Math 162: Calculus IIA

First Midterm Exam, Evening Edition ANSWERS

September 16, 2021

Trigonometric formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

- $u = \sec(\theta) + \tan(\theta)$
- $\sec(\theta)d\theta = \dfrac{du}{u}$
- $\sec(\theta) = \dfrac{u^2+1}{2u}$
- $\tan(\theta) = \dfrac{u^2-1}{2u}$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

1. (30 points) The problem has five parts, each worth 6 points. Each depends on having correct answers to the previous parts. You should not expect to get credit for an answer based on incorrect information.

Consider the integral

$$\int \frac{x^7\,dx}{x^4 - 10x^2 + 9}.$$

(a) Rewrite the integrand (NOT the integral, but the function being integrated) as the sum of a polynomial and a fraction in which the numerator is a polynomial of degree less that 4. SHOW YOUR WORK. You will not get credit for merely writing the correct answer.

Answer:

Here is the relevant long division of polynomials.

$$\begin{array}{r|l} & \qquad\qquad\qquad x^3 + 10x \\ \hline x^4 - 10x^2 + 9 & x^7 \\ & -x^7 + 10x^5 \quad - 9x^3 \\ \hline & \qquad 10x^5 \quad - 9x^3 \\ & \quad -10x^5 + 100x^3 - 90x \\ \hline & \qquad\qquad 91x^3 - 90x \end{array}$$

We see that the quotient is $x^3 + 10x$, and the remainder is $91x^3 - 90x$. This means that the integrand is

$$x^3 + 10x + \frac{91x^3 - 90x}{x^4 - 10x^2 + 9}.$$

(b) Write the denominator of the fraction as a product of linear factors.

Answer:

$$x^4 - 10x^2 + 9 = (x^2 - 1)(x^2 - 9) = (x - 1)(x + 1)(x - 3)(x + 3).$$

(c) Are there any values of x for which the integrand is not defined? If so, what are they and why?

Answer:

The denominator of the fraction is zero when x is ± 1 or ± 3. Since division by zero is not defined, the integrand is not defined at those values of x.

(d) Write the fraction you found in part (a) as a sum of fractions with constant numerators, which you may denote by letters of the alphabet. YOU NEED NOT FIND THE VALUES OF THESE CONSTANTS.

Answer:

$$\frac{91x^3 - 90x}{x^4 - 10x^2 + 9} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-3} + \frac{D}{x+3}.$$

(e) Write the original integral in terms of the constants you named in part (d).

Answer:

$$\begin{aligned}\int \frac{x^7\,dx}{x^4 - 10x^2 + 9} &= (x^3 + 10x)\,dx + \int \frac{(91x^3 - 90x)\,dx}{x^4 - 10x^2 + 9} \\ &= (x^3 + 10x)\,dx + \int \frac{A\,dx}{x-1} + \int \frac{B\,dx}{x+1} + \int \frac{C\,dx}{x-3} + \int \frac{D\,dx}{x+3} \\ &= \frac{x^4}{4} + 5x + A\ln|x-1| + B\ln|x+1| + C\ln|x-3| + D\ln|x+3| + c.\end{aligned}$$

2. (20 points) This a work problem with metric units. Assume that acceleration due to gravity is A meters per second per second. You should give your answer in joules as a multiple of $A\pi$. The density of water is a thousand kilograms per cubic meter.

Consider the region of the xy-plane bounded by the curve $y = x^2$ and the lines defined by $x = 0$ and $y = 3$. Rotate this region about the y-axis to obtain a solid region or bowl, which is filled with water. How much work is needed to pump the water about over the top of the bowl?

Answer:

We need to divide the solid region into horizontal layers, one for each value of y. Thus it is convenient to write x as a function of y, namely $x = \sqrt{y}$ for $0 \le y \le 3$.

The radius of such a layer is x, so its area in square meters is $\pi x^2 = \pi y$, and its volume is therefore $\pi y\,dy$ cubic meters. This means its mass is $1000\pi y\,dy$ kilograms, so the gravitational force acting on it is $1000A\pi y\,dy$ newtons. The distance to the top of the solid is $(3 - y)$ meters, so the work needed to lift it is $1000A\pi y(3 - y)\,dy$ joules.

Thus the total amount of work in joules is

$$\begin{aligned}\int_0^3 1000A\pi y(3-y)\,dy &= 1000A\pi\int_0^3 (3y-y^2)\,dy \\ &= 1000A\pi\left(\frac{3y^2}{2}-\frac{y^3}{3}\right)\Big|_0^3 \\ &= 1000A\pi\left(\frac{27}{2}-\frac{27}{3}\right) \\ &= 4500A\pi.\end{aligned}$$

3. (30 points)

(a) (15 points) Use integration by parts twice to find a formula for

$$\int x^n e^{-x^2}\,dx \qquad \text{in terms of} \qquad \int x^{n-4}e^{-x^2}\,dx \qquad \text{for } n>4.$$

Hint: start with $x^n e^{-x^2} = \left(-\frac{1}{2}x^{n-1}\right)\left(-2xe^{-x^2}\right)$, then use

$$u=-\frac{1}{2}x^{n-1}, \quad dv=-2xe^{-x^2}\,dx.$$

Answer:

Using integration by parts twice we have

$$\begin{aligned}\int x^n e^{-x^2}\,dx &= \int\left(-\frac{1}{2}x^{n-1}\right)\left(-2xe^{-x^2}\right)\,dx \\ &= -\frac{1}{2}\int x^{n-1}\left(e^{-x^2}\right)'\,dx \\ &= -\frac{1}{2}\left(x^{n-1}e^{-x^2}-\int(n-1)x^{n-2}e^{-x^2}\,dx\right) \\ &= -\frac{1}{2}x^{n-1}e^{-x^2}+\frac{n-1}{2}\int x^{n-2}e^{-x^2}\,dx \\ &= -\frac{1}{2}x^{n-1}e^{-x^2}+\frac{n-1}{2}\int\left(-\frac{1}{2}x^{n-3}\right)\left(e^{-x^2}\right)'\,dx \\ &= -\frac{1}{2}x^{n-1}e^{-x^2}-\frac{n-1}{4}\left(x^{n-3}e^{-x^2}-\int(n-3)x^{n-4}e^{-x^2}\,dx\right)\end{aligned}$$

$$= -\frac{1}{2}x^{n-1}e^{-x^2} - \frac{n-1}{4}x^{n-3}e^{-x^2} + \frac{(n-1)(n-3)}{4}\int x^{n-4}e^{-x^2}\,dx$$

(b) (15 points) Let

$$I_n = \int x^n e^{-x^2}\,dx.$$

Use your formula of part (a) to find I_9. You should not expect to get credit for an answer based on an incorrect formula.

Answer:

$$\begin{aligned}
I_9 &= \int x^9 e^{-x^2}\,dx \\
&= -\frac{1}{2}x^8e^{-x^2} - \frac{8}{4}x^6e^{-x^2} + \frac{8\cdot 6}{4}\int x^5e^{-x^2}\,dx \\
&= -\frac{1}{2}x^8e^{-x^2} - 2x^6e^{-x^2} + 12\left(-\frac{1}{2}x^4e^{-x^2} - \frac{4}{4}x^2e^{-x^2} + \frac{4\cdot 2}{4}\int xe^{-x^2}\,dx\right) \\
&= -\frac{1}{2}x^8e^{-x^2} - 2x^6e^{-x^2} - 6x^4e^{-x^2} - 12x^2e^{-x^2} + 24\int xe^{-x^2}\,dx \\
&= -\frac{1}{2}x^8e^{-x^2} - 2x^6e^{-x^2} - 6x^4e^{-x^2} - 12x^2e^{-x^2} - 12e^{-x^2} + C
\end{aligned}$$

4. (20 points)

Compute the following integral:

$$\int \frac{\sqrt{4-x^2}}{x^4}dx$$

Answer:

Let $\cos(\theta) = x/2$. So $dx = -\sin(\theta)/2d\theta$. $\sqrt{4-x^2} = \sin(\theta)$.

After trig substitution, the integral is

$$-\int \frac{\sin^2(\theta)}{4\cos^4(\theta)}d\theta.$$

Rewritten, this is

$$-\frac{1}{4}\int \tan^2(\theta)\sec^2(\theta)d\theta.$$

Set $u = \tan(\theta)$, so $du = \sec^2(\theta)d\theta$. After this substitution the integral is

$$-\frac{1}{4}\int u^2 du.$$

This equals

$$-\frac{1}{12}u^3 + C.$$

Substituting back $u = \tan(\theta)$ we get

$$-\frac{1}{12}\tan^3(\theta) + C.$$

In terms of x, $\tan(\theta) = \frac{\sqrt{4-x^2}}{x}$, so we get

$$-\frac{(4-x^2)^{3/2}}{12x^3} + C.$$