Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 13, 2022

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

1. (20 points) If $a \neq 0$, evaluate

$$\int \cos^3(ax+b)\,dx$$

in terms of a and b.

Answer:

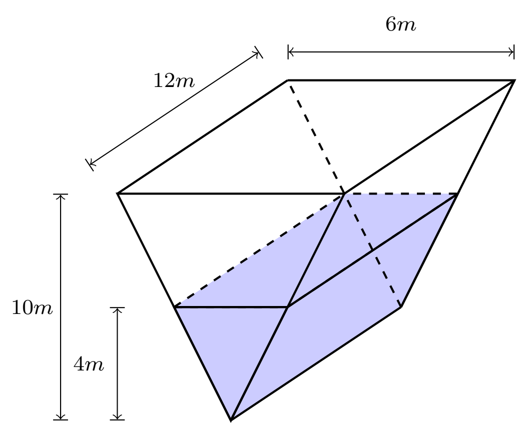
Let $u = \sin(ax+b)$.

Then $du = a\cos(ax+b)dx$,

$$\int \cos^3(ax+b)dx = \int (1-\sin^2(ax+b))\cos(ax+b)dx = (1/a)\int (1-u^2)du$$

$$= (1/a)(u - u^3/3) = (1/a)(\sin(ax+b) - (1/3)\sin^3(ax+b)).$$

2. (20 points) Consider a tank that is 10 meters tall with sides in the shapes of congruent isosceles triangles and a rectangular top that is 6 meters wide and 12 meters in length (see diagram below). The tank is filled with water to a depth of 4 meters. Find the work done pumping the water to a point 1 meter above the top of the tank. (The water density $\rho = 1000kg/m^3$ and the gravity constant is $g = 10m/s^2$). You do not need to simplify your answer.



Answer:

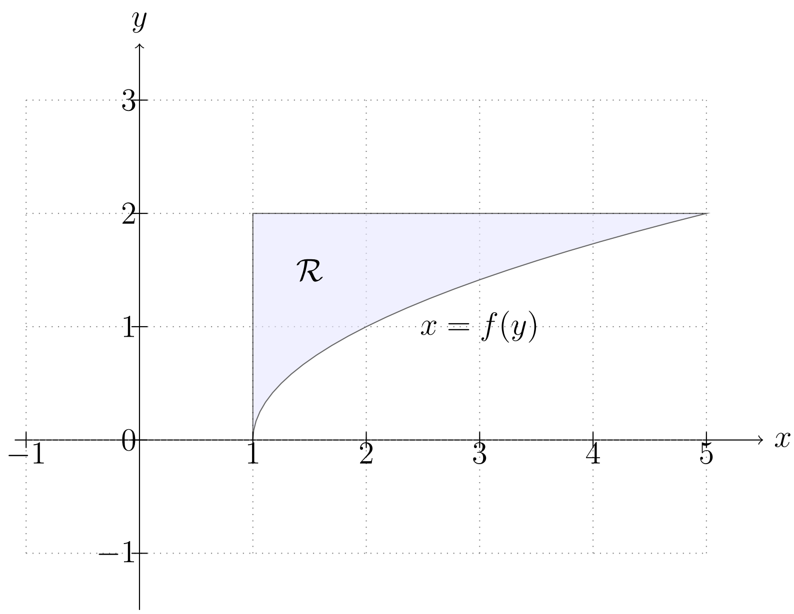
Put the bottom of this tank as the origin. The rectangular cross section at height y is must then have width $w(y) = 2\left(\frac{3}{10}y\right) = \frac{3}{5}y$. This can be seen either using similar triangle or noticing that the slop of the side of the tank with respect to y is $\frac{3}{10}$. It follows that we have

$$W = \int_0^4 (11-y)\cdot g\cdot\rho\cdot w(y)\cdot 12\cdot dy$$

$$= \frac{36\rho g}{5} \int_0^4 11y - y^2 dy$$

$$= \frac{36\rho g}{5} \left[\frac{11y^2}{2} - \frac{y^3}{3} \right]_0^4 = \frac{36}{5}(88 - \frac{64}{3})\rho g\ J.$$

3. (20 points) Set up formulas using integral expressions for the volumes of the following solids related to the region $\mathcal{R}$ where integration is performed with respect to the variable y.



(a) (4 points) The solid resulting from rotating $\mathcal{R}$ about the x-axis.

Answer:

We should use the shell method. radius $= y$ and height $= f(y) - 1$, and

$$\text{Volume} = 2\pi \int_0^2 y\,(f(y) - 1)\ dy.$$

(b) (4 points) The solid resulting from rotating $\mathcal{R}$ about the y-axis.

Answer:

We should use the washer method. $\mathrm{r}_{\text{inner}} = 1$ and $\mathrm{r}_{\text{outer}} = f(y)$, and

$$\text{Volume} = \pi \int_0^2 \left((f(y))^2 - 1^2\right)\ dy.$$

(c) (4 points) The solid resulting from rotating $\mathcal{R}$ about the axis $x = 5$.

Answer:

We should use the washer method. $\mathrm{r}_{\text{inner}} = 5 - f(y)$ and $\mathrm{r}_{\text{outer}} = 4$, and

$$\text{Volume} = \pi \int_0^2 \left(4^2 - (5 - f(y))^2\right)\, dy.$$

(d) (4 points) The solid resulting from rotating $\mathcal{R}$ about the axis $y = 3$.

Answer:

We should use the shell method. radius $= 3 - y$ and height $= f(y) - 1$, and

$$\text{Volume} = 2\pi \int_0^2 (3 - y)\,(f(y) - 1)\, dy.$$

(e) (4 points) The solid with base $\mathcal{R}$ where cross-sections parallel to the x-axis are squares.

Answer:

At the given location y, the corresponding side length of the square is $f(y) - 1$ and hence the cross sectional area is $A(y) = (f(y) - 1)^2$. So, we have

$$\text{Volume} = \int_0^2 (f(y) - 1)^2\, dy.$$

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for

$$\int x^n e^x\, dx \qquad \text{in terms of} \qquad \int x^{n-1} e^x\, dx$$

for any integer $n \geq 0$.

Answer:

Let $u = x^n$ and $dv = e^x\, dx$, so $du = nx^{n-1}\, dx$ and $v = e^x$. Then we have

$$\begin{aligned} \int x^n e^x\, dx = \int u\, dv &= uv - \int v\, du \\ &= x^n e^x - n \int x^{n-1} e^x\, dx. \end{aligned}$$

(b) (10 points) Use your formula repeatedly to find

$$\int x^3 e^x\, dx$$

Answer:

$$\begin{aligned}\int x^3 e^x\,dx &= x^3e^x - 3\int x^2 e^x\,dx \\ &= x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right) \\ &= (x^3-3x^2)e^x + 6\int xe^x\,dx \\ &= (x^3-3x^2)e^x + 6\left(xe^x - \int e^x\,dx\right) \\ &= (x^3-3x^2+6x-6)e^x + C.\end{aligned}$$

5. (20 points) (a) (10 points) Find the integral

$$\int \sqrt{x^2-8x+17}\,dx$$

Answer:

First, complete the square.

$$\begin{aligned}x^2-8x+17 = x^2-8x+16+17-16 \\ = (x-4)^2+1\end{aligned}$$

We then want the substitution $\tan(\theta) = x-4$, so $dx = \sec(\theta)d\theta$. Then

$$\begin{aligned}\int \sqrt{x^2-8x+17}d\theta &= \int \sec^3(\theta)d\theta \\ &= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)d\theta \\ &= \sec(\theta)\tan(\theta) - \int \frac{1-\cos^2(\theta)}{\cos^3(\theta)}d\theta \\ &= \sec(\theta)\tan(\theta) - \int \sec^3(\theta)d\theta + \int \sec(\theta)d\theta\end{aligned}$$

So,

$$2\int \sec^3(\theta)d\theta = \sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)| + C$$

$$\int \sec^3(\theta)d\theta = \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C$$

Thus,

$$\begin{aligned}\int \sqrt{x^2-8x+17}d\theta &= \int \sec^3(\theta)d\theta \\ &= \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C \\ &= \frac{1}{2}(x-4)\sqrt{(x-4)^2+1} + \frac{1}{2}\ln\left|\sqrt{(x-4)^2+1}+x-4\right| + C.\end{aligned}$$

(b) (10 points) Find the integral

$$\int \frac{x+1}{x^3+x}\,dx.$$

Answer:

We have

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Solving for A, B, and C we get

$$A+B=0 \qquad C=1 \qquad A=1.$$

So $B=-1$.

Our integral is then

$$\begin{aligned}\int \frac{x+1}{x^3+x}\,dx &= \int \frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \arctan(x) + K.\end{aligned}$$

We get the $\ln(x^2+1)$ term using a u-substitution with $u = x^2+1$.

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