicate set theory. 1.12

Three examples of Bousfield localization

Let $\mathscr{T}op$ be the category of topological spaces with its usual model structure.

1. Choose an integer $n > 0$. Define a map f to be a weak equivalence if $\pi_k f$ is an isomorphism for $k \leq n$. Then the fibrant objects are the spaces with no homotopy above dimension n. The fibrant replacement functor is the nth Postnikov section. It was originally constructed by attaching cells to kill all homotopy above dimension n.
2. Choose a prime p. Define a map to be a weak equivalence if it induces an isomorphism in mod p homology. On simply connected spaces, the fibrant replacement functor is p-adic completion.
3. Choose a generalized homology theory h_*. Define a map f to be a weak equivalence if $h_* f$ is an isomorphism. The resulting fibrant replacement functor is Bousfield localization with respect to h_*. One can do the same with the category of spectra, once we have the right model structure on it. 1.13

Bousfield localization in a cofibrantly generated model category

Suppose $\mathscr{M}$ is a cofibrantly generated model category with generating sets $\mathscr{I}$ and $\mathscr{J}$. Let $\mathscr{M}'$ denote its Bousfield localization of $\mathscr{M}$ with respect to some expanded class of weak equivalences. What are its generating sets $\mathscr{I}'$? and $\mathscr{J}'$?

Since $\mathscr{M}'$ has the same class of cofibrations as $\mathscr{M}$, we can set $\mathscr{I}' = \mathscr{I}$.

Since $\mathscr{M}'$ has the more trivial cofibrations than $\mathscr{M}$, we need to make $\mathscr{J}'$ bigger tthan $\mathscr{J}$. There is a theorem saying when such a $\mathscr{J}'$ exists, but there is no known general description of it.

We will give such a description in a certain case related to stable homotopy theory. 1.14

5 Enriched category theory

Enriched category theory

A symmetric monoidal structure on a category $\mathscr{V}_0$ is a functor

$$\mathscr{V}_0 \times \mathscr{V}_0 \xrightarrow{\otimes} \mathscr{V}_0$$

sending a pair of objects (X,Y) to a third object $X \otimes Y$. It is required to have suitable properties including

- a natural isomorphism $t : X \otimes Y \to Y \otimes X$ and
- a unit object $\mathbf{1}$ such that $\mathbf{1} \otimes X$ is naturally isomorphic to X.

We denote this by $\mathscr{V} = (\mathscr{V}_0, \otimes, \mathbf{1})$.

Familiar examples include $(\mathscr{S}et, \times, *)$, $(\mathscr{T}op, \times, *)$, $(\mathscr{T}, \wedge, S^0)$, where $\mathscr{T}$ is the category of pointed topological spaces, and $(\mathscr{S}et_\Delta, \times, *)$, where $\mathscr{S}et_\Delta$ is the category of simplicial sets. 1.15

Enriched category theory (continued)

Let $\mathscr{V} = (\mathscr{V}_0, \otimes, \mathbf{1})$ be a symmetric monoidal category as above.

Definition. *A $\mathscr{V}$-category (or a category enriched over $\mathscr{V}$) consists of*

- *a collection of objects,*
- *for each pair of objects (X,Y) a morphism object $\mathscr{C}(X,Y)$ in $\mathscr{V}_0$ (instead of a set of morphisms $X \to Y$),*
- *for each triple of objects (X,Y,Z) a composition morphism in $\mathscr{V}_0$*

$$c_{X,Y,Z} : \mathscr{C}(Y,Z) \otimes \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$$

(replacing the usual composition) and

- *for each object X, an identity morphism in $\mathscr{V}_0$ $\mathbf{1} \to \mathscr{C}(X,X)$, replacing the usual identity morphism $X \to X$.*

There is an underlying ordinary category $\mathscr{C}_0$ with the same objects as $\mathscr{C}$ and morphism sets

$$\mathscr{C}_0(X,Y) = \mathscr{V}_0(\mathbf{1}, \mathscr{C}(X,Y)).$$

1.16

Enriched category theory (continued)

One can define enriched functors ($\mathscr{V}$-functors) between $\mathscr{V}$-categories and enriched natural transformations ($\mathscr{V}$-natural transformations) between them.

In this language, an ordinary category is enriched over $\mathscr{S}et$.

A topological category is one that is enriched over $\mathscr{T}op$.

A simplicial category is one that is enriched over $\mathscr{S}et_\Delta$, the category of simplicial sets.

A symmetric monoidal category $\mathscr{V}_0$ is closed if it enriched over itself. This means that for each pair of objects (X,Y) there is an internal Hom object $\mathscr{V}(X,Y)$ with natural isomorphisms

$$\mathscr{V}_0(X \otimes Y, Z) \cong \mathscr{V}_0(X, \mathscr{V}(Y,Z)).$$

The symmetric monoidal categories $\mathscr{S}et$, $\mathscr{T}op$, $\mathscr{T}$ and $\mathscr{S}et_\Delta$ are each closed.

1.17

6 Spectra as enriched functors

Spectra as enriched functors

Recall that a spectrum X was originally defined to be a sequence of pointed spaces $\{X_n\}$ with structure maps $\Sigma X_n \to X_{n+1}$. We will redefine it to be an enriched $\mathscr{T}$-valued functor on a small $\mathscr{T}$-category $J^{\mathbf{N}}$. This will make the structure maps built in to the functor. Maps between spectra will be enriched natural transformations.

Definition. *The indexing category $J^{\mathbf{N}}$ has natural numbers $n \geq 0$ as objects with*

$$J^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

For $m \leq m \leq p$, the composition morphism

$$j_{m,n,p} : S^{p-n} \wedge S^{n-m} \to S^{p-m}$$

is the standard homeomorphism.

1.18

Spectra as enriched functors (continued)

We can define a spectrum X to be an enriched functor $X : J^{\mathbf{N}} \to \mathscr{T}$. We denote its value at n by X_n. Functoriality means that for each $m,n \geq 0$ there is a continuous structure map

$$\varepsilon^X_{m,n} : J^{\mathbf{N}}(m,n) \wedge X_m \to X_n.$$

Since

$$J^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise,} \end{cases}$$

for $m \leq n$ we get the expected map $\Sigma^{n-m} X_m \to X_n$.

Definition. *For $m \geq 0$, the Yoneda spectrum $\text{よ}^m = S^{-m}$ is given by*

$$(S^{-m})_n = J^{\mathbf{N}}(m,n) = \begin{cases} S^{n-m} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

In particular, S^{-0} is the sphere spectrum, and S^{-m} is its formal mth desuspension.

1.19

Spectra as enriched functors (continued)

Warning The catgeory $J^{\mathbf{N}}$ is monoidal (under addition) but not symmetric monoidal. It admits an embedding functor into $\mathscr{T}$, namely the Yoneda functor よ^0 given by

$$n \mapsto J^{\mathbf{N}}(0,n) = S^n$$

$\mathscr{T}$ is symmetric monoidal, and there is a twist isomorphism

$$t : S^m \wedge S^n \to S^n \wedge S^m.$$

However this morphism is not in the image of the functor よ^0. There is no twist isomorphism in $J^{\mathbf{N}}$, so its monoidal structure is not symmetric.

This is the reason that the category of spectra $\mathscr{S}p$ defined in this way does not have a convenient smash product. This was a headache in the subject for decades! 1.20

Spectra as enriched functors (continued)

However we can define the smash product of a spectrum X and a pointed space K by

$$(X \wedge K)_n = X_n \wedge K.$$

The categorical term for this is that $\mathscr{S}p$ is tensored over $\mathscr{T}$.

The category of spectra is also cotensored over $\mathscr{T}$, meaning we can define a spectrum X^K by

$$(X^K)_n = X_n^K.$$

More generally when a $\mathscr{V}$-category is both tensored and cotensored over $\mathscr{V}$, we say it is bitensored over $\mathscr{V}$. 1.21

7 The projective model structure

The projective model structure on the category of spectra

We can define the category of spectra to be $[J^{\mathbf{N}}, \mathscr{T}]$, the category of $\mathscr{T}$-valued $\mathscr{T}$-functors on the $\mathscr{T}$-category $J^{\mathbf{N}}$. We define the projective model structure on it as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$. In other words, weak equivalences and fibrations are strict weak equivalences and fibrations.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with the following generating sets.

$$\mathscr{I}^{proj} = \left\{S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) : m,n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathscr{I}_+$$
$$\mathscr{J}^{proj} = \left\{S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) : m,n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathscr{J}_+$$

where $f_+ : X_+ \to Y_+$ denotes $f : X \to Y$ with disjoint base points added to X and Y. $\mathscr{I}_+$ and $\mathscr{J}_+$ are generating sets for $\mathscr{T}$. They are the pointed analogs of $\mathscr{I}$ and $\mathscr{J}$, the generating sets for $\mathscr{T}op$. 1.22

A generalization

The above can be generalized as follows.

- Replace $\mathscr{T}$ by a pointed cofibrantly generated model category $\mathscr{M}$ with a closed symmetic monoidal structure (sometimes called a cofibrantly generated Quillen ring) and generating sets $\mathscr{I}$ an $\mathscr{J}$. For example, $\mathscr{M}$ could be $\mathscr{T}^G$, the category of pointed G-spaces with the Bredon model structure.
- Replace the suspension functor $\Sigma = S^1 \wedge -$ by the functor $K \wedge -$ for a fixed cofibrant object K, such as S^{ρ_G}, the sphere associated with the regular representation of the finite group G.

- Replace $J^{\mathbf{N}}$ by the $\mathscr{M}$-category $J_K^{\mathbf{N}}$ with morphism objects

$$J_K^{\mathbf{N}}(m,n) = \begin{cases} K^{\wedge(n-m)} & \text{for } n \geq m \\ * & \text{otherwise.} \end{cases}$$

- Replace the Yoneda spectrum S^{-m} by the functor $K^{-m} : J_K^{\mathbf{N}} \to \mathscr{M}$ given by

$$(K^{-m})_n = J_K^{\mathbf{N}}(m,n).$$

1.23

A generalization (continued)

Then we can define the projective model structure on the enriched functor category $[J_K^{\mathbf{N}}, \mathscr{M}]$ as follows.

- A map $f : X \to Y$ is a weak equivalence or fibration if $f_n : X_n \to Y_n$ is one for each $n \geq 0$.
- Cofibrations are defined in terms of left lifting properties.

This model structure is known to be cofibrantly generated with generating sets

$$\mathscr{I}^{proj} = \left\{K^{-m} \colon m \geq 0\right\} \wedge \mathscr{I}$$
$$\text{and} \qquad \mathscr{J}^{proj} = \left\{K^{-m} \colon m \geq 0\right\} \wedge \mathscr{J}.$$

1.24

8 The stable model structure

More about Bousfield localization

In order to discuss Bousfield localization more precisely, it helps to start with a model category that is enriched over a Quillen ring $\mathscr{M}$ (possibly but not necessarily the category we want to localize), so we can speak of weak equivalences of morphisms objects. Recall that a Quillen ring $\mathscr{M}$ is model category with a closed symmetic monoidal structure. A Quillen $\mathscr{M}$-module is a model category $\mathscr{N}$ that is enriched and bitensored over $\mathscr{M}$. 1.25

More about Bousfield localization (continued)

Definition. *Let $\mathscr{N}$ be a module over Quillen ring $\mathscr{M}$ as above, and let S be a set of morphisms in $\mathscr{N}$.*

An object Z is S-local if for each $f : A \to B$ in S, the map

$$f^* : \mathscr{N}(B,Z) \to \mathscr{N}(A,Z)$$

is a weak equivalence in $\mathscr{M}$.

A morphism $g : X \to Y$ in $\mathscr{N}$ is an S-equivalence if for each S-local object Z the map

$$g^* : \mathscr{N}(Y,Z) \to \mathscr{N}(X,Z)$$

is a weak equivalence in $\mathscr{M}$.

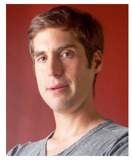
1.26

More about Bousfield localization (continued)

It is easy to verify that every weak equivalence is an S-equivalence, that a retract of an S-equivalence is an S-equivalence, and that S-equivalences have the 2-of-3 property.


Clark Barwick


Phil Hirschhorn


Jacob Lurie


Jeff Smith

The four shown above have shown that under various mild hypotheses on $\mathscr{N}$, the class of S-equivalences leads to a new model structure on $\mathscr{N}$ for any morphism set S. We denote this new model category by $L_S\mathscr{N}$. We also denote its fibrant replacement functor by L_S. The fibrant objects of $L_S\mathscr{N}$ are the S-local objects of $\mathscr{N}$. 1.27

Stabilizing maps and the stable model structure

We will define a set S of morphisms in $\mathscr{S}p = [J^{\mathbf{N}}, \mathscr{T}]$ (and more generally in $[J_K^{\mathbf{N}}, \mathscr{M}]$)such that S-equivalences are stable equivalences.

For each $m \geq 0$, let the mth stabilizing map

$$s_m : S^{-1-m} \wedge S^1 \to S^{-m}$$

be the one whose nth component is

$$\begin{cases} * \to * & \text{for } n < m \\ * \to S^0 & \text{for } n = m \\ S^{n-m-1} \wedge S^1 \to S^{n-m} & \text{otherwise} \end{cases}$$

Since this is a homeomorphism, and hence a weak equivalence, for large n, s_m is a stable equivalence.

The morphism set we want is

$$S = \{s_m : m \geq 0\}.$$

1.28

Stabilizing maps and the stable model structure (continued)

The morphism set we want is

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\}.$$

What are the S-local objects? Now for the fun part! The Yoneda lemma implies that for any space K and spectrum Z,

$$\mathscr{S}p(S^{-n} \wedge K, Z) \cong (Z_n)^K.$$

This means that s_m^* is the map

$$\eta_m^Z : Z_m \to \Omega Z_{m+1},$$

the adjoint of the structure map $\varepsilon_m^Z : \Sigma Z_m \to Z_{m+1}$.

The spectrum Z is S-local iff the map η_m^Z is a weak equivalence for each $m \geq 0$, i.e., Z is an Ω-spectrum as classically defined. The observation that the fibrant objects are the Ω-spectra is originally due to Bousfield-Friedlander, 1978. 1.29

Stabilizing maps and the stable model structure (continued)

For

$$S = \left\{ s_m : S^{-1-m} \wedge S^1 \to S^{-m} : m \geq 0 \right\},$$

a spectrum Z is S-local iff it is an Ω-spectrum.

What are the S-equivalences? A map $g : X \to Y$ is an S-equivalence if

$$g^* : \mathscr{S}p(Y,Z) \to \mathscr{S}p(X,Z)$$

is a weak equivalence for every Ω-psectrum Z, if g induces an isomorphism in every generalized cohomology theory. This coincides with a classical definition of stable equivalence.

This means that the Bousfield localization $L_S \mathscr{S}p$ is the category of classically define spectra in which weak equivalences are stable equivalences. Its homotopy cetegory is the one described long ago by Boardman and Vogt. 1.30

9 Stable cofibrant generating sets

Cofibrant generating sets for the stable category

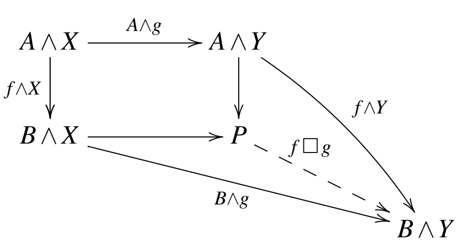
Recall that the projective (or strict) model structure on $\mathscr{S}p$ has cofibrant generating sets

$$\mathscr{I}^{proj} = \left\{ S^{-m} \wedge (i_{n_+} : S^{n-1}_+ \to D^n_+) : m,n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathscr{I}_+$$
$$\mathscr{J}^{proj} = \left\{ S^{-m} \wedge (j_{n+} : I^n_+ \to I^{n+1}_+) : m,n \geq 0 \right\} = \left\{ S^{-m} \right\} \wedge \mathscr{J}_+$$

We can define $\mathscr{I}^{stable}$ to be $\mathscr{I}^{proj}$, but we must enlarge $\mathscr{J}^{proj}$ in some way to get $\mathscr{J}^{stable}$. To describe this we need the following. 1.31

Cofibrant generating sets for the stable category (continued)

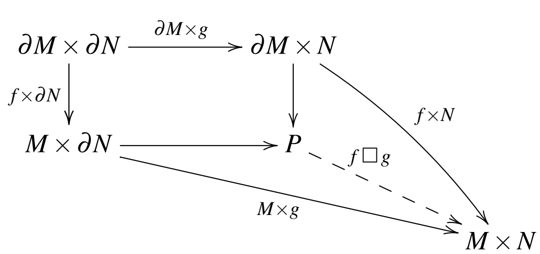
Definition. *Let $\mathscr{M}$ be a Quillen ring with a morphism $g : X \to Y$, and $\mathscr{N}$ a Quillen $\mathscr{M}$-module with a morphism $f : A \to B$. Consider the diagram*



where P is the pushout of the two maps from $A \wedge X$. Then the pushout corner map (or pushout smash product) $f \square g$ is the unique map $P \to B \wedge Y$ that makes the diagram commute. 1.32

Cofibrant generating sets for the stable category (continued)

An easy example of a pushout corner map. Let $\mathscr{M} = \mathscr{N} = \mathscr{T}op$, let M and N be manifolds with boundary, and consider the morphisms $f : \partial M \to M$ and $g : \partial N \to N$, the inclusions of the boundaries. Then the diagram is



In this case the pushout is

$$P = (\partial M \times N) \cup_{\partial M \times \partial N} (M \times \partial N) = \partial(M \times N),$$

and $f \Box g$ is the inclusion $\partial(M \times N) \to M \times N$. 1.33

Cofibrant generating sets for the stable category (continued)

Now we can describe the cofibrant generating sets for $L_s\mathscr{S}p$.

Recall again that

$$\mathscr{I}^{proj} = \left\{S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m,n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathscr{I}_+$$
$$\mathscr{J}^{proj} = \left\{S^{-m} \wedge (j_{n+} : I_+^n \to I_+^{n+1}) \colon m,n \geq 0\right\} = \left\{S^{-m}\right\} \wedge \mathscr{J}_+$$

Theorem. *The following are cofibrant generating sets for $L_S\mathscr{S}p$.*

$$\begin{aligned} \mathscr{I}^{stable} &= \mathscr{I}^{proj} \\ \mathscr{J}^{stable} &= \mathscr{J}^{proj} \cup \{s_m \Box i_{n+} \colon m,n \geq 0\} \\ &= \mathscr{J}^{proj} \cup (S \Box \mathscr{I}_+). \end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem. Most of it is routine. 1.34

Cofibrant generating sets for the stable category (continued)

Theorem. *The following are cofibrant generating sets for $L_S\mathscr{S}p$.*

$$\begin{aligned} \mathscr{I}^{stable} &= \mathscr{I}^{proj} \\ \mathscr{J}^{stable} &= \mathscr{J}^{proj} \cup \{s_m \Box i_{n+} \colon m,n \geq 0\} \\ &= \mathscr{J}^{proj} \cup (S \Box \mathscr{I}_+). \end{aligned}$$

The proof consists of showing that these two sets satisfy the four (unnamed) technical conditions of the Kan Recognition Theorem. Most of it is routine.

The most difficult point is to show that a stable equivalence with the right lifting property with respect to $\mathscr{J}^{stable}$ also has it with respect to $\mathscr{I}^{stable}$, which means it is a trivial fibration. 1.35

Cofibrant generating sets for the stable category (continued)

Again, the key point is to show that a stable equivalence $p : X \to Y$ with the right lifting property with respect to

$$\begin{aligned} \mathscr{J}^{stable} = &\left\{S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m,n \geq 0\right\} \\ &\cup \{s_m \Box i_{n+} \colon m,n \geq 0\} \end{aligned}$$

also has it with respect to

$$\mathscr{I}^{proj} = \left\{S^{-m} \wedge (i_{n_+} : S_+^{n-1} \to D_+^n) \colon m,n \geq 0\right\}.$$

Hence we are looking at a strict fibration that has the right lifting property with respect to each pushout corner map $s_m \Box i_{n+}$.

The latter condition is equivalent to the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & Y_m \\ {\scriptstyle \eta_m^X}\downarrow & & \downarrow{\scriptstyle \eta_m^Y} \\ \Omega X_{m+1} & \xrightarrow{\Omega p_{m+1}} & \Omega Y_{m+1} \end{array}$$

being homotopy Cartesian. 1.36

Cofibrant generating sets for the stable category (continued)

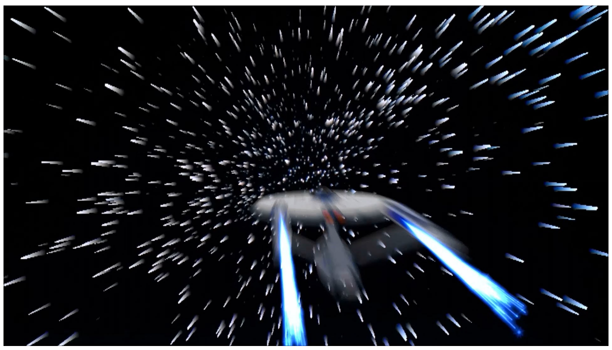
Recall the functor $\Lambda : \mathscr{S}p \to \mathscr{S}p$ for which $(\Lambda X)_m$ is the homotopy colimit of

$$X_m \xrightarrow{\eta_m^X} \Omega X_{m+1} \xrightarrow{\Omega\eta_{m+1}^X} \Omega^2 X_{m+2} \xrightarrow{\Omega^2\eta_{m+2}^X} \dots$$

We know that the corner map condition on our strict fibration $p : X \to Y$ implies that the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & Y_m \\ \downarrow & & \downarrow \\ (\Lambda X)_m & \xrightarrow{\Lambda p_m} & (\Lambda Y)_m \end{array}$$

is homotopy Cartesian. It is known that Λ converts stable equivalences to strict ones, so p_m is a weak equivalence, which makes p a trivial fibration as desired. 1.37



Thank you! 1.38