Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 9, 2011

1. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the curve $y = \sqrt{4 - x^2}$ around the x-axis.

(b) Do the same for the curve $y = 1 - |x|$, $-1 \le x \le 1$.

Solution:

(a)

$$\begin{aligned} A &= 2\pi \int_{-2}^{2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_{-2}^{2} \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx \\ &= 2\pi \int_{-2}^{2} 2\, dx = 16\pi \end{aligned}$$

(b)

$$\begin{aligned} A &= 2\pi \int_{-1}^{0} (1 + x)\sqrt{1 + 1}\, dx + 2\pi \int_{0}^{1} (1 - x)\sqrt{1 + 1}\, dx \\ &= 2\pi\sqrt{2} \left(\left[x + \frac{x^2}{2} \right]_{-1}^{0} + \left[x - \frac{x^2}{2} \right]_{0}^{1} \right) \\ &= 2\pi\sqrt{2} \left(\frac{1}{2} + \frac{1}{2} \right) = 2\pi\sqrt{2} \end{aligned}$$

2. (20 points)

Consider the parametric curve

$x = \cos(t)$, $y = \sin(2t)$, $t \in [0, 2\pi]$.

(a) At what points is the tangent horizontal or vertical?

(b) The curve passes through the origin twice. What are the slopes of the two tangent lines to the curve at the origin?

(c) Find the equation of the form $y = mx + b$ for the tangent at $t = \frac{\pi}{6}$.

Solution: (a) We have

$$\frac{dy}{dx} = -\frac{2\cos 2t}{\sin t}$$

The tangent line is horizontal when this derivative is 0, namely when $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. The corresponding four Cartesian points are $(\pm\sqrt{2}/2, \pm 1)$.

The tangent line is vertical when the derivative is undefined, namely at $t = 0$ and $t = \pi$. The corresponding two Cartesian points are $(\pm 1, 0)$.

Solution: (b) The curve passes through the origin at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. The two slopes are 2 and -2, respectively.

Solution: (c) At $t = \pi/6$ we have $x = y = \frac{\sqrt{3}}{2}$ and $dy/dx = -2$, so the equation for the tangent line is

$$\begin{aligned} \frac{y - \frac{\sqrt{3}}{2}}{x - \frac{\sqrt{3}}{2}} &= -2 \\ y - \frac{\sqrt{3}}{2} &= -2\left(x - \frac{\sqrt{3}}{2}\right) \\ &= -2x + \sqrt{3} \\ y &= -2x + \frac{3\sqrt{3}}{2}. \end{aligned}$$

3. (20 points)

Find the arc-length of the parametric curve

$$x = 3\cos t - \cos 3t\,,\ y = 3\sin t - \sin 3t\,,\ 0 \le t \le \pi\,.$$

Solution: We have

$$dx/dt = -3(\sin t - \sin 3t) \qquad \text{and} \qquad dy/dt = 3(\cos t - \cos 3t).$$

Therefore

$$\begin{aligned}
(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\
&= 9(\sin t - \sin 3t)^2 + 9(\cos t - \cos 3t)^2 \\
&= 9(\sin^2 t - 2\sin t \sin 3t + \sin^2 3t + \cos^2 t - 2\cos t\cos 3t + \cos^2 3t) \\
&= 9(2 - 2\cos 2t) \\
&\qquad \text{since } \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= 36\frac{1 - \cos 2t}{2} \\
&= 36\sin^2 t,
\end{aligned}$$

so

$$\frac{ds}{dt} = 6|\sin t|.$$

By the arc length formula, we have

$$\begin{aligned}
L &= \int_0^\pi ds \\
&= \int_0^\pi 6|\sin t| dt \\
&= 6\int_0^\pi \sin t dt \\
&= -6\cos t\Big|_0^\pi \\
&= 12.
\end{aligned}$$

4. (20 points)

(a) Calculate the arc-length of the curve $r = \cos^2(\theta/2)$.

(b) Calculate the area enclosed by the curve $r^2 = \sin(2\theta)$.

Solution: (a) As $\cos^2(\theta/2) = \frac{1+\cos(\theta)}{2}$ has a period of 2π, we just need to find the arc length of this curve for $\theta \in [-\pi, \pi]$ where $\cos(\theta/2)$ is positive. Therefore

$$\begin{aligned} L &= \int_{-\pi}^{\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta \\ &= \int_{-\pi}^{\pi} \sqrt{\cos^4(\theta/2) + \cos^2(\theta/2)\sin^2(\theta/2)} d\theta \\ &= \int_{-\pi}^{\pi} \sqrt{\cos^2(\theta/2)} d\theta \\ &= \int_{-\pi}^{\pi} \cos(\theta/2) d\theta \\ &= 2\sin(\theta/2)\Big|_{-\pi}^{\pi} \\ &= 4 \end{aligned}$$

Solution: (b) Firstly we need to identity the domain of θ. As $\sin(2\theta) = r^2 \geq 0$, $2\theta \in [2k\pi, 2k\pi + \pi]$ for any integer k. Therefore $\theta \in [k\pi, k\pi + \pi/2]$ for any integer k. Due to the periodicity, we just need to consider the area enclosed by the curve when $\theta \in [0, \pi/2] \cup [\pi, 3\pi/2]$. Thus

$$\begin{aligned} A &= \frac{1}{2} \int r^2 d\theta \\ &= \frac{1}{2} \left(\int_0^{\pi/2} \sin 2\theta d\theta + \int_{\pi}^{3\pi/2} \sin 2\theta d\theta \right) \\ &= \frac{1}{2}(-\frac{\cos 2\theta}{2})\Big|_0^{\pi/2} + \frac{1}{2}(-\frac{\cos 2\theta}{2})\Big|_{\pi}^{3\pi/2} \\ &= 1 \end{aligned}$$

5. (20 points)

(a) (5 points) Does the sequence $\{a_n : n \geq 1\}$ with $a_n = 1/\sqrt{n}$ converge? Why or why not?

(b) (5 points) Use L'Hospital's Rule to show that for $k > 0$,

$$\lim_{x\to\infty} x^k \, \mathrm{e}^{-x} = k \lim_{x\to\infty} x^{k-1} \, \mathrm{e}^{-x} \, .$$

(c) (5 points) Let $a_n = n^4 \, \mathrm{e}^{-n}$. Show that the sequence $\{a_n : n \geq 1\}$ converges. What is the limit?

(d) (5 points) Does the sequence $\{b_n : n \geq 1\}$ with $b_n = \sin(\frac{n\pi}{2})(-\frac{1}{3})^n$ converge? Why or why not?

Solution:

(a) Since $\lim_{n\to\infty}\sqrt{n} = \infty$, $\lim_{n\to\infty} b_n = 0$ and the sequence converges to 0.

(b) We have

$$\begin{aligned} \lim_{x\to\infty} x^k \, \mathrm{e}^{-x} &= \lim_{x\to\infty} \frac{x^k}{\mathrm{e}^x} \\ &= \lim_{x\to\infty} \frac{kx^{k-1}}{x \, \mathrm{e}^x} \\ &= k \lim_{x\to\infty} \frac{x^{k-1}}{\mathrm{e}^x} \\ &= k \lim_{x\to\infty} x^{k-1} \, \mathrm{e}^{-x} \, . \end{aligned}$$

(c) From (b) we see that

$$\lim_{x\to\infty} x^4 \, \mathrm{e}^{-x} = 4 \lim_{x\to\infty} x^3 \, \mathrm{e}^{-x} = 12 \lim_{x\to\infty} x^2 \, \mathrm{e}^{-x} = 24 \lim_{x\to\infty} x \, \mathrm{e}^{-x} = 24 \lim_{x\to\infty} \mathrm{e}^{-x} = 0,$$

so the sequence converges to 0.

(d) Since $-1 \leq \sin(n\pi/2) \leq 1$, $-1/3^n \leq b_n \leq 1/3^n$, so $\lim_{n\to\infty} b_n = 0$.