Math 162: Calculus IIA

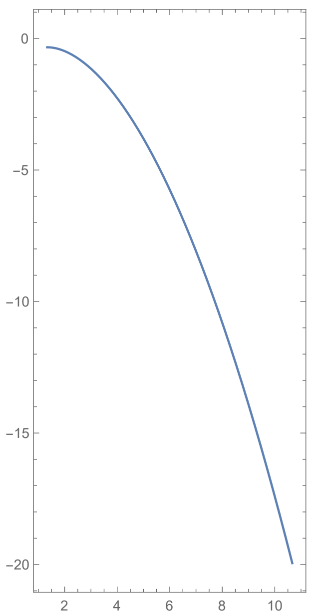
Second Midterm Exam ANSWERS

April 2, 2019

1. (20 points) Consider the curve C defined by the parametric equations

$$x = \frac{4\sqrt{t^3}}{3} \qquad \text{and} \qquad y = \ln(t) - \frac{t^3}{3} \qquad \text{for } t > 0.$$

Find the length of the curve C between the points $\left(\frac{4}{3}, -\frac{1}{3}\right)$ and $\left(\frac{32}{3}, 2\ln(2) - \frac{64}{3}\right)$.



Answer:

Note that

$$\frac{dx}{dt} = 2\sqrt{t} \qquad \text{and} \qquad \frac{dy}{dt} = t^{-1} - t^2,$$

hence

$$\Big(\frac{dx}{dt}\Big)^2 = 4t \qquad \text{and} \qquad big(\frac{dy}{dt}\Big)^2 = t^{-2} - 2t + t^4.$$

Also, the endpoints correspond to $t = 1$ and $t = 4$. Then the arc length is

$$\int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^4 \sqrt{4t + t^{-2} - 2t + t^4} dt$$

$$\begin{aligned}
&= \int_1^4 \sqrt{t^{-2} + 2t + t^4}dt \\
&= \int_1^4 \sqrt{(t^{-1} + t^2)^2}dt \\
&= \int_1^4 (t^{-1} + t^2)dt \\
&= (\ln|t| + \frac{1}{3}t^3)\ |_1^4 \\
&= \Big(\ln(4) + \frac{4^3}{3}\Big) - \Big(\ln(1) + \frac{1}{3}\Big) \\
&= 2\ln(2) + \frac{63}{3} = 2\ln(2) + 21
\end{aligned}$$

2. (20 points) Determine if the following sequences are convergent or divergent. If it is convergent, give its limit.

(a)

$$\{\cos(n^2\pi) \colon n \geq 0\}.$$

Answer:

Since

$$\cos(n^2\pi) = \begin{cases} -1 & n \text{ is odd.} \\ 1 & n \text{ is even,} \end{cases}$$

we know that this sequence is divergent.

(b)

$$\left\{ \frac{1 - \cos(1/n)}{\sin^2(1/n)} \colon n \geq 1 \right\}.$$

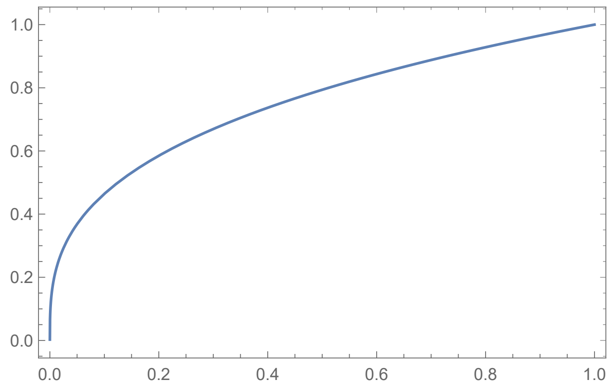
Answer:

We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1 - \cos(1/n)}{\sin^2(1/n)} &= \lim_{n\to\infty} \frac{1 - \cos(1/n)}{1 - \cos^2(1/n)} \\
&= \lim_{n\to\infty} \frac{1}{1 + \cos(1/n)} \\
&= 1/2
\end{aligned}$$

Therefore, this sequence it convergent with limit 1/2.

3. (20 points) (a) Find the area of the surface obtained by rotating the curve $y = \sqrt[3]{x}$ about the y-axis for $0 \leq y \leq 1$.



Answer:

The curve being rotated can be described as $x = y^3$, $0 \leq y \leq 1$.

The surface area is

$$S = \int_0^1 2\pi x ds,$$

where

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

We have

$$\frac{dx}{dy} = 3y^2,$$

so

$$\sqrt{1 + (\frac{dx}{dy})^2} = \sqrt{1 + 9y^4}.$$

Hence

$$S = \int_0^1 2\pi y^3 \sqrt{1 + 9y^4} dy$$

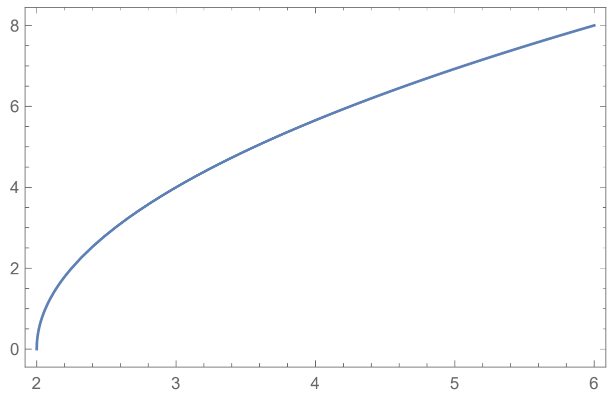
.

Make the substitution $u = 1 + 9y^4$. Then $du = 36y^3 dy$. When $y = 0$, $u = 1$, and when $y = 1$, $u = 10$.

So

$$S = \int_1^{10} 2\pi (1/36) u^{1/2} du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{\pi}{27}\left[10^{3/2} - 1\right].$$

(b) The curve $16x = y^2 + 32$ is rotated about the x-axis from $x = 2$ to $x = 6$. Find the area S of the resulting surface.



Answer:

$$S = \int_2^6 2\pi y\sqrt{1 + (dy/dx)^2}dx$$

.

$$16x = y^2 + 32,$$

so $16dx = 2ydy$, $8dx = ydy$,

$\frac{dy}{dx} = 8/y$, $1 + (\frac{dy}{dx})^2 = 1 + 64/y^2 = \frac{y^2+64}{y^2}$,

$\sqrt{1 + (dy/dx)^2} = \sqrt{y^2 + 64}/y$.

The curve is $16x = y^2 + 32$, so $y^2 + 64 = 16x + 32 = 16(x + 2)$. Hence

$\sqrt{1 + (dy/dx)^2} = \sqrt{16(x + 2)}/y = 4\sqrt{x + 2}/y$.

Hence

$$S = 2\pi \int_2^6 y \cdot 4 \cdot \sqrt{x + 2}/y dx = 8\pi \int_2^6 \sqrt{x + 2}dx.$$

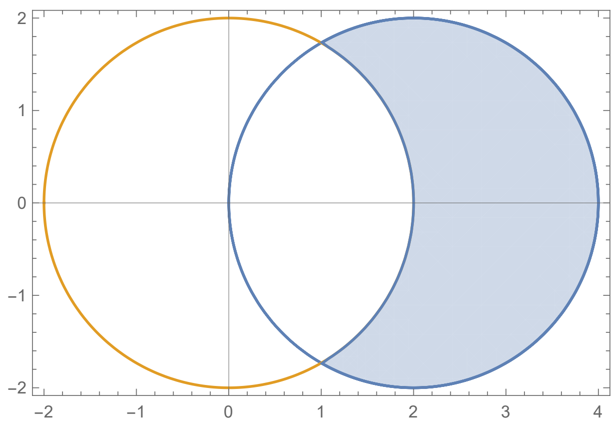
Make the substitution $u = x + 2$. Then $du = dx$. When $x = 2$, $u = 4$, and when $x = 6$, $u = 8$.

So

$$\begin{aligned} S &= 8\pi \int_4^8 u^{1/2} du \\ &= 8\pi(2/3)[u^{3/2}]_4^8 \\ &= (16/3)\pi(8^{3/2} - 4^{3/2}) \\ &= (16/3)\pi(2^{9/2} - 2^{6/2}) \\ &= (16/3)\pi(16\sqrt{2} - 8). \end{aligned}$$

4. (20 points)

(a) Find the area inside the polar curve $r = 4\cos(\theta)$ and outside the polar curve $r = 2$.



Answer:

The curves intersect when $4\cos(\theta) = 2$ or $\cos(\theta) = \frac{1}{2}$. We know $\cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$ so the points of intersection are $\theta_1 = -\frac{\pi}{3}$ and $\theta_2 = \frac{\pi}{3}$. Thus,

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}[(4\cos\theta)^2 - (2)^2]\, d\theta = 2\int_0^{\pi/3} \frac{1}{2}[(4\cos\theta)^2 - (2)^2]\, d\theta \\ &= \int_0^{\pi/3} [16\cos^2\theta - 4]\, d\theta = \int_0^{\pi/3} [8(1+\cos 2\theta) - 4]\, d\theta \\ &= \int_0^{\pi/3} [4 + 8\cos 2\theta)]\, d\theta = [4\theta + 4\sin 2\theta]_0^{\pi/3} = \frac{4\pi}{3} + 2\sqrt{3}. \end{aligned}$$

(b) Find the arc length of the boundary of the region inside the polar curve $r = 4\cos(\theta)$ and outside the polar curve $r = 2$ (the region from part (a)).

Answer:

The points of intersection are the same as in part (a). Thus,

$$AL = \int_{-\pi/3}^{\pi/3} \sqrt{2^2 + 0^2}\, d\theta + \int_{-\pi/3}^{\pi/3} \sqrt{(4\cos\theta)^2 + (4\sin\theta)^2}\, d\theta = [2\theta + 4\theta]_{-\pi/3}^{\pi/3} = 4\pi.$$

5. (20 points)

(a) Let $a > 0$ be a fixed positive number. Compute the definite integral

$$\int_{a\sqrt{2}}^{2a} \frac{dx}{\sqrt{x^2 - a^2}}.$$

Answer:

We set $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2 - a^2} = a\tan\theta$. Also when $x = a\sqrt{2}$, $\sec\theta = \sqrt{2}$ so that $\theta = \pi/4$, and when $x = 2a$, $\sec\theta = 2$ so that $\theta = \pi/3$. The definite integral becomes

$$\int_{a\sqrt{2}}^{2a} \frac{dx}{\sqrt{x^2 - a^2}} = \int_{\pi/4}^{\pi/3} \sec\theta\, d\theta.$$

Now let $u = \sec\theta + \tan\theta$ so $\sec\theta d\theta = du/u$. When $\theta = \pi/4, u = 1 + \sqrt{2}$ and when $\theta = \pi/3$, $u = 2 + \sqrt{3}$, so the definite integral becomes

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \sec\theta\, d\theta &= \int_{1+\sqrt{2}}^{2+\sqrt{3}} \frac{du}{u} \\
&= \ln u\Big|_{1+\sqrt{2}}^{2+\sqrt{3}} \\
&= \ln(2+\sqrt{3}) - \ln(1+\sqrt{2}) = \ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right) \\
&= \ln\left(\frac{(2+\sqrt{3})(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right) = \ln\left(\frac{(\sqrt{3}+2)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right) \\
&= \ln\left(\sqrt{6} + 2\sqrt{2} - \sqrt{3} - 2\right).
\end{aligned}$$

(b) Find the integral

$$\int \frac{1}{\sqrt{x^2 + 6x + 10}}\, dx.$$

Answer:

We complete the square $x^2 + 6x + 10 = (x+3)^2 + 1$. Then consider the substitution $u = x + 3$, so that $du = dx$, and we find

$$\int \frac{1}{\sqrt{x^2+6x+10}}\,dx = \int \frac{1}{\sqrt{(x+3)^2+1}}\,dx = \int \frac{1}{\sqrt{u^2+1}}\,du.$$

Next we use a trig substitution. Let $u = \tan\theta$. Then $du = \sec^2\theta d\theta$ and $\sqrt{u^2+1} = 1\sec\theta$, so that

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+6x+10}}\,dx = \int \frac{1}{\sqrt{u^2+1}}\,du = \int \frac{1}{\sec\theta}\sec^2\theta\,d\theta = \int \sec\theta\,d\theta\\ = \ln|\sec\theta + \tan\theta| + C = \ln|\sqrt{u^2+1} + u| + C\\ = \ln|\sqrt{(x+3)^2+1} + x + 3| + C\\ = \ln|x + 3 + \sqrt{x^2+6x+10}| + C.\end{aligned}$$

Scratch paper