Math 162: Calculus IIA

Final Exam ANSWERS

December 17, 2023

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution for integrals of the form

$$\int \tan^m x\sec^n x\,dx \qquad \text{with } n > 0,$$

known in Doug's section as *the rabbit trick.*

$$u = \sec x + \tan x \qquad \sec x\,dx = \frac{du}{u}$$

$$\sec x = \frac{u^2+1}{2u} \qquad \tan x = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan(y/x) \qquad \text{for } x > 0$$
$$\pi + \arctan(y/x) \text{for } x < 0$$
$$\pi/2 \text{for } x = 0 \text{ and } y > 0$$
$$3\pi/2 \text{for } x = 0 \text{ and } y < 0$$
$$\text{undefined} \text{for } (x, y) = (0, 0)$$
$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \leq t \leq b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Infinite series formulas

The Maclaurin series for $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor series for $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

The nth Taylor polynomial is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i,$$

and the nth Taylor remainder is

$$R_n(x) = f(x) - T_n(x).$$

Taylor's inequality says that if $|f^{(n+1)}(x)| \leq M$ for suitable x, then

$$|R_n(x)| \leq \frac{|x-a|^{n+1} M}{(n+1)!}.$$

Part A

1. (10 points)

Evaluate the integral

$$\int \arctan(2x)dx.$$

Answer:

Using integration by parts with $u = \arctan(2x)$ and $dv = dx$ yields $du = \frac{2}{1+4x^2}$ and $v = x$, so we have

$$\int \arctan(2x)dx = x\arctan(2x) - \int \frac{2x}{1+4x^2}dx$$

then a substitution of $w = 1 + 4x^2$, $dw = 8xdx$ yields

$$\int \frac{2x}{1+4x^2}dx = \frac{1}{4}\int \frac{dw}{w} = \frac{1}{4}\ln|w| - C = \frac{1}{4}\ln(1+4x^2) - C$$

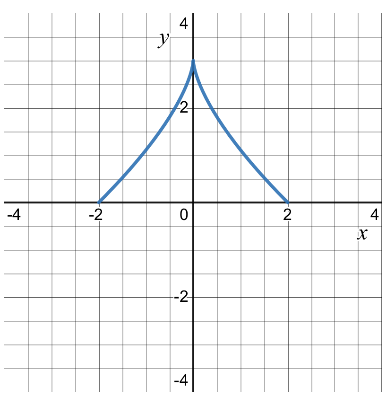
thus

$$\int \arctan(2x)dx = x\arctan(2x) - \frac{1}{4}\ln(1+4x^2) + C.$$

2. (20 points) Find the arc length of the following parametric curve.

$$x = 2\cos^3(t), \quad y = 3\sin^2(t), \quad 0 \le t \le \pi.$$

Hint: Find the arc-length for $0 \le t \le \pi/2$ and then multiply your result by 2.



Answer:

We have

$$\frac{dx}{dt} = 6\cos^2(t)(-\sin(t)) \quad \text{and} \quad \frac{dy}{dt} = 6\sin(t)\cos(t)$$

and therefore

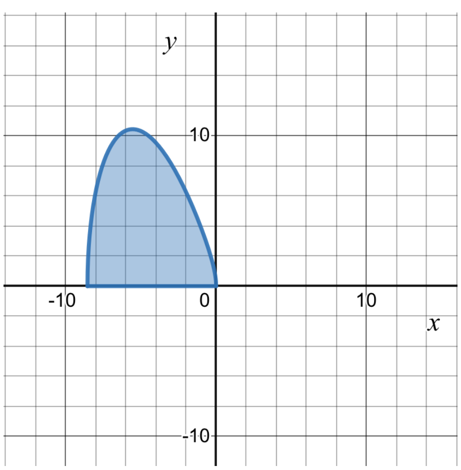
$$ds = \sqrt{36\cos^4(t)\sin^2(t) + 36\sin^2(t)\cos^2(t)} = 6|\cos(t)\sin(t)|\sqrt{1+\cos^2(t)}dt.$$

For $0 \leq t \leq \pi/2$, $\cos(t)\sin(t) \geq 0$, so

$$\begin{aligned}
\text{arc-length} &= 2\int_0^{\pi/2} ds \\
&= 2\int_0^{\pi/2} 6|\cos(t)\sin(t)|\sqrt{1+\cos^2(t)}\,dt \\
&= 2\int_0^{\pi/2} 6\cos(t)\sin(t)\sqrt{1+\cos^2(t)}\,dt \\
&= 2\int_2^1 -3\sqrt{u}\,du \\
&\quad \text{where } u = 1+\cos^2(t) \\
&\quad \text{and } du = -2\sin(t)\cos(t)dt \\
&= 6\int_1^2 \sqrt{u}\,du \\
&= 6\frac{2}{3}u^{3/2}\Big]_1^2 \\
&= 4(2\sqrt{2}-1)
\end{aligned}$$

3. (20 points)

(a) (10 points) Fix a positive number t. Compute the volume of the solid generated by rotating the region bounded by the curves $y = \sqrt{x(x-1)(x+t)}$, $y = 0$, about the x-axis. Your answer should be a function of t.

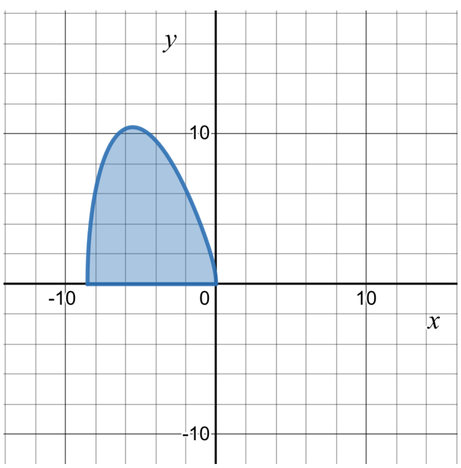


Answer:

We use the washer method. First note that $x = -t$ and $x = 0$ are the two x-values where the curve $y = \sqrt{x(x-1)(x+t)}$ meets the x-axis. So

$$\begin{aligned}
\text{volume} &= \int_{-t}^{0} \pi y^2 \, dx \\
&= \int_{-t}^{0} \pi x(x-1)(x+t) \, dx \\
&= \pi \left[\frac{x^4}{4} + \frac{tx^3}{3} - \frac{x^3}{3} - \frac{tx^2}{2} \right]_{-t}^{0} \\
&= \pi \left(0 - \left(\frac{t^4}{4} - \frac{t^4}{3} + \frac{t^3}{3} - \frac{t^3}{2} \right) \right) \\
&= \pi \frac{t^4}{12} + \pi \frac{t^3}{6}.
\end{aligned}$$

(b) (10 points) Fix a positive number t. Set up the integral for the volume of the region bounded by $y = \sqrt{x(x-1)(x+t)}$, $y = 0$ and rotated around the line $x = 1$. Your integral should depend on t. Do not evaluate the integral.

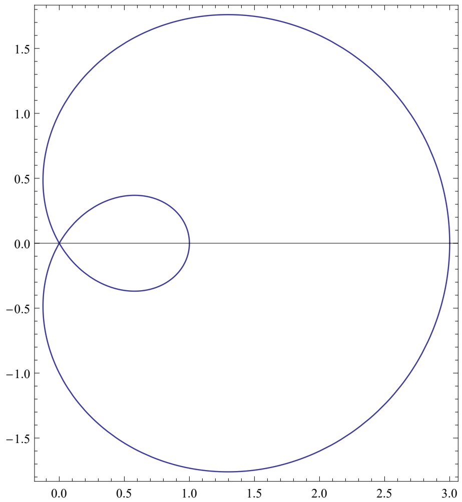


Answer:

Using the shell method we have shells of radius $(1-x)$, thickness dx and height $\sqrt{x(x-1)(x+t)}$. Thus the volume is

$$V = \int_{-t}^{0} 2\pi(1-x)\sqrt{x(x-1)(x+t)} \, dx.$$

4. (20 points) Find the area inside the outer (larger) loop but outside the inner (smaller) loop of the limaçon $r = 1 + 2\cos(\theta)$.



Answer:

The curve intersects itself when the radius equals zero, or $2\cos(\theta) = -1$, which means $\cos(\theta) = \frac{-1}{2}$. We know $\cos^{-1}(\frac{-1}{2}) = \frac{2\pi}{3}$ so the points of intersection are $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{4\pi}{3}$. The outer loop is traced out from $\frac{-2\pi}{3}$ to $\frac{2\pi}{3}$ and contains area A_1, while the inner loop is traced out from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ (with negative radius) and contains area A_2. The desired area is then $A = A_1 - A_2$. First, we compute the indefinite integral

$$\begin{aligned}\int (1 + 2\cos(\theta))^2\, d\theta &= \int (1 + 4\cos(\theta) + 4\cos^2(\theta))\, d\theta \\ &= \int (1 + 4\cos(\theta) + 2(1 + \cos(2\theta)))\, d\theta \\ &= \int (3 + 4\cos(\theta) + 2\cos(2\theta))\, d\theta \\ &= 3\theta + 4\sin(\theta) + \sin(2\theta).\end{aligned}$$

Then we compute the two separate areas (since they are traced out for different intervals)

$$\begin{aligned}
A_1 &= \int_{-2\pi/3}^{2\pi/3} \frac{1}{2} r^2 \, d\theta = 2 \int_0^{2\pi/3} \frac{1}{2}(1 + 2\cos(\theta))^2 \, d\theta \\
&= [3\theta + 4\sin(\theta) + \sin(2\theta)]_0^{2\pi/3} = 2\pi + \frac{3\sqrt{3}}{2} \\
A_2 &= \int_{2\pi/3}^{4\pi/3} \frac{1}{2} r^2 \, d\theta = 2 \int_{2\pi/3}^{\pi} \frac{1}{2}(1 + 2\cos(\theta))^2 \, d\theta \\
&= [3\theta + 4\sin(\theta) + \sin(2\theta)]_{2\pi/3}^{\pi} = \pi - \frac{3\sqrt{3}}{2} \\
A &= A_1 - A_2 = \pi + 3\sqrt{3}.
\end{aligned}$$

5. (15 points) Compute the following indefinite integral:

$$\int \frac{x^2 + 3x}{x^2 - 1} \, dx$$

Answer:

We need long division:

$$\begin{array}{r|l}
 & \quad\quad\quad 1 \\ \hline
x^2 - 1 & x^2 + 3x \\
 & -x^2 \quad\quad + 1 \\ \hline
 & \quad\quad 3x + 1
\end{array}$$

Then, we can rewrite the integrand as

$$\frac{x^2 + 3x}{x^2 - 1} = 1 + \frac{3x + 1}{x^2 - 1}$$

We can apply partial fraction to the later term and get

$$\frac{x^2 + 3x}{x^2 - 1} = 1 + \frac{2}{x - 1} + \frac{1}{x + 1}$$

and we are ready to integrate

$$\begin{aligned}
\int \frac{x^2 + 3x}{x^2 - 1} \, dx &= \int dx + \int \frac{2}{x - 1} dx + \int \frac{1}{x + 1} dx \\
&= x + 2\ln|x - 1| + \ln|x + 1| + C
\end{aligned}$$

6. (15 points) Compute the following indefinite integral:

$$\int \frac{x^2 \, dx}{(1 - x^2)^{3/2}}$$

Answer:

We will apply a trig substitution: $x = \sin\theta$, then $dx = \cos\theta d\theta$ and

$$\begin{aligned}\int \frac{x^2}{(1-x^2)^{3/2}}\,dx &= \int \frac{\sin^2(\theta)}{(1-\sin^2(\theta))^{3/2}}\cos(\theta)d\theta \\ &= \int \frac{\sin^2\theta}{\cos^2\theta}d\theta \\ &= \int \tan^2\theta d\theta \\ &= \int \sec^2\theta - 1 d\theta \\ &= \tan\theta - \theta + C.\end{aligned}$$

From $\sin\theta = x$, by drawing a right triangle with one angle θ, we can check that

$$\tan\theta = \frac{x}{\sqrt{x^2-1}},$$

so the answer becomes

$$\int \frac{x^2}{(1-x^2)^{3/2}}\,dx = \frac{x}{\sqrt{x^2-1}} - \arcsin x + C$$

Part B

7. (20 points)

(a) (10 points) Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5}$$

is absolutely convergent, conditionally convergent, or divergent.

Answer:

The series converges by the alternating series test. It converges absolutely by the intgeral test or the p-test.

(b) (10 points) Estimate the sum of the series with an accuracy of $.01 = 1/100$.

Answer:

The alternating series is

$$1 - \frac{1}{2^5} + \frac{1}{3^5} + \cdots = 1 - \frac{1}{32} + \frac{1}{243} + \cdots$$

Its third terms is less that $.005 = 1/200$, so the sum of the first two terms will give the desired precision. That sum is

$$1 - \frac{1}{32} = \frac{31}{32} = .96875.$$

8. (20 points)

(a) (10 points) Find a power series representation centered at 1 as well as the radius and interval of convergence for the function

$$f(x) = \frac{x-1}{x+2}.$$

Answer:

Write $f(x)$ as the sum $a/(1-r)$ of a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ [which converges iff $|r| < 1$]

$$f(x) = \frac{x-1}{x+2} = \frac{x-1}{3+(x-1)} = \frac{\frac{1}{3}(x-1)}{1+\left(\frac{x-1}{3}\right)} = \sum_{n=1}^{\infty} \tfrac{1}{3}(x-1)\left(\frac{x-1}{3}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}(x-1)^n$$

This converges if and only if:

$$|r| = \frac{|x-1|}{3} < 1 \iff |x-1| < 3$$

So the radius of convergence is $R = 3$ and the interval of convergence is $(-2, 4)$.

(b) (10 points) Write the following integral as a power series in $x-1$. What is the radius of convergence of this power series?

$$\int \frac{x-1}{x+2} dx$$

Answer:

By the integration theorem:

$$\begin{aligned} \int \frac{x-1}{x-2} dx &= \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}(x-1)^n dx \quad \text{for } |x-1| < 3 \\ &= \sum_{n=1}^{\infty} \int \frac{(-1)^{n-1}}{3^n}(x-1)^n dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}(n+1)(x-1)^{n+1} \end{aligned}$$

with the same radius of convergence $R = 3$.

9. (20 points)

(a) (10 points) Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

Answer:

Applying Ratio test, we have:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{1 \cdot 3 \cdots (2n-1) \cdot (2n+1)} \cdot \frac{1 \cdot 3 \cdots (2n-1)}{n!x^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(n+1)x}{2n+1} \right| = |x| \cdot \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{|x|}{2}.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$ has the radius of convergence $R = 2$.

(b) (10 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$.

Answer:

We know that the series is convergent for $x \in (-2, 2)$, but we are uncertain about its convergence at the endpoints.

If $|x| = 2$, then we see that the absolute value of the nth term of the series is equal to

$$\frac{n! \cdot 2^n}{1 \cdot 3 \cdots (2n-1)} = \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} = \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2n}{2n-1} > 1,$$

because it is a product of n numbers, each of which is greater than 1.

Therefore, the nth term does not go to zero as $n \to \infty$, and hence the series is divergent for $x = \pm 2$. Therefore, the interval of convergence is $(-2, 2)$.

10. (20 points)

Decide whether the following series are absolutely convergent, conditionally convergent, or divergent. Give reasoning for your answers.

(a) (10 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$$

Answer:

This is an alternating series. To show that it is convergent, it suffices to demonstrate that

$$\frac{e^{1/n}}{n^3}$$

monotonically goes to zero.

We note that

$$0 < \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3}.$$

Hence, by the squeeze theorem,

$$\lim_{n\to\infty} \frac{e^{1/n}}{n^3} = 0.$$

Since $n < n+1$, then $1/n^3 > 1/(n+1)^3$. Additionally, $e^{1/n} > e^{1/(n+1)}$, and hence

$$\frac{e^{1/n}}{n^3} > \frac{e^{1/(n+1)}}{(n+1)^3}.$$

This shows that the sequence

$$\left\{ \frac{e^{1/n}}{n^3} \right\}_{n=1}^{\infty}$$

is monotonic. Therefore, by the alternating series test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$$

is convergent.

Moreover, the series is absolutely convergent because

$$\left| \frac{(-1)^n e^{1/n}}{n^3} \right| = \frac{e^{1/n}}{n^3} \leq \frac{e}{n^3},$$

and the series

$$\sum_{n=1}^{\infty} \frac{e}{n^3}$$

is convergent. Hence, by the comparison test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{1/n}}{n^3}$$

is also convergent.

(b) (10 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^2}$$

Answer:

This is an alternating series, and to show that it is convergent, it suffices to demonstrate that the sequence

$$\left\{ \frac{\ln n}{n^2} \right\}_{n=1}^{\infty}$$

monotonically goes to zero. To establish its monotonicity, one can consider the function $f(x) = (\ln x)/x^2$. Note that

$$f'(x) = \frac{1-2}{x^3} < 0 \qquad \text{for } x \geq 2.$$

Hence, the sequence

$$\left\{ \frac{\ln n}{n^2} \right\}_{n=1}^{\infty}$$

is decreasing as well. Applying L'Hopital's rule, we can show that $f(x) \to 0$ as $x \to \infty$, which implies that the sequence goes to zero as well.

Therefore, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^2}$$

is convergent.

It is absolutely convergent because the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

is convergent by the comparison test. Indeed, for sufficiently large n, we have $\ln n \leq \sqrt{n}$, and hence

$$\frac{\ln n}{n^2} \leq \frac{1}{n^{3/2}}$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

is convergent.

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