Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 17, 2011

1. (20 points)

(a) (10 points) Find a partial fraction expansion for the function

$$\frac{1}{x^3 - x^2 + 2x - 2}.$$

1. (b) (10 points) Calculate the integral

$$\int \frac{dx}{x^3 - x^2 + 2x - 2}.$$

Solution: (a) One notices that 1 is a root of the denominator. Polynomial division yields $x^3 - x^2 + 2x - 2 = (x-1)(x^2+2)$, so

$$\begin{aligned}
\frac{1}{x^3 - x^2 + 2x - 2} &= \frac{1}{(x-1)(x^2+2)} \\
&= \frac{A}{x-1} + \frac{Bx + C}{x^2+2} \\
&= \frac{A(x^2+2) + (Bx+C)(x-1)}{(x-1)(x^2+2)} \\
&= \frac{Ax^2 + 2A + Bx^2 + Cx - Bx - C}{(x-1)(x^2+2)} \\
&= \frac{(A+B)x^2 + (C-B)x + 2A - C}{(x-1)(x^2+2)}
\end{aligned}$$

By comparing numerators we must have $A + B = 0$, $C - B = 0$ and $2A - C = 1$. From this we get $A = 1/3$, $B = -1/3$ and $C = -1/3$.

(b) To calculate this integral, use the partial fraction expansion from (a). One gets:

$$\begin{aligned}
\int \frac{dx}{x^3 - x^2 + 2x - 2} &= \int \left(\frac{1}{3(x-1)} - \frac{x+1}{3(x^2+2)} \right) dx \\
&= \frac{1}{3} \int \left(\frac{1}{x-1} - \frac{x}{x^2+2} - \frac{1}{x^2+2} \right) dx \\
&= \frac{1}{3} \left(\int \frac{dx}{x-1} - \int \frac{x\,dx}{x^2+2} - \int \frac{dx}{x^2+2} \right).
\end{aligned}$$

The first two integrals are done by substitution, $u = x+1$ in the first integral and $u = x^2+2$ in the second integral. For the last summand we use the trigonometric substitution $x = \sqrt{2}\tan(\theta)$ and hence $dx = \sqrt{2}\sec^2(\theta)d\theta$. The last integral then becomes

$$\int \frac{dx}{x^2+2} = \int \frac{\sqrt{2}\sec^2(\theta)\,d\theta}{2\sec^2(\theta)} = \int \frac{d\theta}{\sqrt{2}}.$$

substituting back $\theta = \arctan\left(\frac{x}{\sqrt{2}}\right)$ we get

$$\int \frac{dx}{x^2+2} = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right).$$

Combining the three calculations we get

$$\int \frac{dx}{x^3-x^2+2x-2} = \frac{1}{3}\left(\ln|x+1| + \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\right) + C.$$

2. (20 points) Consider the curve $y = x^{3/2}$

(a) (10 points) Calculate the arc length function starting at $x = 0$.

2. (b) (10 points) Calculate the arc length from $x = 4$ to $x = 8$.

Solution: (a) $y' = \frac{3}{2}\sqrt{x}$, so by substituting $u = 1 + \frac{9}{4}x$ one gets

$$\begin{aligned} s(t) &= \int_0^t \sqrt{\left(1+\frac{9}{4}x\right)}dx \\ &= \frac{4}{9}\int_1^{1+9t/4} \sqrt{u}du \\ &= \frac{8}{27}u^{3/2}\Big|_1^{1+9t/4} \\ &= \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2} - \frac{8}{27} \end{aligned}$$

for $t \geq 0$.

(b) By the definition of the arc length function, $s(4)$ is the arclength from $t = 0$ to $t = 4$

and $s(8)$ is the arclength from $t = 0$ to $t = 8$, so the arc length from $t = 4$ to $t = 8$ is

$$s(8) - s(4) = \frac{8}{27}\left(19^{3/2} - 10^{3/2}\right) = \frac{8}{27}\left(19\sqrt{19} - 10\sqrt{10}\right).$$

3. (20 points) Consider region between the curve $y = \sin^2 x$ for $0 \le x \le \pi$ and the x-axis.

(a) Find the volume of the solid of revolution about the x-axis.

3. (b) Find the volume of the solid of revolution about the y-axis. **Solution:** (a) This is a washer method problem. We have

$$\begin{aligned}
V &= \int_0^\pi \pi y^2\,dx \\
&= \pi \int_0^\pi \sin^4 x\,dx \\
&= \pi \int_0^\pi \left(\frac{1-\cos 2x}{2}\right)^2 dx \\
&= \frac{\pi}{4}\int_0^\pi \left(1 - 2\cos 2x + \cos^2 2x\right) dx \\
&= \frac{\pi}{4}\int_0^\pi \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx \\
&= \frac{\pi}{8}\int_0^\pi \left(3 - 4\cos 2x + \cos 4x\right) dx \\
&= \frac{\pi}{8}\left(3x - 2\sin 2x + \frac{\sin 4x}{4}\right)\Bigg|_0^\pi \\
&= \frac{3\pi^2}{8}.
\end{aligned}$$

(b) This is a shell method problem. We have

$$\begin{aligned}
V &= \int_0^\pi 2\pi xy\,dx \\
&= 2\pi \int_0^\pi x\sin^2 x\,dx \\
&= 2\pi \int_0^\pi x\left(\frac{1-\cos 2x}{2}\right) dx \\
&= \pi \int_0^\pi x\,dx - \pi\int_0^\pi x\cos 2x\,dx \\
&= \pi \frac{x^2}{2}\Bigg|_0^\pi - \pi\int_0^\pi x\cos 2x\,dx \\
&= \frac{\pi^3}{2} - \pi\int_0^\pi x\cos 2x\,dx
\end{aligned}$$

The remaining integral requires integration by parts with

$$\begin{aligned} u &= x & dv &= \cos 2x\,dx \\ du &= dx & v &= \frac{\sin 2x}{2} \end{aligned}$$

This gives

$$\begin{aligned} \int_0^\pi x\cos 2x\,dx &= \int_{x=0}^{x=\pi} u\,dv \\ &= uv\big|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v\,du \\ &= \left.\frac{x\sin 2x}{2}\right|_{x=0}^{x=\pi} - \int_0^\pi \frac{\sin 2x}{2}\,dx \\ &= 0, \end{aligned}$$

so

$$V = \frac{\pi^3}{2}.$$

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for

$$\int x^n e^x\,dx \qquad \text{in terms of} \qquad \int x^{n-1}e^x\,dx$$

(b) (10 points) Use this formula to find

$$\int x^3 e^x\,dx.$$

Solution: (a) Let $u = x^n$ and $dv = e^x dx$, so $du = nx^{n-1}$ and $v = e^x$. Then we have

$$\begin{aligned} \int x^n e^x\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^n e^x - n\int x^{n-1}e^x\,dx. \end{aligned}$$

(b)

$$\begin{aligned}
\int x^3 e^x \, dx &= x^3 e^x - 3 \int x^2 e^x \, dx \\
&= x^3 e^x - 3 \left(x^2 e^x - 2 \int x e^x \, dx. \right) \\
&= (x^3 - 3x^2) e^x + 6 \int x e^x \, dx \\
&= (x^3 - 3x^2) e^x + 6 \left(x e^x - \int e^x \, dx \right) \\
&= (x^3 - 3x^2 + 6x) e^x - 6 \int e^x \, dx \\
&= (x^3 - 3x^2 + 6x - 6) e^x + C
\end{aligned}$$

5. (20 points) Consider the integral

$$\int \frac{dx}{\sqrt{4x^2 - 12x}}$$

(a) (5 points) Write the quantity under the square root sign as a sum or difference of two squares.

(b) (5 points) Draw a right triangle in which one of the sides is the square root in the integer and another is a constant.

5. (c) (10 points) Evaluate

$$\int_3^4 \frac{dx}{\sqrt{4x^2 - 12x}}.$$

Solution: (a) $(2x-3)^2 = 4x^2 - 12x + 9$ so $4x^2 - 12x = (2x-3)^2 - 3^2$.

(a) The triangle has hypotenuse $2x - 3$, adjacent sdie 3 and opposite side $\sqrt{4x^2 - 12x}$.

(c) We have

$$\begin{aligned}
\sqrt{4x^2 - 12x} &= 3 \tan \theta \\
2x - 3 &= 3 \sec \theta \\
2dx &= 3 \sec \theta \tan \theta d\theta
\end{aligned}$$

so the indefinite integral is

$$\begin{aligned}
\int \frac{dx}{\sqrt{4x^2-12x}} &= \frac{3}{2}\int \frac{\sec\theta\tan\theta}{3\tan\theta}d\theta \\
&= \frac{1}{2}\int \sec\theta d\theta \\
&= \frac{1}{2}\log(\sec\theta+\tan\theta)+C \\
&= \frac{1}{2}\log\left(\frac{2x-3}{3}+\frac{\sqrt{4x^2-12x}}{3}\right)+C
\end{aligned}$$

and

$$\begin{aligned}
\int_3^4 \frac{dx}{\sqrt{4x^2-12x}} &= \frac{1}{2}\log\left(\frac{2x-3}{3}+\frac{\sqrt{4x^2-12x}}{3}\right)\Bigg|_3^4 \\
&= \frac{1}{2}\left(\log\left(\frac{5}{3}+\frac{4}{3}\right)-\log\left(\frac{3+0}{3}\right)\right) \\
&= \frac{\log(3)}{2}.
\end{aligned}$$