Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 19, 2010

1. (20 points) Evaluate the following integrals:
(a) (10 points)

$$\int \frac{48}{x^4 - 16} dx.$$

(b) (10 points)

$$\int_0^\pi \sin^2 x \cos^2 x dx.$$

Solution: (a) By partial fractions we have

$$\begin{aligned}
\frac{48}{x^4-16} &= \frac{48}{(x-2)(x+2)(x^2+4)} \\
&= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4} \\
&= \frac{A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x^2-4)}{(x-2)(x+2)(x^2+4)} \\
&= \frac{(A+B+C)x^3 + (2A-2B+D)x^2 + (4A+4B-4C)x + (8A-8B-4D)}{(x-2)(x+2)(x^2+4)}
\end{aligned}$$

By comparing numerators we must have $A+B+C=0$, $2A-2B+D=0$, $4A+4B-4C=0$ and $8A-8B-4D=48$. From this we get $A=3/2$, $B=-3/2$, $C=0$ and $D=-6$. There one gets

$$\int \frac{48}{x^4-16} dx = \frac{3}{2}\int \frac{dx}{x-2} - \frac{3}{2}\int \frac{dx}{x+2} - 6\int \frac{dx}{x^2+4}$$

The first two integrals are done by substitution, $u = x-2$ in the first integral and $u = x+2$ in the second integral. For the last integral, we observe that

$$\frac{1}{x^2+4} = \frac{1}{4}\frac{1}{(x/2)^2+1}.$$

Therefore we use substitution $u = x/2$ and $2du = dx$, then the last integral is

$$6\int \frac{dx}{x^2+4} = \frac{6}{4}\int \frac{2du}{u^2+1} = 3\tan^{-1} u = 3\tan^{-1}(x/2)\,.$$

Thus we have

$$\int \frac{48}{x^4-16}dx = \frac{3}{2}\ln|x-2| - \frac{3}{2}\ln|x+2| - 3\tan^{-1}(x/2)\,.$$

(b) We apply the trigonometric identities

$$\sin(2\theta) = 2\sin\theta\cos\theta \quad \text{and} \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

to the integrand, which is

$$\begin{aligned}
\int_0^\pi \sin^2 x\cos^2 x dx &= \int_0^\pi (\sin x\cos x)^2\, dx \\
&= \int_0^\pi \frac{1}{4}\sin^2(2x)dx \\
&= \frac{1}{4}\int_0^\pi \frac{1-\cos(4x)}{2}dx \\
&= \frac{1}{8}\left(\int_0^\pi 1dx - \int_0^\pi \cos(4x)dx\right) \\
&= \frac{1}{8}\left(x\Big|_0^\pi - \frac{1}{4}\sin(4x)\Big|_0^\pi\right) \\
&= \frac{1}{8}(\pi - 0) \\
&= \frac{\pi}{8}
\end{aligned}$$

2. (20 points) Consider the curve

$$y = f(x) = \frac{e^{2x}+e^{-2x}}{4}.$$

(a) (10 points) Calculate the arc length function $s(x)$ starting at $x = 0$, the length of the curve from $(0, f(0))$ to $(x, f(x))$.

(b) (10 points) Calculate the arc length from $x = 1$ to $x = 2$.

Solution: (a) $y' = e^{2x}/2 - e^{-2x}/2$, so

$$\begin{aligned} s(t)w &= \int_0^t \sqrt{1 + (e^{2x}/2 - e^{-2x}/2)^2} dx \\ &= \int_0^t \sqrt{1 + e^{4x}/4 + e^{-4x}/4 - 1/2} dx \\ &= \int_0^t \sqrt{e^{4x}/4 + e^{-4x}/4 + 1/2} dx \\ &= \int_0^t \sqrt{(e^{2x}/2 + e^{-2x}/2)^2} dx \\ &= \int_0^t e^{2x}/2 + e^{-2x}/2 dx \\ &= \frac{1}{4} e^{2x} \Big|_0^t - \frac{1}{4} e^{-2x} \Big|_0^t \\ &= \frac{1}{4} e^{2t} - \frac{1}{4} e^{-2t} \end{aligned}$$

for $t \geq 0$.

(b) By the definition of the arc length function, $s(2)$ is the arclength from $t = 0$ to $t = 2$ and $s(1)$ is the arc length from $t = 0$ to $t = 1$, so the arc length from $t = 1$ to $t = 2$ is

$$s(2) - s(1) = \frac{1}{4} e^4 - \frac{1}{4} e^{-4} - \frac{1}{4} e^2 + \frac{1}{4} e^{-2}$$

3. (20 points) Consider region between the curves $y = x$ and $y = \sqrt{x}$.

(a) Find the volume of the solid of revolution about the x-axis.

(b) Find the volume of the solid of revolution about the y-axis.

Solution: (a) This is a washer method problem. The region bounded by the two graphs sits between $x = 0$ and $x = 1$ and in that interval $\sqrt{x}$ is the bigger function. We have

$$\begin{aligned}
V &= \int_0^1 \pi[(\sqrt{x})^2 - x^2]\,dx \\
&= \pi \int_0^1 (x - x^2)\,dx \\
&= \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \\
&= \pi\left(\frac{1}{2} - \frac{1}{3}\right) \\
&= \frac{\pi}{6}
\end{aligned}$$

(b) This is a shell method problem. We have

$$\begin{aligned}
V &= \int_0^1 2\pi x(\sqrt{x} - x)\,dx \\
&= 2\pi \int_0^1 (x^{3/2} - x^2)\,dx \\
&= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 \\
&= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) \\
&= \frac{2\pi}{15}
\end{aligned}$$

OR:

Reversing the roles of the x and y axes, we can use the washer method again:

$$\begin{aligned} V &= \int_0^1 \pi[y^2 - (y^2)^2]\,dx \\ &= \pi \int_0^1 (y^2 - y^4)\,dx \\ &= \pi \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 \\ &= \pi\left(\frac{1}{3} - \frac{1}{5}\right) \\ &= \frac{2\pi}{15} \end{aligned}$$

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for

$$\int x^{2n} \sin x\,dx \qquad \text{in terms of} \qquad \int x^{2n-2} \sin x\,dx$$

(b) (10 points) Use this formula to find

$$\int x^4 \sin x\,dx.$$

Solution:

(a) Using integration by parts twice we have

$$\begin{aligned} \int x^{2n} \sin x\,dx &= \int x^{2n}(-\cos x)'\,dx \\ &= -x^{2n}\cos x + 2n\int x^{2n-1}\cos x\,dx \\ &= -x^{2n}\cos x + 2n\int x^{2n-1}(\sin x)'\,dx \\ &= -x^{2n}\cos x + 2nx^{2n-1}\sin x - 2n(2n-1)\int x^{2n-2}\sin x\,dx \end{aligned}$$

(b)

$$\begin{aligned}
\int x^4 \sin x\, dx &= -x^4 \cos x + 4x^3 \sin x - 4\cdot 3\int x^2 \sin x\, dx \\
&= -x^4 \cos x + 4x^3 \sin x - 12\left(-x^2 \cos x + 2x \sin x - 2\int \sin x\, dx\right) \\
&= -x^4 \cos x + 4x^3 \sin x - 12\left(-x^2 \cos x + 2x \sin x + 2\cos x\right) + C \\
&= (-x^4 + 12x^2 - 24)\cos x + (4x^3 + 24x)\sin x + C
\end{aligned}$$

5. (20 points) (a) (10 points) Find the integral

$$\int_{-3}^{1} \frac{dx}{\sqrt{x^2+6x+25}}$$

(b) (10 points) Find the integral

$$\int_0^3 \sqrt{9-x^2}\,dx.$$

Solution: (a) We have

$$x^2+6x+25 = (x+3)^2 + 4^2$$

Therefore $\sqrt{x^2+6x+25}$ is the hypotenuse of a right triangle with sides 4 and $x+3$. We denote the angle adjacent to 4 by θ. Then we have

$$\begin{aligned}
x+3 &= 4\tan\theta \\
dx &= 4\sec^2\theta d\theta \\
\sqrt{x^2+6x+25} &= 4\sec\theta
\end{aligned}$$

so our integral is

$$\begin{aligned}
\int_{-3}^{1} \frac{dx}{\sqrt{x^2+6x+25}} &= \int_0^{\pi/4} \frac{4\sec^2\theta d\theta}{4\sec\theta} \\
&= \int_0^{\pi/4} \sec\theta d\theta \\
&= \ln|\sec\theta + \tan\theta|_0^{\pi/4} \\
&= \ln(\sqrt{2}+1)
\end{aligned}$$

(b) $\sqrt{9-x^2}$ is the adjacent side of a right triangle in which the opposite side is x and the hypotenuse is 3. From this we get

$$\begin{aligned} x &= 3\sin\theta \\ dx &= 3\cos\theta d\theta \\ \sqrt{9-x^2} &= 3\cos\theta \end{aligned}$$

so

$$\begin{aligned} \int_0^3 \sqrt{9-x^2}dx &= \int_0^{\pi/2} (3\cos\theta)3\cos\theta d\theta \\ &= 9\int_0^{\pi/2} \cos^2\theta d\theta \\ &= \frac{9}{2}\int_0^{\pi/2} (1+\cos 2\theta)d\theta \\ &= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} \\ &= \frac{9\pi}{4}. \end{aligned}$$