Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 18, 2022

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution (known in Doug's section as *the rabbit trick.*)) for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad \text{for } x > 0$$

$$\pi + \arctan(y/x) \text{for } x < 0$$

$$\pi/2 \text{for } x = 0 \text{ and } y > 0$$

$$3\pi/2 \text{for } x = 0 \text{ and } y < 0$$

$$\text{undefined} \text{for } (x, y) = (0, 0)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

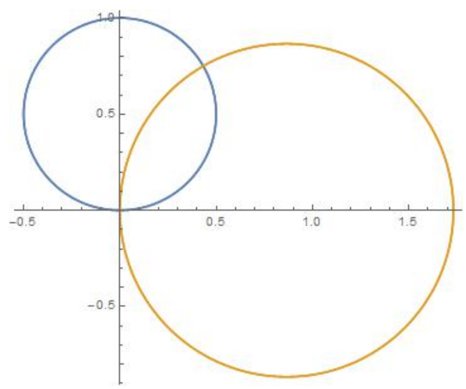
$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \leq t \leq b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. (20 points)

(a) Find the area of the region both inside the circle $r = \sin\theta$ and outside the circle $r = \sqrt{3}\cos\theta$ (both equations are in polar coordinates). The two circles are shown below. THEY INTERSECT AT THE ORIGIN AND THE POLAR POINT $(\theta, r) = (\pi/3, \sqrt{3}/2)$.



Answer:

Find the area of the region inside the first circle and outside the second by integrating:

$$\int_{\pi/3}^{\pi} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{\pi/3}^{\pi} \sin^2\theta\,d\theta = \tfrac{1}{4}\int_{\pi/3}^{\pi}\big(1-\cos 2\theta\big)\,d\theta = \frac{\pi}{6} + \frac{\sqrt{3}}{16}$$

and subtracting:

$$\int_{\pi/3}^{\pi/2} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{\pi/3}^{\pi/2} 3\cos^2\theta\,d\theta = \tfrac{3}{4}\int_{\pi/3}^{\pi/2}\big(1+\cos 2\theta\big)\,d\theta = \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

So the area of the region is $\dfrac{\pi}{24} + \dfrac{\sqrt{3}}{4} \approx 0.563912$.

(b) Compute the equation (in Cartesian coordinates x, y) of the tangent line to the circle $r = \sin\theta$ at the points where it intersects the circle $r = \sqrt{3}\cos\theta$

Answer:

Convert the curve to Cartesian coordinates:

$$x = r\cos\theta = \sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$$
$$y = r\sin\theta = \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Thus:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan(2\theta)$$

So at the points of intersection $\theta = 0$ and $\theta = \pi/3$:

$$\frac{dy}{dx} = \tan(0) = 0 \qquad\qquad \frac{dy}{dx} = \tan(2\pi/3) = -\sqrt{3}$$

Since $(r, \theta) = (\sqrt{3}/2, \pi/2)$ corresponds to $(x, y) = (\sqrt{3}/4, 3/4)$ (scale the 1-2-$\sqrt{3}$ triangle by $\sqrt{3}/4$), the equations of the tangents at those points are:

$$y = 0 \qquad\qquad y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \tfrac{\sqrt{3}}{4}\right)$$

2. (20 points)

(a) (10 points) Find

$$\int_a^\infty \frac{dx}{(x+4)^{3/2}} \qquad \text{for } a \geq 0.$$

Answer:

Use the substitution $u = x + 4$, making $dx = du$. Then we have

$$\int_a^\infty \frac{dx}{(x+4)^{3/2}} = \int_{a+4}^\infty \frac{du}{u^{3/2}} = \left.\frac{u^{-1/2}}{-1/2}\right|_{a+4}^\infty = \left.\frac{-2}{\sqrt{u}}\right|_{a+4}^\infty = \frac{2}{\sqrt{a+4}}$$

(b) (10 points) Find

$$\int_4^\infty e^{-x/4} dx$$

Answer:

Let $u = x/4$, so $dx = 4du$. Then we have

$$\int_4^\infty e^{-x/4} dx = 4\int_1^\infty e^{-u} du = -4e^{-u}\Big|_1^\infty = \frac{4}{e}.$$

3. (20 points)

Determine if the following sequences are convergent or divergent and explain why. If it is convergent, give its limit.

(a) (10 points)

$$\left\{ \frac{3n\sin(n)}{2n^3+5} \;\middle|\; n \geq 0 \right\}.$$

Answer:

We use the Squeeze Theorem. Because $-1 \leq \sin(n) \leq 1$, the sequence is bounded below by $\left\{ -\frac{3n}{2n^3+5} \;\middle|\; n \geq 0 \right\}$ and above by $\left\{ \frac{3n}{2n^3+5} \;\middle|\; n \geq 0 \right\}$. The limit of both of these sequences is zero, so the limit of $\left\{ \frac{3n\sin(n)}{2n^3+5} \;\middle|\; n \geq 0 \right\}$ must also be zero by the Squeeze Theorem.

(b) (10 points)

$$\left\{ \frac{n}{\ln(n)} \;\middle|\; n \geq 2 \right\}$$

Answer:

Let

$$f(x) = \frac{x}{\ln(x)}.$$

Now as $x \to \infty$, L'Hopital's rule applies. We have

$$\lim_{x\to\infty} \frac{x}{\ln(x)} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Because this limit diverges, so does the limit of the sequence.

4. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the following curve around the x-axis:

$$y = \sqrt{1+e^x} \quad 0 \leq x \leq 1$$

Answer:

$$
\begin{aligned}
A &= 2\pi \int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\pi \int_0^\pi \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\,dx \\
&= 2\pi \int_0^1 \sqrt{e^{2x}+4e^x+4}\,dx \\
&= 2\pi \int_0^1 e^x+2\,dx \\
&= (2\pi)\left[e^x+2x\right]_0^1 \\
&= 2\pi(e+1)
\end{aligned}
$$

(b) Compute the area of surface of revolution obtained by rotating the following curve around the y-axis:

$$y=\frac{x^2}{2} \quad 0\le x\le 1$$

Answer:

$$
\begin{aligned}
A &= 2\pi \int_0^1 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\pi \int_0^1 x\sqrt{1+x^2}\,dx \\
&= \pi \int_1^2 \sqrt{u}\,du \\
&= \frac{2\pi}{3}\left[u^{3/2}\right]_1^2 \\
&= \frac{2\pi}{3}\left(2^{3/2}-1\right)
\end{aligned}
$$

5. (20 points)

Find the arc length of the curve described by the parametric equations

$$x=\cos(t^2), \quad y=\sin(t^2)$$

between the points with Cartesian coordinates $(1,0)$ and $(-1,0)$.

Answer:

The points on the curve with Cartesian coordinates $(1, 0)$ and $(-1, 0)$ are the points when the parameter t equals 0 and $\sqrt{\pi}$ respectively.

We have that

$$\begin{aligned} dx/dt &= -\sin(t^2)2t, \quad dy/dt = \cos(t^2)2t \\ (dx/dt)^2 &= 4t^2\sin^2(t^2), \quad (dy/dt)^2 = 4t^2\cos^2(t^2) \\ (dx/dt)^2 + (dy/dt)^2 &= 4t^2(\sin^2(t^2) + \cos^2(t^2) = 4t^2 \\ \sqrt{(dx/dt)^2 + (dy/dt)^2} &= 2t \end{aligned}$$

So the arc length L is

$$L = \int_0^{\sqrt{\pi}} \sqrt{(dx/dt)^2 + (dy/dt)^2}dt = \int_0^{\sqrt{\pi}} 2tdt = t^2\Big|_0^{\sqrt{\pi}} = \pi.$$

Scratch paper

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