Math 162: Calculus IIA

First Midterm Exam ANSWERS

September 30, 2021

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

1. (20 points)

Compute the following integral:

$$\int \cos^{2021}(x)\tan^3(x)dx$$

Answer:

$$\begin{aligned}
&\int \cos^{2021}(x)\tan^3(x)dx \\
=&\int \cos^{2021}(x)\frac{\sin^3(x)}{\cos^3(x)}dx \\
=&\int \cos^{2018}(x)\sin^3(x)dx \\
=&\int \cos^{2018}(x)\left(1-\cos^2(x)\right)\sin(x)dx \\
=&\int \cos^{2018}(x)\sin(x)dx - \cos^{2020}(x)\sin(x)dx.
\end{aligned}$$

Set $u=\cos(x)$, so $du=-\sin(x)dx$. So this becomes

$$\begin{aligned}
&\int -u^{2018}du + u^{2020}du \\
=&-\frac{u^{2019}}{2019}+\frac{u^{2021}}{2021}+C \\
=&-\frac{\cos^{2019}(x)}{2019}+\frac{\cos^{2021}(x)}{2021}+C.
\end{aligned}$$

2. (20 points)

The average value of a function $f(x)$ for $a \leq x \leq b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Find the average value of the function $f(x)=2+\sin x$ for $0 \leq x \leq 5\pi$. Hint: You may use the fact that

$$\int_0^\pi \sin x\,dx = 2 \qquad \text{and} \int_a^{a+2\pi} \sin x\,dx = 0 \qquad \text{for any number } a.$$

Answer:

The relevant integral is

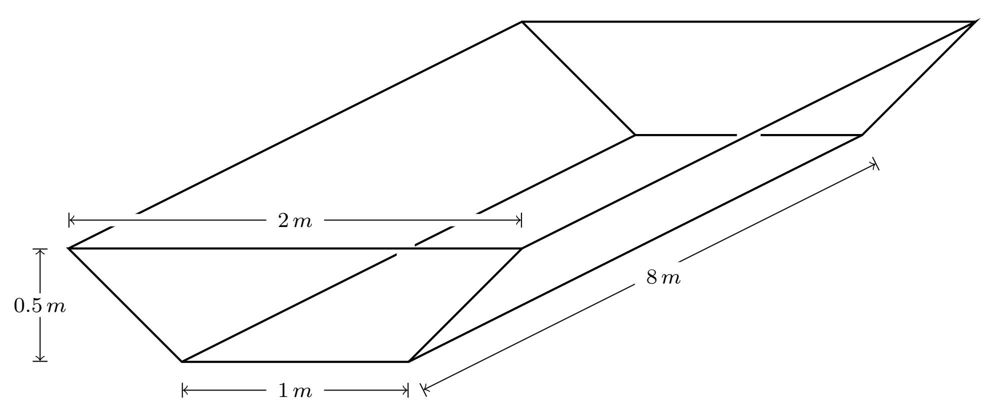
$$\begin{aligned}\int_0^{5\pi} (2+\sin x)\,dx &= 2\int_0^{5\pi} dx + \int_0^{5\pi} \sin x\,dx \\ &= 10\pi + \int_0^{5\pi} \sin x\,dx \\ \int_0^{5\pi} \sin x\,dx &= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{3\pi} \sin x\,dx + \int_{3\pi}^{5\pi} \sin x\,dx \\ &= \int_0^{\pi} \sin x\,dx \qquad \text{by the hint} \\ &= 2,\end{aligned}$$

so the average value is

$$\frac{1}{5\pi}\int_0^{5\pi} (2+\sin x)\,dx = \frac{2+10\pi}{5\pi} = \frac{2}{5\pi} + 2$$

3. (20 points)

A trough is 4 meters long and half a meter tall, with vertical cross-sections parallel to the ends in the shape of isosceles trapezoids which are 1 meter wide at the bottom and 2 meters wide at the top. The trough is full of water. Find work done pumping the water to the top of the trough. Assume that the water density is $\rho = 1000\,kg/m^3$ and the gravity constant is $g = 10\,m/s^2$.



Answer:

Put the bottom of the trough as the origin. Taking a horizontal "slice" of water at height y, with $0 \le y \le \frac{1}{2}$, we have that the slice is $1 + 2y$ meters across (using similar triangles), 4 meters long, and dy thick. Therefore, the volume of the slice is $4(1+2y)dy$ meters, so the mass of the slice is $4000(1+2y)dy$. The slice needs to be lifted $1/2-y$ meters to reach the top of the tank, so, incorporating the acceleration due to gravity, the work done to the slice is $4000(1+2y)(10)(1/2-y)dy\,J = 40000(1+2y)(1/2-y)dy\,J$. Therefore, the total work done is

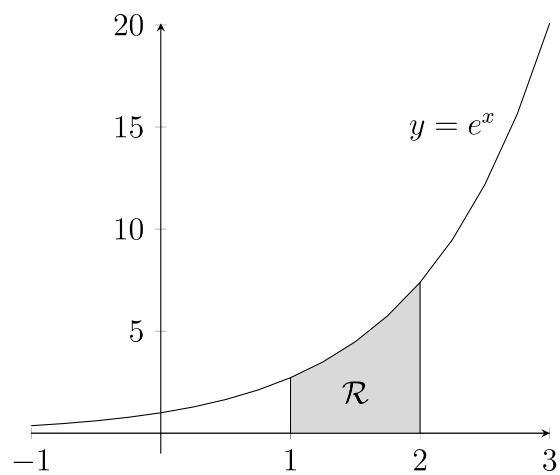
$$\int_0^{\frac{1}{2}} 40000(1+2y)(1/2-y)dy\,J = \frac{1}{2}\int_0^{\frac{1}{2}} 40000(1+2y)(1-2y)dy\,J = 20000\int_0^{\frac{1}{2}} 1-4y^2dy\,J$$
$$= 40000\left(y-\frac{4y^3}{3}\right)\Bigg|_0^{\frac{1}{2}} J$$
$$= 40000\left(\frac{1}{2}-\frac{1}{6}\right) J$$
$$= 40000\cdot\frac{28}{48} J$$
$$= \frac{20000}{3} J.$$

4. (20 points)

The region $\mathcal{R}$ in the plane is bounded by the graph of $y=e^x$, the x-axis, and the lines $x=1$ and $x=2$.

(a) Find the volume of the solid $\mathcal{S}$ obtained by revolving $\mathcal{R}$ about the x-axis.

(b) Find the volume of the solid $\mathcal{T}$ obtained by revolving $\mathcal{R}$ about the y-axis.



Answer:

(a) Using the disk/washer method, we have that

$$\mathrm{Vol}(\mathcal{S}) = \int_1^2 \pi (e^x)^2\,dx = \pi \int_1^2 e^{2x}\,dx = \pi \left(\frac{1}{2}e^{2x}\right)\Big|_1^2 = \pi\left(\frac{1}{2}e^4 - \frac{1}{2}e^2\right).$$

(b) Using the shell method, we have that

$$\mathrm{Vol}(\mathcal{T}) = \int_1^2 2\pi x e^x\,dx = 2\pi \int_1^2 xe^x\,dx.$$

We integrate by parts, letting $u = x$ and $dv = e^x\,dx$, so $du = dx$ and $v = e^x$. Therefore,

$$\mathrm{Vol}(\mathcal{T}) = 2\pi\left(xe^x\Big|_1^2 - \int_1^2 e^x\,dx\right) = 2\pi\left((2e^2 - e) - e^x\Big|_1^2\right) = 2\pi(2e^2 - e - (e^2 - e)) = 2\pi e^2.$$

5. (20 points)

(a) Use integration by parts to find a formula for

$$\int x^n e^x\,dx \qquad \text{in terms of} \qquad \int x^{n-1}e^x\,dx$$

for any integer $n \geq 0$.

(b) Use your formula repeatedly to find

$$\int x^3 e^x\,dx$$

Answer:

(a) Let $u = x^n$ and $dv = e^x\,dx$, so $du = nx^{n-1}\,dx$ and $v = e^x$. Then we have

$$\begin{aligned}\int x^n e^x\,dx = \int u\,dv &= uv - \int v\,du \\ &= x^n e^x - n\int x^{n-1}e^x\,dx.\end{aligned}$$

(b)

$$\begin{aligned}\int x^3 e^x\,dx &= x^3 e^x - 3\int x^2 e^x\,dx \\ &= x^3 e^x - 3\left(x^2 e^x - 2\int xe^x\,dx\right)\end{aligned}$$

$$\begin{aligned}
&= (x^3 - 3x^2)e^x + 6\int xe^x\,dx \\
&= (x^3 - 3x^2)e^x + 6\left(xe^x - \int e^x\,dx\right) \\
&= (x^3 - 3x^2 + 6x - 6)e^x + C.
\end{aligned}$$

Scratch paper

More scratch paper