Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 11, 2013

1. (20 points)

A swimming pool is 20 feet wide, 50 feet long and 5 feet deep. It is partly filled with water to a depth of 4 feet. How much work is needed to pump all the water out of the pool by lifting it to the top of the pool, which is five feet above the bottom? *Your answer should be expressed in foot-pounds. Assume that the density of water is 60 pounds per cubic foot.*

Solution: Let x be the distance in feet to the top of the pool, so $1 \leq x \leq 5$. For each horizontal layer of water we have

$$\begin{aligned}
\text{area} &= 20 \text{ feet } \times 50 \text{ feet } = 1000 \text{ square feet} \\
\text{volume} &= 1000\, dx \text{ cubic feet} \\
\text{weight} &= 60,000\, dx \text{ pounds} \\
\text{work need to lift} &= 60,000x\, dx \text{ foot-pounds,}
\end{aligned}$$

so the total amount of work required is

$$\begin{aligned}
\int_1^5 60,000x\, dx &= 60,000 \int_1^5 x\, dx = 60,000 \frac{x^2}{2}\bigg|_1^5 \\
&= 30,000(25-1) = 720,000 \text{ foot-pounds.}
\end{aligned}$$

Alternatively, let x be the distance from the bottom of the pool, so $0 \leq x \leq 4$. Then the work needed to lift a horizontal layer of water is $60,000(5-x)\, dx$ foot-pounds, so the total work required is

$$\begin{aligned}
\int_0^4 60,000(5-x)\, dx &= 60,000 \int_0^4 (5-x)\, dx = 60,000 \left(5x - \frac{x^2}{2}\right)\bigg|_0^4 \\
&= 60,000(20-8) = 720,000 \text{ foot-pounds.}
\end{aligned}$$

2. (20 points)

(a) Find the integral

$$\int \frac{4x^2}{(x^2+1)(x^2-1)}\,dx.$$

(b) Find the integral

$$\int \sec^3 x \tan^3 x\,dx.$$

Solution: (a)

First, find A, B, C and D such that

$$\frac{4x^2}{(x^2+1)(x^2-1)} = \frac{4x^2}{(x^2+1)(x-1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}.$$

To find C, multiply $x-1$ both sides and obtain

$$\frac{4x^2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1}(x-1) + C + \frac{D}{x+1}(x-1).$$

Then by letting $x = 1$ we get $C = 1$. Similarly by multiplying $x+1$ and then letting $x = -1$, we obtain $D = -1$. To find A and B, observe

$$\begin{aligned}\frac{4x^2}{(x^2+1)(x^2-1)} &= \frac{Ax+B}{x^2+1} + \frac{1}{x-1} + \frac{-1}{x+1}\\ &= \frac{Ax+B}{x^2+1} + \frac{2}{x^2-1}\\ &= \frac{(Ax+B)(x^2-1)+2(x^2+1)}{(x^2+1)(x^2-1)}\end{aligned}$$

Comparing coefficients of numerators both sides, we have $A = 0$ and $B = 2$. Hence,

$$\frac{4x^2}{(x^2+1)(x^2-1)} = \frac{2}{x^2+1} + \frac{1}{x-1} + \frac{-1}{x+1}$$

and

$$\int \frac{4x^2}{(x^2+1)(x^2-1)}\,dx = 2\arctan x + \ln|x-1| - \ln|x+1| + C.$$

Note : We may do this by observing

$$\frac{4x^2}{(x^2+1)(x^2-1)} = \frac{2}{x^2-1} + \frac{2}{x^2+1}$$

and

$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}.$$

(b)

Let $u = \sec x$, then $du = \sec x \tan x \, dx$. Since $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, we obtain

$$\int \sec^3 x \tan^3 x \, dx = \int u^2(u^2 - 1) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$$

3. (20 points)

(a) Find the integral

$$\int_0^{\sqrt{3}} \frac{dx}{(9 + x^2)^{3/2}}.$$

(b) Find the integral

$$\int \frac{\cos x}{\sqrt{2\sin x + 1 - \cos^2 x}} \, dx.$$

Solution: (a)

Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta \, d\theta$, and $9 + x^2 = 9 + 9\tan^2\theta = 9\sec^2\theta$. Also, when $x = 0$ we have $\tan\theta = 0$, so that $\theta = 0$ (recall $-\pi/2 < \theta < \pi/2$), and when $x = \sqrt{3}$ we have $\sqrt{3} = 3\tan\theta$ so that $\tan\theta = \frac{1}{\sqrt{3}}$, and $\theta = \pi/6$. We find

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{dx}{(9 + x^2)^{3/2}} &= \int_0^{\pi/6} \frac{3\sec^2\theta \, d\theta}{(9\sec^2\theta)^{3/2}} = \frac{1}{9}\int_0^{\pi/6} \frac{d\theta}{\sec\theta} \\ &= \frac{1}{9}\int_0^{\pi/6} \cos\theta \, d\theta = \frac{\sin\theta}{9}\bigg|_0^{\pi/6} = \frac{1}{18}.\end{aligned}$$

(b)

Let $u = \sin x$. Then $du = \cos x \, dx$, and with $1 - \cos^2 x = \sin^2 x$ we have

$$\int \frac{\cos x}{\sqrt{2\sin x + 1 - \cos^2 x}} \, dx = \int \frac{du}{\sqrt{2u + u^2}} = \int \frac{du}{\sqrt{(u+1)^2 - 1}}.$$

Next we set $u + 1 = \sec\theta$. Then $du = \sec\theta\tan\theta \, d\theta$. Also $(u+1)^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$.

Then

$$\begin{aligned}\int \frac{du}{\sqrt{(u+1)^2-1}} &= \int \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} = \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta| + C \\ &= \ln\left|u+1+\sqrt{(u+1)^2-1}\right| + C \\ &= \ln\left|\sin x+1+\sqrt{(\sin x+1)^2-1}\right| + C \\ &= \ln\left|\sin x+1+\sqrt{(\sin^2 x+2\sin x}\right| + C.\end{aligned}$$

4. (20 points)

(a) Use integration by parts to prove the reduction formula

$$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx \qquad \text{for } n \geq 2.$$

(b) Use the formula to find

$$\int_0^{\pi/2} \cos^3 x\,dx.$$

Solution:
a.) Let $u = \cos^{n-1}x$, so $du = (n-1)\cos^{n-2}x(-\sin x)\,dx$. Let $dv = \cos x\,dx$, so $v = \sin x$. Then:

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx \\ &= \int u\,dv \ = \ uv - \int v\,du \\ &= \cos^{n-1}x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx \\ &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x - \cos^n x\,dx \\ &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx \\ n\int \cos^n x\,dx &= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx\end{aligned}$$

Now divide by n.

b.) Let $n = 3$.

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\sin x \cos^{3-1} x}{3} + \frac{3-1}{3}\int \cos^{3-2} x\,dx \\ &= \frac{\sin x \cos^2 x}{3} + \frac{2}{3}\sin x + C\end{aligned}$$

So

$$\int_0^{\pi/2} \cos^3 x\,dx = \frac{\sin x \cos^2 x}{3} + \frac{2}{3}\sin x \Big|_0^{\pi/2} = \frac{2}{3}$$

5. (20 points) Consider the region bounded by the x-axis and the curve $y = \sin x$ for $0 \le x \le \pi$.

(a) Find the volume of the solid obtained by rotating it about the x-axis.

(b) Find the volume of the solid obtained by rotating the same region about the y-axis.

Solution:

a) Integrating with respect to x makes this a washer method problem with

$$\begin{aligned}V &= \int_0^{\pi} \pi \sin^2 x\,dx \\ &= \pi \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx \qquad \text{by the half angle formula} \\ &= \pi \left(\int_0^{\pi} \frac{dx}{2} - \int_0^{2\pi} \frac{\cos u}{4}\,du\right) \qquad \text{where } u = 2x \\ &= \left(\frac{\pi x}{2}\right)\Big|_0^{\pi} - \left(\frac{\sin u}{4}\right)\Big|_0^{2\pi} \\ &= \frac{\pi^2}{2}.\end{aligned}$$

b) Integrating with respect to x makes this a shell method problem with

$$V = \int_0^{\pi} 2\pi x \sin x\,dx = 2\pi \int_0^{\pi} x \sin x\,dx$$

For this we need integration by parts with

$$\begin{aligned}u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x\end{aligned}$$

Then we have

$$\begin{aligned} V &= 2\pi \int_{x=0}^{x=\pi} u\,dv \\ &= 2\pi \left(uv\Big|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v\,du \right) \\ &= 2\pi \left(-x\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x\,dx \right) \\ &= 2\pi \left(-x\cos x + \sin x\right)\Big|_0^{\pi} \\ &= 2\pi(\pi - 0) \;=\; 2\pi^2. \end{aligned}$$