The Hill-Lawson spectral sequence and the telescope conjecture

Doug Ravenel
University of Rochester

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$$P(v_n : n \geq 0) \otimes E(h_{i,j} : i > 0, j \geq 0) \otimes P(b_{i,j} : i > 0, j \geq 0)$$

where $n \geq 0$, $i > 0$, and $j \geq 0$ with

$$v_n \in E_1^{2p^n-2,1}, \quad h_{i,j} \in E_1^{2p^i(p^i-1)-1,1}, \quad b_{i,j} \in E_1^{2p^{j+1}(p^i-1)-2,2}.$$
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Here the superscripts are topological dimension and filtration, the \((x,y)\) convention. For \( p = 2 \), there is a similar description with \( b_{i,j} = h_{i,j}^2 \). In general it is a \( p \)-fold Massey product.
Again, for the Adams spectral sequence,

$$E_1 = P(v_n : n \geq 0) \otimes E(h_{i,j} : i > 0, j \geq 0) \otimes P(b_{i,j} : i > 0, j \geq 0)$$

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Note that the Hill-Lawson filtration is higher than that of Adams.
The MRS spectrum $\gamma(m)$

The following can be found in a paper I wrote with Mark Mahowald and Paul Shick in 1999. For each prime $p$, there is a $p$-local spherical fibration $\lambda_p$ over the space $\Omega S^3$ whose Thom spectrum is $\mathbb{H}/p$, the mod $p$ Eilenberg-Mac Lane spectrum. We can restrict $\lambda_p$ to various spaces over $\Omega S^3$ to get more Thom spectra. In the 1950s Ioan James showed that $\Omega S^3$ is homotopy equivalent to a certain CW-complex with a single cell in every even dimension. We denote its $2k$-skeleton by $J_k S^2$, the $k$th James construction on $S^2$, which is a certain quotient of $(S^2)^k$. 

**Inverting $v_m$**

**The Adams-Novikov spectral sequence**

**The localized Adams spectral sequence**

**The localized Hill-Lawson spectral sequence for $\gamma(m)$**

**Internal Steenrod operations for $\gamma(m)$**

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**A possible $E_2$ structure**

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Definition

For a prime $p$ and positive integer $m$,
The MRS spectrum $y(m)$ (continued)

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For a prime $p$ and positive integer $m$, let $y(m)$ denote the Thom spectrum of the restriction of $\lambda_p$ induced by the map

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This spectrum has some pleasant properties. Recall that

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H_* H/p \cong E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots)
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with \( |\tau_i| = 2p^i - 1 \) and \( |\xi_i| = 2p^i - 2 \).
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It turns out that

$$H_* y(m) \simeq E(\tau_0, \ldots, \tau_{m-1}) \otimes P(\xi_1, \ldots, \xi_m).$$
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This implies that the Adams spectral sequence for $y(m)$ has

$$E_1 = P(v_{m+n} : n \geq 0) \otimes E(h_{m+i,j} : i > 0, j \geq 0) \otimes P(b_{m+i,j} : i > 0, j \geq 0)$$

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where $n \geq 0$, $i > 0$, and $j \geq 0$.

We have added $m$ to each (first) subscript. There is a Hill-Lawson spectral sequence having a similar $E_1$-term in which each element has filtration divisible by $p^m$. 
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When $k = p^m$, its $p$-local homotopy theoretic fiber is our friend $J_{p^{m-1}} S^2$. It follows that there is a fiber sequence

$$\Omega^3 S^{2p^m+1} \to \Omega J_{p^{m-1}} S^2 \to \Omega^2 S^3,$$

which Thomifies to

$$\Sigma^\infty \Omega^3 S^{2p^m+1} \to y(m) \to H/p.$$
The composite

\[ S^{2p^m-2} \to \sum_\infty \Omega^3 S^{2p^m+1} \to y(m) \]
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$$y(m) \xrightarrow{v_m} \Sigma^{-|v_m|}y(m) \xrightarrow{v_m} \Sigma^{-2|v_m|}y(m) \xrightarrow{} \ldots,$$

which we denote by $Y(m)$.
The telescope $Y(m)$ is the homotopy colimit of

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There is a well understood Adams-Novikov spectral sequence converging to $\pi_\ast L_{K(m)} y(m)$. 

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There is a well understood Adams-Novikov spectral sequence converging to $\pi_* L_{K(m)} y(m)$.

There are localized forms of both the Adams and Hill-Lawson spectral sequences that converge to $\pi_* Y(m)$.
Inverting $v_m$ (continued)

The telescope $Y(m)$ is the homotopy colimit of

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There are localized forms of both the Adams and Hill-Lawson spectral sequences that converge to $\pi_* Y(m)$. The latter is a new tool for studying the telescope conjecture.
The Adams-Novikov spectral sequence

The Adams-Novikov $E_2$-term for $L_{K(m)}y(m)$ is

$$R_m \otimes E(h_{m+i,j} : 1 \leq i, j + 1 \leq m),$$

where $R_m = v_m^{-1}P(v_m, \ldots, v_{2m})$. 

This is an exterior algebra on $m$ odd dimensional generators tensored with an even dimensional localized polynomial ring. Each $v_m+i$ has filtration 0, and each $h_{m+i,j}$ has filtration 1.

The spectral sequence collapses for large primes. The exterior algebra is the cohomology of a certain open subgroup of the $m$th Morava stabilizer group. It is cofinite with index $p^m 2^{p^m - m(p^m - 1)}$. 

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The exterior algebra is the cohomology of a certain open subgroup of the $m$th Morava stabilizer group.
The Adams-Novikov spectral sequence

The Adams-Novikov $E_2$-term for $L_{K(m)}y(m)$ is

$$R_m \otimes E(h_{m+i,j} : 1 \leq i, j + 1 \leq m),$$

where $R_m = v_m^{-1}P(v_m, \ldots, v_{2m})$.

This is an exterior algebra on $m^2$ odd dimensional generators tensored with an even dimensional localized polynomial ring. Each $v_{m+i}$ has filtration 0, and each $h_{m+i,j}$ has filtration 1. The spectral sequence collapses for large primes.

The exterior algebra is the cohomology of a certain open subgroup of the $m$th Morava stabilizer group. It is cofinite with index $\rho^{m^2-m}(\rho^m - 1)$. 

Internal Steenrod operations for $y(m)$

Some Hill-Lawson $d_1$'s

Conclusion
To repeat, the Adams-Novikov $E_2$-term for $L_{K(m)}y(m)$ is

$$R_m \otimes E(h_{n+i,j} : 1 \leq i, j + 1 \leq h).$$
To repeat, the Adams-Novikov $E_2$-term for $L_{K(m)}y(m)$ is

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The localized Adams $E_2$-term for $Y(m)$ is

$$R_m \otimes E(h_{m+i,j}) \otimes P(b_{m+i,j})$$

where $i > 0$ and $0 \leq j \leq m - 1$

and $R_m = v_m^{-1} P(v_m, \ldots, v_{2m}).$
To repeat, the Adams-Novikov $E_2$-term for $L_{K(m)}y(m)$ is

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The Adams filtration of each $v_{m+i}$ is 1 instead of 0.
To repeat, the Adams-Novikov $E_2$-term for $L_{K(m)}y(m)$ is

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The Adams filtration of each $v_{m+i}$ is 1 instead of 0. Unlike the Adams-Novikov $E_2$-term, it is infinitely generated over $R_m.$
The localized Adams spectral sequence (continued)

To repeat, the localized Adams $E_2$-term for $Y(m)$ is

$$R_m \otimes E(h_{m+i,j}) \otimes P(b_{m+i,j}).$$
The localized Adams spectral sequence (continued)

To repeat, the localized Adams $E_2$-term for $Y(m)$ is

$$R_m \otimes E(h_{m+i,j}) \otimes P(b_{m+i,j}).$$

We conjectured that there are differentials

$$d_{2p^i}h_{m+i,j} = v_mb_{i+j,m-1-j}^{p^i}$$

for $0 \leq j \leq m-1$ and $i+j > m$.

and no others.
To repeat, the localized Adams $E_2$-term for $Y(m)$ is

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and no others. This would leave

$$E_\infty = R_m \otimes E(h_{m+i,j} : i + j \leq m) \otimes P(b_{m+i,j})/(b_{m+i,j}^{p^{m-1-j}}).$$
The localized Adams spectral sequence (continued)

The conjectured localized Adams $E_\infty$-term is

$$R_m \otimes E(h_{m+i,j} : i + j \leq m)$$

$$\otimes P(b_{m+i,j} : i > 0, 0 \leq j \leq m - 2)/(b^{p^{m-1}-j}_{m+i,j}).$$
The localized Adams spectral sequence (continued)

The conjectured localized Adams $E_\infty$-term is

$$R_m \otimes E(h_{m+i,j} : i + j \leq m)$$

$$\otimes P(b_{m+i,j} : i > 0, 0 \leq j \leq m - 2) / (b_b^{m-1-j})$$.

For $m = 1$ this reads $R_1 \otimes E(h_{2,0})$, 

$$\otimes P(b_{2+i,j} : i > 0, 0 \leq j \leq m - 2) / (b_b^{m-1-j})$$.

Unfortunately we were unable to prove that the expected differentials all occur or that the $b_b^{m+i,j}$ are all permanent cycles.
The localized Adams spectral sequence (continued)

The conjectured localized Adams $E_\infty$-term is

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For $m = 1$ this reads $R_1 \otimes E(h_{2,0})$, which is also the Adams-Novikov $E_2$-term.
The localized Adams spectral sequence (continued)

The conjectured localized Adams $E_\infty$-term is

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For $m > 1$ the number of exterior generators is $(m^2 + m)/2$, which is fewer than the $m^2$ generators predicted by the telescope conjecture.
The localized Adams spectral sequence (continued)

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For $m > 1$ the number of exterior generators is $(m^2 + m)/2$, which is fewer than the $m^2$ generators predicted by the telescope conjecture. For $m = 2$, the above reads

$$R_2 \otimes E(h_{3,0}, h_{3,1}, h_{4,0}) \otimes P(b_{2+i,0} : i > 0)/(b_{2+i,0}^{2i}).$$

Unfortunately we were unable to prove that the expected differentials all occur or that the $b_{m+i,j}$s are all permanent cycles.
The localized Adams spectral sequence (continued)

The conjectured localized Adams $E_\infty$-term is

$$R_m \otimes E(h_{m+i,j} : i + j \leq m) \otimes P(b_{m+i,j} : i > 0, 0 \leq j \leq m-2)/(b_{m+i,j}^{m-1-j}).$$

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Unfortunately we were unable to prove that the expected differentials all occur or that the $b_{m+i,j}$s are all permanent cycles.
The localized Hill-Lawson spectral sequence

The Hill-Lawson $E_1$-term for the spectrum $y(m)$ is

$$E_1 = P(v_{m+n} : n \geq 0) \otimes E(h_{m+i,j} : i > 0, j \geq 0) \otimes P(b_{m+i,j})$$

$$v_{m+n} \in E_1^{2p^{M+n-2},p^{n}}, \quad h_{m+i,j} \in E_1^{2p^i(p^{m+i}-1)-1,p^{i+j}}$$

$$b_{m+i,j} \in E_1^{2p^{i+1}(p^{m+i}-1)-2,p^{i+j+1}}$$
The localized Hill-Lawson spectral sequence

The Hill-Lawson $E_1$-term for the spectrum $y(m)$ is

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\[ v_{m+n} \in E_1^{2p^{M+n-2},p^n}, \quad h_{m+i,j} \in E_1^{2p^i(p^{m+i-1}-1),p^i+j} \]

\[ b_{m+i,j} \in E_1^{2p^{i+1}(p^{m+i-1}-2),p^{i+j+1}}. \]

As we did for the (May) Adams $E_1$-term, we added $m$ to all of the subscripts.
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As we did for the (May) Adams $E_1$-term, we added $m$ to all of the subscripts. Here we have divided the previously defined filtrations by $p^m$.

Before discussing differentials we need to describe some internal structure of $y(m)$. 
Recall that

\[ H_\ast y(m) \cong E(\tau_0, \ldots, \tau_{m-1}) \otimes P(\xi_1, \ldots, \xi_m) \subseteq A_\ast, \]

where \( A_\ast \) is the dual Steenrod algebra.
Recall that

\[ H_*y(m) \cong E(\tau_0, \ldots, \tau_{m-1}) \otimes P(\xi_1, \ldots, \xi_m) \subseteq A_*, \]

where \( A_* \) is the dual Steenrod algebra. This leads to a splitting

\[ y(m) \wedge y(m) \cong \bigvee_\alpha \Sigma |\alpha| y(m) \]

with one summand for each monomial \( \alpha \) in \( H_*y(m) \),
The Hill-Lawson spectral sequence and the telescope conjecture

Doug Ravenel

The Hill-Lawson spectral sequence

The MRS spectrum $y(m)$

Inverting $v_m$

The Adams-Novikov spectral sequence

The localized Adams spectral sequence

The localized Hill-Lawson spectral sequence for $y(m)$

Internal Steenrod operations for $y(m)$

Recall that

$$H_*y(m) \cong E(\tau_0, \ldots, \tau_{m-1}) \otimes P(\xi_1, \ldots, \xi_m) \subseteq A_*,$$

where $A_*$ is the dual Steenrod algebra. This leads to a splitting

$$y(m) \wedge y(m) \cong \bigvee_{\alpha} \sum|\alpha| y(m)$$

with one summand for each monomial $\alpha$ in $H_*y(m)$, and to maps (cohomology operations)

$$y(m) \xrightarrow{\theta^\alpha} \sum|\alpha| y(m).$$
Recall that

\[ H_* y(m) \cong E(\tau_0, \ldots, \tau_{m-1}) \otimes P(\xi_1, \ldots, \xi_m) \subseteq A_*, \]

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with one summand for each monomial \( \alpha \) in \( H_* y(m) \), and to maps (cohomology operations)

\[ y(m) \xrightarrow{\theta_{\alpha}} \Sigma^{\vert \alpha \vert} y(m). \]

These lead to right actions of a certain quotient of the Steenrod algebra (the dual of \( H_* y(m) \)).
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\[ y(m) \wedge y(m) \cong \bigvee_{\alpha} \Sigma|\alpha| y(m) \]

with one summand for each monomial $\alpha$ in $H_* y(m)$, and to maps (cohomology operations)

\[ y(m) \xrightarrow{\theta^\alpha} \Sigma|\alpha| y(m) . \]

These lead to right actions of a certain quotient of the Steenrod algebra (the dual of $H_* y(m)$) on each of our spectral sequences.
Internal Steenrod operations for $y(m)$

The action of internal Steenrod operations in the Hill-Lawson $E_1$-term for each $i > 0$ is shown below.

\[
\begin{align*}
\beta : v_{m+i} &\xrightarrow{P^1} v_{m+i-1}^p \xrightarrow{P^p} v_{m+i-2}^{p^2} \xrightarrow{P^{p^2}} \cdots \\
\beta : h_{m+i,0} &\xrightarrow{P^1} h_{m+i-1,1} \xrightarrow{P^p} h_{m+i-2,2} \xrightarrow{P^{p^2}} \cdots \\
h_{m+i,m} &\xrightarrow{\beta} b_{m+i,m-1}^p \xrightarrow{P^1} b_{m+i,m-2}^{p^2} \xrightarrow{P^p} b_{m+i,m-3}^{p^2} \xrightarrow{P^{p^2}} \cdots 
\end{align*}
\]
The action of internal Steenrod operations in the Hill-Lawson $E_1$-term for each $i > 0$ is shown below.

\[
\begin{align*}
\beta \downarrow & \\
\nu_{m+i} & \xrightarrow{P^1} \nu_{m+i-1}^p & \xrightarrow{P^p} \nu_{m+i-2}^{p^2} & \xrightarrow{P^{p^2}} \cdots \\
\beta \downarrow & \\
h_{m+i,0} & \xrightarrow{P^1} h_{m+i-1,1} & \xrightarrow{P^p} h_{m+i-2,2} & \xrightarrow{P^{p^2}} \cdots \\
\beta \downarrow & \\
h_{m+i,m} & \xrightarrow{P^1} b_{m+i,m-1}^p & \xrightarrow{P^p} b_{m+i,m-2}^{p^2} & \xrightarrow{P^{p^2}} \cdots \\
\end{align*}
\]

Elements shown above that are linked by these operations all have the same Hill-Lawson filtration.
The action of internal Steenrod operations in the Hill-Lawson $E_1$-term for each $i > 0$ is shown below.

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  v_{m+i} & \xrightarrow{P^1} v_{m+i-1}^p & \xrightarrow{P^p} v_{m+i-2}^{p^2} & \xrightarrow{P^{p^2}} \cdots \\
  h_{m+i,0} & \xrightarrow{P^1} h_{m+i-1,1} & \xrightarrow{P^p} h_{m+i-2,2} & \xrightarrow{P^{p^2}} \cdots \\
  h_{m+i,m} & \xrightarrow{\beta} b_{m+i,m-1}^p & \xrightarrow{P^p} b_{m+i,m-2}^{p^2} & \xrightarrow{P^{p^2}} \cdots \\
\end{align*}
\]

Elements shown above that are linked by these operations all have the same Hill-Lawson filtration. This is not true for the Adams and Novikov filtrations.
**Internal Steenrod operations for \( y(m) \)**

(continued)

The action of internal Steenrod operations in the Hill-Lawson \( E_1 \)-term for each \( i > 0 \) is shown below.

\[
\begin{align*}
& v_{m+i} \xrightarrow{P_1} v_{m+i-1} \xrightarrow{P^0} v_{m+i-2} \xrightarrow{P^0} \cdots \\
& h_{m+i,0} \xrightarrow{P_1} h_{m+i-1,1} \xrightarrow{P^0} h_{m+i-2,2} \xrightarrow{P^0} \cdots \\
& h_{m+i,m} \xrightarrow{\beta} b_{m+i,m-1} \xrightarrow{P_1} b_{m+i,m-2} \xrightarrow{P^0} b_{m+i,m-3} \xrightarrow{P^0} \cdots
\end{align*}
\]

Elements shown above that are linked by these operations all have the same Hill-Lawson filtration. This is not true for the Adams and Novikov filtrations. Each sequence has finite length because one of the subscripts in it eventually gets too small.
Some Hill-Lawson $d_1$s

It is easy to show that $d_1 v_{2m+i} = v_m h_{m+i,m}$. 

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It is easy to show that $d_1 v_{2m+i} = v_m h_{m+i,m}$. Differentials must commute with internal Steenrod operations, so for each $i > 0$ we get a diagram
Some Hill-Lawson $d_1$s

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$$
\begin{align*}
  v_{m} h_{m+i,m} \xrightarrow{\beta} v_{m} b_{m+i,m-1} & \xrightarrow{p^1} v_{m} b_{m+i,m-2}^p & \xrightarrow{p^p} \cdots \\
  d_1 \uparrow & \quad d_1 \uparrow & \quad d_1 \uparrow \\
  v_{2m+i} \xrightarrow{\beta} h_{2m+i,0} & \xrightarrow{p^1} h_{2m+i-1,1} & \xrightarrow{p^p} \cdots 
\end{align*}
$$

These $d_1$s correspond to the $d_p j$s that Mahowald, Shick and I wanted in the localized Adams spectral sequence! I will call them Steenrod differentials. This is why I like the Hill-Lawson spectral sequence.
Some Hill-Lawson $d_1s$

It is easy to show that $d_1v_{2m+i} = v_m h_{m+i,m}$. Differentials must commute with internal Steenrod operations, so for each $i > 0$ we get a diagram

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\begin{array}{cccccc}
  v_m h_{m+i,m} & \xrightarrow{\beta} & v_m b_{m+i,m-1} & \xrightarrow{p^1} & v_m b_{m+i,m-2} & \xrightarrow{p^p} \\
  d_1 \uparrow & & d_1 \uparrow & & d_1 \uparrow & \\
  v_{2m+i} & \xrightarrow{\beta} & h_{2m+i,0} & \xrightarrow{p^1} & h_{2m+i-1,1} & \xrightarrow{p^p} \\
\end{array}
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Some Hill-Lawson $d_1$s

It is easy to show that $d_1 v_{2m+i} = v_m h_{m+i,m}$. Differentials must commute with internal Steenrod operations, so for each $i > 0$ we get a diagram

$$
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&v_{2m+i} \xrightarrow{\beta} h_{2m+i,0} \xrightarrow{P^1} h_{2m+i-1,1} \xrightarrow{P^p} \cdots
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Some Hill-Lawson $d_1$s

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& \uparrow d_1 & \uparrow d_1 & \uparrow d_1 \\
v_{2m+i} & \xrightarrow{\beta} h_{2m+i,0} & \xrightarrow{p^1} h_{2m+i-1,1} & \xrightarrow{p^p} \\
& & & \\
\end{align*}
$$

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This is why I like the Hill-Lawson spectral sequence.
A possible $E_2$ structure

Since $y(m)$ is the Thom spectrum associated with a loop map (but not a double loop map),
A possible $E_2$ structure

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It is known that any $E_1$ ring spectrum $R$ has an $E_2$ center $Z(R)$, AKA its topological Hochschild cohomology.
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$$\mathcal{Z}(y(m)) \cong F(J_{p^m-1}S^2, y(m)),$$

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a certain function spectrum. Its Hill-Lawson filtration may or may not be compatible with its \( E_2 \) structure. If it is, the spectral sequence has certain Dyer-Lashof operations.
A possible $E_2$ structure (continued)

If we have the desired $E_2$ structure, we get the following diagram for each $i > 0$, where the horizontal arrows are Dyer-Lashof operations.
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$$v_m h_{m+i, m} \xrightarrow{Q_1} v_m^p h_{m+i, m+1} \xrightarrow{Q_1} v_m^{p^2} h_{m+i, m+2} \xrightarrow{Q_1} \cdots$$

$$d_1$$

$$v_{2m+i} \xrightarrow{Q_0} v_{2m+i}^p$$

$$d_p$$

$$v_{2m+i}^p \xrightarrow{Q_0} v_{2m+i}^{p^2} \xrightarrow{Q_0} \cdots$$

$$d_{p^2}$$

These longer Hill-Lawson differentials correspond to $d_1$'s in both the Adams and Adams-Novikov spectral sequences. I will call them Dyer-Lashof differentials.
If we have the desired $E_2$ structure, we get the following diagram for each $i > 0$, where the horizontal arrows are Dyer-Lashof operations.

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A possible $\mathbb{E}_2$ structure (continued)

If we have the desired $\mathbb{E}_2$ structure, we get the following diagram for each $i > 0$, where the horizontal arrows are Dyer-Lashof operations.

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Conclusion

After taking these Steenrod and Dyer-Lashoff differentials into account, we would be left with

\[ R_m \otimes E(h_{m+i,j} : i + j \leq m) \otimes P(b_{m+i,j})/\left( b_{m+i,j}^{p-1-j} \right). \]
Conclusion

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This is similar to the answer Mahowald, Shick and I were hoping for.
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THANK YOU!
The Hill-Lawson spectral sequence and the telescope conjecture

Doug Ravenel

The Hill-Lawson spectral sequence

The MRS spectrum $y(m)$

Inverting $v_m$

The Adams-Novikov spectral sequence

The localized Adams spectral sequence

The localized Hill-Lawson spectral sequence for $y(m)$

Internal Steenrod operations for $y(m)$

Some Hill-Lawson $d_1$'s

A possible $E_2$ structure

Table of spectral sequence filtrations and dimensions

<table>
<thead>
<tr>
<th>Spectral sequence</th>
<th>$v_{m+n}$</th>
<th>$h_{m+i,j}$</th>
<th>$b_{m+i,j}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adams-Novikov</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Adams</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Hill-Lawson</td>
<td>$p^n$</td>
<td>$p^{i+j}$</td>
<td>$p^{i+j+1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_{m+n}$</td>
<td>$2p^{m+n} - 2$</td>
</tr>
<tr>
<td>$h_{m+i,j}$</td>
<td>$2p^i(p^{m+i} - 1) - 1$</td>
</tr>
<tr>
<td>$b_{m+i,j}$</td>
<td>$2p^{i+1}(p^{m+i} - 1) - 2$</td>
</tr>
</tbody>
</table>
**Hill-Lawson differentials**

**Steenrod differentials:**

\[
\begin{align*}
v_m h_{m+i, m} &\xrightarrow{\beta} v_m b_{m+i, m-1} \xrightarrow{\beta} v_m b_{m+i, m-2} \xrightarrow{P^1} \cdots \\
v_{2m+i} &\xrightarrow{\beta} h_{2m+i, 0} \xrightarrow{P^1} h_{2m+i-1, 1} \xrightarrow{P^p} \cdots
\end{align*}
\]

**Dyer-Lashof differentials:**

\[
\begin{align*}
v_m h_{m+i, m} &\xrightarrow{Q_1} v_m b_{m+i, m+1} \xrightarrow{Q_1} v_m b_{m+i, m+2} \xrightarrow{Q_0} \cdots \\
v_{2m+i} &\xrightarrow{Q_0} v_{2m+i} \xrightarrow{Q_0} v_{2m+i} \xrightarrow{Q_0} \cdots
\end{align*}
\]