Series Summary

Monotonic Sequence Thm If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an increasing (monotonic) sequence of numbers, always smaller than some number M (i.e. $a_n < M < \infty$), then the sequence converges. That is, $\lim_{n\to\infty} a_n$ exists.

Test for divergence If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} \qquad \begin{array}{l} |r| < 1 \ : \ \text{The series converges to } \frac{a}{1-r}. \\ |r| \geq 1 \ : \ \text{The series diverges.} \end{array}$$

(Note, if we replace r^{n-1} with r^n, the series will still converge when $|r| < 1$, but now to $\frac{ar}{1-r}$.)

p-series We used the integral test to see that when

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \begin{array}{l} p > 1 \ , \text{ the series converges.} \\ p \leq 1 \ , \text{ the series diverges.} \end{array}$$

Integral Test Applies when $a_n = f(n)$, and $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$. In this case the series $\sum a_n$ converges *if and only if* the integral $\int_1^\infty f(x)\,dx$ does.

Comparison Test Applies so long as a_n and b_n are always positive.

(i) If $a_n \leq b_n$ and $\sum b_n$ *converges*, then so does $\sum a_n$.

(ii) If $a_n \geq b_n$ and $\sum b_n$ *diverges*, then so does $\sum a_n$.

Limit Comparison Test Applies so long as a_n and b_n are always positive. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$$

(c can't be ∞), then $\sum a_n$ converges *if and only if* $\sum b_n$ converges.

Alternating Series Test

$$\sum_{n=1} (-)^{n-1} b_n$$

Applies when $b_n \geq 0$: If

(i) $b_{n+1} \leq b_n$, and

(ii) $\lim_{n\to\infty} b_n = 0$;

then the series converges. (O.K. to replace $(-1)^{n-1}$ with $(-1)^n$.)

Ratio Test Study this limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

- When the limit exists and is *less than* 1, the series $\sum a_n$ is absolutely convergent (and convergent).
- When the limit exists and is *greater than* 1 (or if the limit diverges to infinity) the series diverges.
- When the limit is equal to 1, the ratio test is useless.

RootTest Study this limit:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}$$

- When the limit exists and is *less than* 1, the series $\sum a_n$ is absolutely convergent (and convergent).
- When the limit exists and is *greater than* 1 (or if the limit diverges to infinity) the series is divergent.
- When the limit is equal to 1, the root test is useless.

Which test when?

(A rough guide)

(1) If you can see easily that $\lim_{n\to\infty} a_n \neq 0$, apply the test for divergence.

(2) Is $\sum a_n$ a p-series or geometric series? If yes, apply those tests.

(3) If $\sum a_n$ is close to a p-series or geometric series, try one of the comparison tests.

(4) If $a_n = f(n)$ and $\int_1^\infty f(x)\,dx$ is easily evaluated, use the integral test.

(5) If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$ try the alternating series test.

(6) Series involving factorials (e.g. $n!$) or n^{th} powers of a constant (e.g. 4^n) can often be studied with the ratio test.

(7) When a_n looks like $(\cdots)^n$, and the term inside the parenthesis *also* involves n, try the root test.