Math 162: Calculus IIA

Second Midterm Exam, Evening Edition ANSWERS

November 17, 2020

Trig formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \frac{1 + \cos(2x)}{2}$
- $\sin^2(x) = \frac{1 - \cos(2x)}{2}$

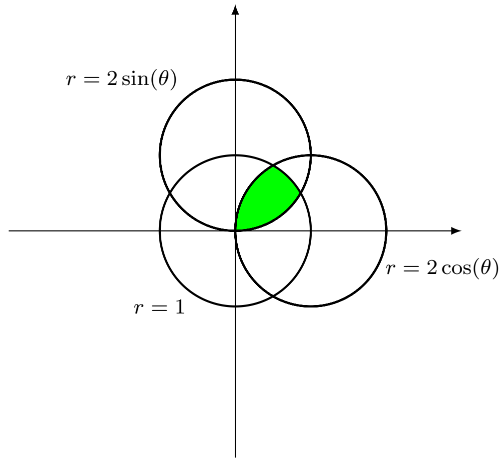
Polar coordinate formulas:

- Area:
$$\frac{1}{2}\int r^2 d\theta$$
- Arc length:
$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Parametric equation formulas:

- Newton's notation: $\dot{x} = dx/dt \qquad \dot{y} = dy/dt$
- Slope of tangent line: $dy/dx = \dot{y}/\dot{x}$.
- Second derivative
$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{\dot{x}}.$$
Curve is concave up/down when this is positive/negative.
- Arc length:
$$\int \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

1. (25 points) Compute the area inside the polar curves $r = 1, r = \cos(\theta)$ and $r = \sin(\theta)$.



Solution: The region occurs in the first quadrant, which is the interval θ in $[0, \frac{\pi}{2}]$. The first θ where two of the functions intersect is where $1 = 2\sin(\theta)$, or $\theta = \frac{\pi}{6}$. For θ in $[0, \frac{\pi}{6}]$, the region is bound by the function $r = 2\sin(\theta)$. The next θ where 2 of the functions intersect is where $1 = 2\cos(\theta)$, or $\theta = \frac{\pi}{3}$. For θ in $[\frac{\pi}{6}, \frac{\pi}{3}]$, the region is bound by the function $r = 1$. For θ in $[\frac{\pi}{3}, \frac{\pi}{2}]$, the region is bound by the function $r = 2\cos(\theta)$.

So the area of this region is

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\sin(\theta))^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos(\theta))^2 d\theta \\
=&\int_0^{\frac{\pi}{6}} 1 - \cos(2\theta) d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1 + \cos(2\theta) d\theta \\
=&\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{6}} + \left[\frac{\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \left[\theta + \frac{\sin(2\theta)}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
=&\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\pi}{6} - \frac{\pi}{12} + \frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4} \\
=&\frac{5\pi}{12} - \frac{\sqrt{3}}{2}.
\end{aligned}
$$

2. (25 points)

Let C be the upper half of the circle centered at the origin with radius 6. The arc on C between $(0, 6)$ and $(\sqrt{11}, 5)$ is rotated about the x-axis to produce a surface S.

(a) (15 points) Use

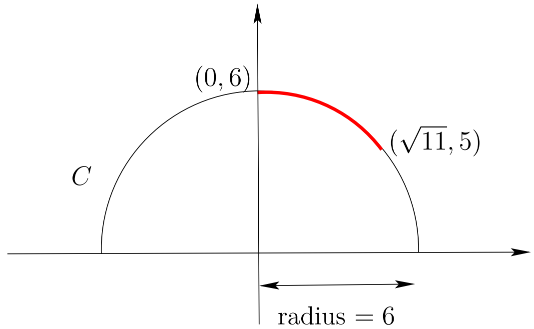


Figure 1:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

to find the surface area.

Answer:

The formula for C is $y = \sqrt{36 - x^2}$. Then

$$\begin{aligned}
S &= \int_0^{\sqrt{11}} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{\sqrt{11}} 2\pi \sqrt{36 - x^2} \sqrt{1 + \left(-\frac{x}{\sqrt{36 - x^2}}\right)^2}\, dx \\
&= \int_0^{\sqrt{11}} 2\pi \sqrt{36 - x^2} \sqrt{\frac{36}{36 - x^2}}\, dx \\
&= \int_0^{\sqrt{11}} 12\pi\, dx \\
&= 12\sqrt{11}\pi
\end{aligned}$$

(b) (10 points) Consider the same surface as in part (a). This time, use

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

to find the surface area.

Answer:

Since the arc lies on the right half of C, we use the formula $x = \sqrt{36 - y^2}$, so that

$$\begin{aligned} S &= \int_5^6 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= \int_5^6 2\pi y \sqrt{1 + \left(-\frac{y}{\sqrt{36 - y^2}}\right)^2}\, dy \\ &= \int_5^6 2\pi y \sqrt{\frac{36}{36 - y^2}}\, dy \\ &= \int_5^6 12\pi y \, \frac{1}{\sqrt{36 - y^2}}\, dy \end{aligned}$$

Make the substitution $u = 36 - y^2$, so that

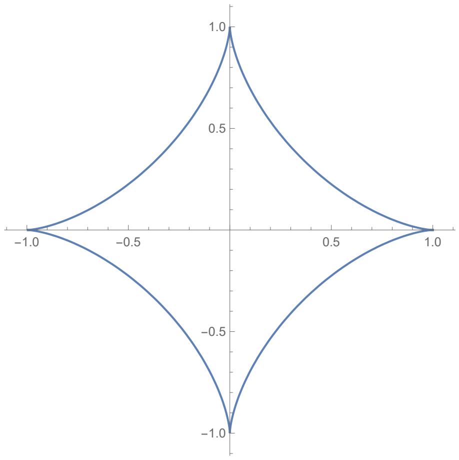
$$-du = 2y\, dy, \quad \text{when} \quad y = 5, u = 11, \quad \text{when} \quad y = 6, u = 0.$$

The integral becomes

$$\begin{aligned} S &= -\int_{11}^{0} 6\pi \, \frac{1}{\sqrt{u}}\, du \\ &= \int_0^{11} 6\pi \, \frac{1}{\sqrt{u}}\, du \\ &= 12\pi \sqrt{u} \,\Big|_0^{11} \\ &= 12\sqrt{11}\pi \end{aligned}$$

3. (25 points)

Consider the parametric curve (an astroid or 4 pointed hypocycloid) $x = \cos^3(t)$, $y = \sin^3(t)$, $t \in [0, 2\pi]$.



(a) (9 points) At what points is the tangent horizontal or vertical?

Answer:

We have

$$\begin{aligned} \frac{dx}{dt} &= -3\sin t\cos^2 t \\ \frac{dy}{dt} &= 3\cos t\sin^2 t \\ \frac{dy}{dx} &= -\frac{\cos t\sin^2 t}{\sin t\cos^2 t} = -\tan t \end{aligned}$$

The tangent line is horizontal when this derivative is 0, namely when $t = 0$ and $t = \pi$. The tangent line is vertical when the derivative is undefined, namely at $t = \pi/2$ and $t = 3\pi/2$. (b) (8 points) At what points does it have slope ± 1?

Answer:

The slope of the tangent line is ± 1 when $t = \pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$.

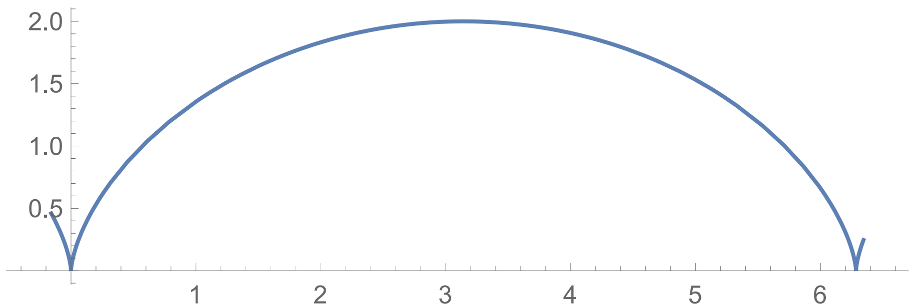
(c) (8 points) Find the equation of the form $y = mx + b$ for the tangent at $t = \frac{\pi}{4}$.

Answer:

At $t = \pi/4$ we have $x = y = \sqrt{2}/4$ and $dy/dx = -1$, so the equation for the tangent line is

$$\begin{aligned} \frac{y - \sqrt{2}/4}{x - \sqrt{2}/4} &= -1 \\ y - \sqrt{2}/4 &= -(x - \sqrt{2}/4) \\ &= -x + \sqrt{2}/4 \\ y &= -x + \sqrt{2}/2. \end{aligned}$$

4. (25 points) Find the arc length of the cycloid $x = r(t - \sin(t))$ and $y = r(1 - \cos(t))$, for $0 \le t \le 2\pi$.



Answer:

We have

$$\begin{aligned}
\frac{dx}{dt} &= r(1 - \cos t) \\
\frac{dy}{dt} &= r\sin t) \\
\left(\frac{ds}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
&= r^2\left((1 - \cos t)^2 + \sin^2 t\right) \\
&= r^2\left(1 - 2\cos t + \cos^2 t + \sin^2 t\right) \\
&= 2r^2\left(1 - \cos t\right) \\
\frac{ds}{dt} &= 2r\sqrt{\frac{1 - \cos t}{2}} \\
&= 2r\sin(t/2),
\end{aligned}$$

so the arc length is

$$\begin{aligned}
s &= \int 02\pi 2r\sin(t/2)dt \\
&= 4r\int 0\pi \sin u du \qquad \text{where } u = t/2 \\
&= -4r\cos u\big|_0^\pi \\
&= 8r
\end{aligned}$$