Math 162: Calculus IIA

Final Exam, Saturday Edition ANSWERS

December 15, 2020

Trig formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

- $u = \sec(\theta) + \tan(\theta)$
- $\sec(\theta)d\theta = \dfrac{du}{u}$
- $\sec(\theta) = \dfrac{u^2+1}{2u}$
- $\tan(\theta) = \dfrac{u^2-1}{2u}$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Polar coordinate formulas:

- Area:
$$\frac{1}{2}\int r^2 d\theta$$
- Arc length:
$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Parametric equation formulas:

- Newton's notation: $\dot{x} = dx/dt \qquad \dot{y} = dy/dt$
- Slope of tangent line: $dy/dx = \dot{y}/\dot{x}$.
- Second derivative
$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{\dot{x}}.$$
Curve is concave up/down when this is positive/negative.
- Arc length:
$$\int \sqrt{\dot{x}^2 + \dot{y}^2}dt.$$

Power series formulas:

- MACLAURIN series for $f(x)$:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}.$$
- Maclaurin series for specific functions:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \qquad \sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$\arctan x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} \qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$
$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad \text{where} \qquad \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$
- TAYLOR series for $f(x)$ about a:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}.$$
- The nth partial sum of the above, also called the nth Taylor polynomial, is
$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!},$$
and the nth Taylor remainder is $R_n(x) = f(x) - T_n(x)$. Taylor's inequality says that
$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!},$$
when x is in an interval centered at a in which $|f^{(n+1)}| \leq M$.

1. (25 points) Consider the solid formed by taking the area between $y = \sqrt[3]{x}$, $y = 2$ and the y-axis, and revolving it about the y-axis.

Suppose x and y measured meters, and a cup in the shape of the solid were filled with water of density 1000kg/m^3 from the bottom to $y = 1$ meter. How much energy would it take to lift the water over the top of the cup? (You can approximate gravity as $9.8\ \text{m/s}^2$.)

Solution:

For a layer of water at height y, the distance it needs to travel is $2 - y$.

For the force, let us consider the area of a layer of water. The radius of this layer will be the distance from the y-axis to the curve $y = \sqrt[3]{x}$. In terms of y, this distance is y^3. So the area of a layer is πy^6. For a small thickness dy, the volume of a layer is $\pi y^6 dy$. To find the force, we multiply this volume by the density and gravity to get $9800\pi y^6 dy$.

To find the energy, we compute

$$\begin{aligned} & \int_0^1 9800\pi y^6(2-y)dy \\ =& \int_0^1 9800\pi(2y^6 - y^7)dy \\ =& 9800\pi \left[\frac{2y^7}{7} - \frac{y^8}{8}\right]_0^1 \\ =& 9800\pi \left[\frac{2}{7} - \frac{1}{8}\right]. \end{aligned}$$

2. (25 points)

Compute the following integral:

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx.$$

Answer:

Split the integrand using partial fractions:

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{x(x^2+4)} = \frac{a}{x} + \frac{bx + c}{x^2 + 4}$$

Using Heaviside's Method, we multiply through by x and set $x = 0$ to obtain

$$\frac{2x^2 - x + 4}{x^2 + 4} = a + \frac{x(bx + x)}{x^2 + 4}$$
$$\frac{4}{4} = a$$

so $a = 1$. It follows that

$$\begin{aligned}\frac{bx + c}{x^2 + 4} &= \frac{2x^2 - x + 4}{x(x^2 + 4)} - \frac{1}{x} \\ &= \frac{2x^2 - x + 4 - (x^2 + 4)}{x(x^2 + 4)} = \frac{x^2 - x}{x(x^2 + 4)} \\ &= \frac{x - 1}{x^2 + 4}\end{aligned}$$

so $b = 1$ and $c = -1$. It follows that

$$\begin{aligned}\int \frac{2x^2 - x + 4}{x^3 + 4x} dx &= \int \frac{a\,dx}{x} + \int \frac{bx\,dx}{x^2 + 4} + \int \frac{c\,dx}{x^2 + 4} \\ &= \int \frac{dx}{x} + \int \frac{x\,dx}{x^2 + 4} - \int \frac{dx}{x^2 + 4}.\end{aligned}$$

For these three integrals, we have

- $\displaystyle\int \frac{1}{x} dx = \ln |x|$.

- For $\displaystyle\int \frac{x}{x^2 + 4} dx$, set $u = x^2 + 4, \Rightarrow du = 2xdx$. So

$$\int \frac{x}{x^2 + 4} dx = \int \frac{du}{2u} = \frac{1}{2} \ln |u| = \frac{1}{2} \ln |x^2 + 4|.$$

- For $\displaystyle\int \frac{1}{x^2 + 4} dx$, using trig substitution $x = 2\tan(\theta)$, $x^2 + 4 = 4\sec^2(\theta)$, and $dx = 2\sec^2(\theta)d\theta$. So

$$\int \frac{1}{x^2 + 4} dx = \int \frac{2\sec^2(\theta)d\theta}{4\sec^2(\theta)} = \int \frac{1}{2} d\theta = \frac{\theta}{2} = \frac{\arctan(x/2)}{2}.$$

Adding these together, we get the full integral is

$$= \ln |x| + \frac{1}{2} \ln \left|x^2 + 4\right| - \frac{\arctan(x/2)}{2} + C.$$

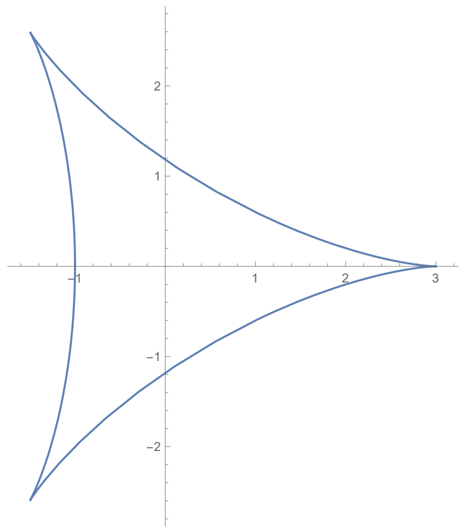
3. (25 points) Find the arc-length of the parametric curve

$$x = 2\cos t + \cos 2t\,,\ y = 2\sin t - \sin 2t\,,\ 0 \le t \le 2\pi\,.$$

by doing it for $0 \le t \le 2\pi/3$ and multiplying your answer by 3.

You may want to use the trig identities $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin^2\theta = (1-\cos 2\theta)/2$.

The curve for $0 \le t \le 2\pi$ is pictured below.



Answer:

We have

$$dx/dt = -2(\sin t + \sin 2t) \qquad \text{and} \qquad dy/dt = 2(\cos t - \cos 2t).$$

Therefore

$$\begin{aligned}
(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\
&= 4(\sin t + \sin 2t)^2 + 4(\cos t - \cos 2t)^2 \\
&= 4(\sin^2 t + 2\sin t\sin 2t + \sin^2 2t + \cos^2 t - 2\cos t\cos 2t + \cos^2 2t) \\
&= 4(2 - 2\cos 3t) = 8(1-\cos 3t) \\
&\qquad \text{since } \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&= 16\left(\frac{1-\cos 3t}{2}\right) \\
&= 16\sin^2(3t/2),
\end{aligned}$$

so

$$\frac{ds}{dt} = 4|\sin(3t/2)|.$$

By the arc length formula, we have

$$\begin{aligned} L &= 3\int_0^{2\pi/3} ds = 12\int_0^{2\pi/3} \sin(3t/2)\,dt \\ &= 8\int_0^{\pi} \sin u\,du, \qquad \text{where } u = 3t/2, \text{ so } dt = 2du/3 \\ &= -8\cos u\Big|_0^{\pi} = 16. \end{aligned}$$

4. (25 points)

Let $g : (0, \infty) \to \mathbb{R}$ be a positive, increasing function that satisfies

$$\lim_{x\to\infty} \frac{g(x)}{x} = 1.$$

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{g(n)+n}$$

Answer:

First we check absolute convergence. Let $a_n = \frac{1}{g(n)+n}$. We use the limit comparison test with the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$:

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n}{g(n)+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{1}{\frac{g(n)}{n} + 1} \end{aligned}$$

Using the assumption that $\lim_{x\to\infty} \frac{g(x)}{x} = 1$, the above limit is $\frac{1}{2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series that diverges, the limit comparison implies that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+g(n)}$ diverges as well.

Next we check convergence. Since the function $g(x)$ is positive, we have $0 < \frac{1}{g(n)+n} < \frac{1}{n}$, then it is clear that $\lim_{n\to\infty} \frac{1}{g(n)+n} = 0$. Moreover, since $g(n)$ are n are both increasing, $\frac{1}{g(n)+n}$ is decreasing. By the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{g(n)+n}$ is convergent. Combine with above, the series is conditionally convergent.

5. (20 points) Let r be a real number with $0 < r < 1$. Determine whether the series is absolutely convergent, conditonally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^r}.$$

Answer:

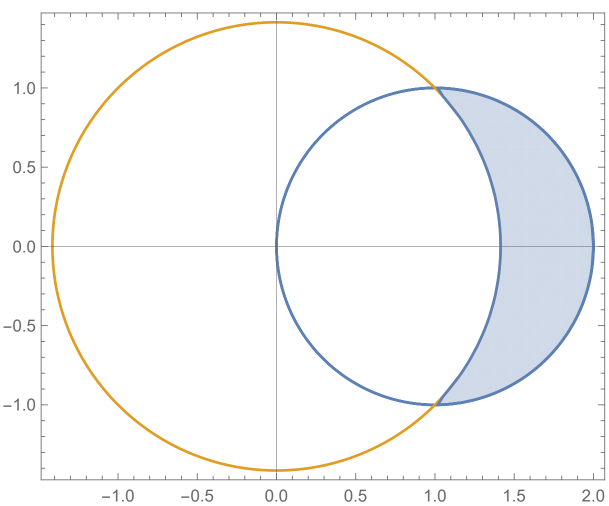
We compute the improper integral

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln x)^r}\,dx &= \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln x)^r}\,dx \\
&= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^r} \qquad \text{where } u = \ln x \\
&= \lim_{t\to\infty} \frac{1}{-r+1}[u^{-r+1}]_{\ln 2}^{\ln t} \\
&= \lim_{t\to\infty} \frac{1}{-r+1}\left((\ln t)^{-r+1} - (\ln 2)^{-r+1}\right) \\
&= \infty \qquad \text{since } 0 < r < 1
\end{aligned}$$

Since the improper integral diverges, by the integral test the series is divergent.

6. (25 points)

(10 points) (a) (10 points) Find the area inside the polar curve $r = 2\cos(\theta)$ and outside the polar curve $r = \sqrt{2}$, as shown below.



Answer:

The curves intersect when $2\cos(\theta) = \sqrt{2}$ or $\cos(\theta) = \frac{\sqrt{2}}{2}$. We know $\cos^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$ so the points of intersection are $\theta_1 = -\frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{4}$. Thus,

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}[(2\cos\theta)^2 - (\sqrt{2})^2]\, d\theta = 2\int_0^{\pi/4} (2\cos^2\theta - 1)\, d\theta \\ &= 2\int_0^{\pi/4} \cos 2\theta\, d\theta = \sin 2\theta|_0^{\pi/4} = \sin\pi/2 = 1. \end{aligned}$$

(b) (10 points) Find the arc length of the boundary of the region of part (a).

Answer:

The points of intersection are the same as in part (a). Thus,

$$\begin{aligned} s &= \int_{-\pi/4}^{\pi/4} \sqrt{(\sqrt{2})^2 + 0^2}\, d\theta + \int_{-\pi/4}^{\pi/4} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta \\ &= 2\int_0^{\pi/4} \sqrt{2}\, d\theta + 4\int_0^{\pi/4} d\theta \\ &= \frac{\pi\sqrt{2}}{2} + \pi = \frac{\pi(2+\sqrt{2})}{2}. \end{aligned}$$

7. (25 points) Find the radius of convergence and interval of convergence of the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{3^n n(\ln n)^2}\,.$$

Answer:

Solution: Using ratio test, we have

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{3^{n+1}(n+1)(\ln(n+1))^2}\frac{3^n n(\ln n)^2}{x^n}\right| \\ &= \left|\frac{x}{3}\right| \cdot \left|\lim_{n\to\infty}\frac{n(\ln(n))^2}{(n+1)(\ln(n+1))^2}\right| = \left|\frac{x}{3}\right| \end{aligned}$$

because by the l'Hospital rule

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = 1.$$

To have an absolute convergence series, we need to have $\left|\frac{x}{3}\right| < 1$, so $|x| < 3$ and the radius of convergence is 2. Consider the end points at $x = 3$ and $x = -3$, we have

- $x = -3$: The series is equal to
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$
By the integral test, the convergence of the series is equivalent to the convergence of the integral
$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx.$$
By letting $\ln x = y$, we can calculate the integral as follows:
$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{dy}{y^2} = \frac{1}{\ln 2}.$$
Therefore, the series is also convergent.

- $x = 3$: The series is equal to
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}.$$
This is absolutely convergent from the case $x = -3$. So, the above series is convergent.

(Note : You may also use the alternating series test for the convergence.)

8. (25 points)

(a) (15 points) Find the Taylor series centered at 0 of the function $\ln(1 - (x/2)^2)$, as well as the radius and interval of convergence.

Answer:

The Taylor series of $\ln(1 + x)$ is
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots ,$$
which converges for $|x| < 1$. Therefore, replacing x by $-(x/2)^2$,
$$\ln(1 - (x/2)^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{-x^2}{4}\right)^n = -\sum_{n=1}^{\infty} \frac{x^{2n}}{4^n n}$$
$$= -\frac{x^2}{4} - \frac{x^4}{2 \cdot 4^2} - \frac{x^6}{3 \cdot 4^3} - \cdots$$

It converges for $(x/2)^2 < 1$, so its radius of convergence is 2.

We check the convergence at the end points $x = \pm 2$. When $x = 2$, the series is

$$-\sum_{n=1}^{\infty} \frac{1}{n}$$

and it is diverges because it is the harmonic series. When $x = -2$, the same thing happens.

Hence the interval of convergence is $(-2, 2)$.

(b) (10 points) Write the integral

$$\int_0^x \ln(1 - (t/2)^2)dt$$

as a power series in x.

Answer:

$$\begin{aligned}
\int_0^x \ln(1 - (t/2)^2)dt &= -\int_0^x \sum_{n=1}^{\infty} \frac{t^{2n}}{4^n n} dt \\
&= -\sum_{n=1}^{\infty} \frac{1}{4^n n} \int_0^x t^{2n} dt \\
&= -\sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n n(2n+1)} \\
&= -\frac{x^3}{4 \cdot 3} - \frac{x^5}{4^2 \cdot 10} - \frac{x^7}{4^3 \cdot 21} - \cdots
\end{aligned}$$

The equation holds for $|x| < 2$.