# Math 162: Calculus IIA

Second Midterm Exam, Evening Edition

November 5, 2020

**NAME (please print legibly):** ______________________________

**Your University ID Number:** ______________________________

**Your University email** ______________________________

**Write the name of your proctor here.**

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## Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

**Signature:** ______________________________

**Instructions**

- **You may not consult the textbook, your notes, the internet, your classmates, friends or any other external source of information. YOUR WEBCAM MUST BE ON AT ALL TIMES.**
- **If you have access to a printer, you may print this exam and write your answers in the spaces provided. Otherwise, write the answers to each problem on a separate sheet of paper. YOU MUST ALSO WRITE AND SIGN THE PLEDGE OF HONESTY AND GIVE ALL OF THE INFORMATION REQUESTED ABOVE.**
- **Show your work and justify your answers. You may use the formulas on the next page. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **You must finish work on this exam by 9:15, and then scan and upload it to Gradescope as previously instructed by 9:30. Exams received after that time will be subject to a penalty.**

Trig formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

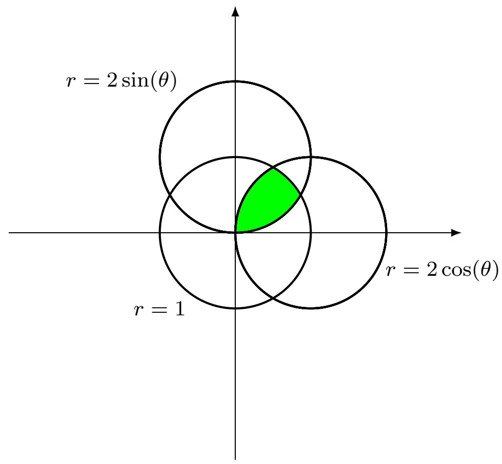
Polar coordinate formulas:

- Area:
$$\frac{1}{2}\int r^2 d\theta$$
- Arc length:
$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Parametric equation formulas:

- Newton's notation: $\dot{x} = dx/dt \qquad \dot{y} = dy/dt$
- Slope of tangent line: $dy/dx = \dot{y}/\dot{x}$.
- Second derivative
$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{\dot{x}}.$$
Curve is concave up/down when this is positive/negative.
- Arc length:
$$\int \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

**1. (25 points)** Compute the area inside the polar curves $r = 1, r = \cos(\theta)$ and $r = \sin(\theta)$.




**Solution:** The region occurs in the first quadrant, which is the interval $\theta$ in $[0, \frac{\pi}{2}]$. The first $\theta$ where two of the functions intersect is where $1 = 2\sin(\theta)$, or $\theta = \frac{\pi}{6}$. For $\theta$ in $[0, \frac{\pi}{6}]$, the region is bound by the function $r = 2\sin(\theta)$. The next $\theta$ where 2 of the functions intersect is where $1 = 2\cos(\theta)$, or $\theta = \frac{\pi}{3}$. For $\theta$ in $[\frac{\pi}{6}, \frac{\pi}{3}]$, the region is bound by the function $r = 1$. For $\theta$ in $[\frac{\pi}{3}, \frac{\pi}{2}]$, the region is bound by the function $r = 2\cos(\theta)$.

So the area of this region is

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\sin(\theta))^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos(\theta))^2 d\theta \\
=&\int_0^{\frac{\pi}{6}} 1 - \cos(2\theta) d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 + \cos(2\theta) d\theta \\
=&\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{6}} + \left[\frac{\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \left[\theta + \frac{\sin(2\theta)}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
=&\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\pi}{6} - \frac{\pi}{12} + \frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} \\
=&\frac{5\pi}{12} - \frac{\sqrt{3}}{2}.
\end{aligned}
$$

**2. (25 points)**

Let $C$ be the upper half of the circle centered at the origin with radius 6. The arc on $C$ between $(0, 6)$ and $(\sqrt{11}, 5)$ is rotated about the $x$-axis to produce a surface $S$.

(a) (15 points) Use

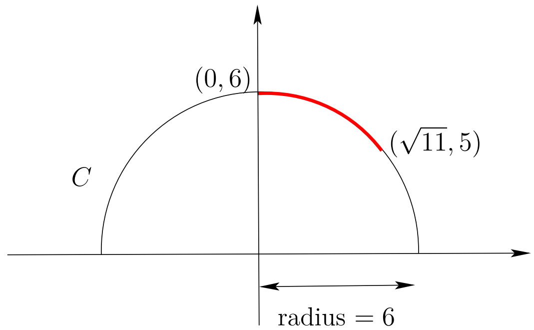



Figure 1:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

to find the surface area.

ANSWER:

(b) (10 points) Consider the same surface as in part (a). This time, use

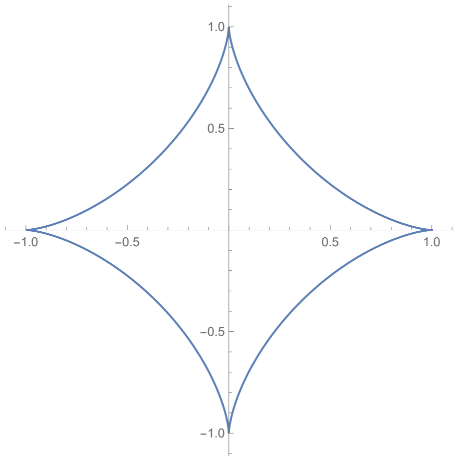
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

to find the surface area.

ANSWER:

**3. (25 points)**

Consider the parametric curve (an astroid or 4 pointed hypocycloid) $x = \cos^3(t)$, $y = \sin^3(t)$, $t \in [0, 2\pi]$.




(a) (9 points) At what points is the tangent horizontal or vertical?

ANSWER:

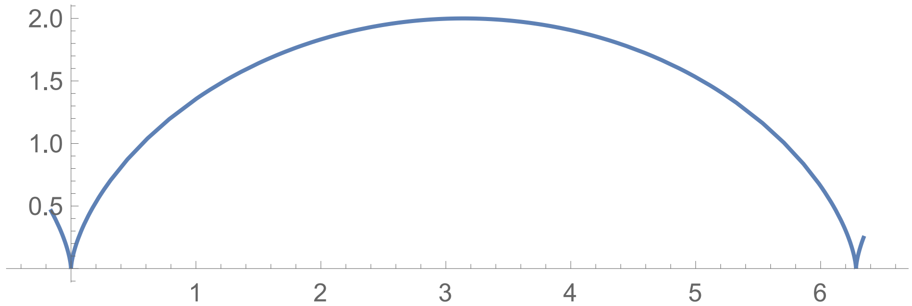
(b) (8 points) At what points does it have slope $\pm 1$?

ANSWER:

(c) (8 points) Find the equation of the form $y = mx + b$ for the tangent at $t = \frac{\pi}{4}$.

ANSWER:

**4. (25 points)** Find the arc length of the cycloid $x = r(t - \sin(t))$ and $y = r(1 - \cos(t))$, for $0 \leq t \leq 2\pi$.




ANSWER: