Math 162: Calculus IIA

Final Exam ANSWERS

December 19, 2016

Part A

1. (15 points) Evaluate the integral

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx$$

Answer:

Substitute $x = 2\cos\theta$:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{8\cos^3\theta}{2\sin\theta}(-2\sin\theta)\,d\theta \\
&= -8\int \cos^3\theta\,d\theta \\
&= -8\int \cos^2\theta\cdot\cos\theta\,d\theta \\
&= -8\int (1-\sin^2\theta)\cdot\cos\theta\,d\theta \\
&= -8\int (1-u^2)\,du \qquad \text{(using the substitution } u=\sin\theta\text{)} \\
&= -8\left(u-\frac{u^3}{3}\right)+C \\
&= -8\left(\sin\theta-\frac{\sin^3\theta}{3}\right)+C \\
&= -8\left(\frac{\sqrt{4-x^2}}{2}-\frac{(\sqrt{4-x^2})^3}{24}\right)+C
\end{aligned}$$

since $\sin\theta = \sqrt{4-x^2}/2$.

2. (20 points)

(a) Compute the volume of a region bounded by the curves $y = x^3+1$, $y=1$ and $x=1$ and rotated around the y-axis.

Answer:

Using the shell method we have shells of radius x, thickness dx and height $(x^3+1)-1 = x^3$. Therefore

$$V = \int_0^1 2\pi x \cdot x^3 dx = 2\pi \frac{x^5}{5}\bigg|_0^1 = \frac{2\pi}{5}$$

(b) Set up the integral for the volume of the region bounded by $y = x^4$, $y = 0$ and $x = 2$ and rotated around line $x = 2$. Use the shell method. Do not evaluate the integral.

Answer:

Using the shell method we have shells of radius $(2-x)$, thickness dx and height x^4. Thus the volume is

$$V = \int_0^2 2\pi(2-x)x^4 \, dx.$$

3. (10 points)

Evaluate the integral

$$\int \arcsin x \, dx.$$

Answer:

Integrating by parts with $u = \arcsin x$ and $dv = dx$, we get

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad \text{and} \qquad v = x,$$

so that the integral becomes

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x\, dx}{\sqrt{1-x^2}}$$

Now make the substitution $w = 1 - x^2$, so $dw = -2x\, dx$ and $x\, dx = -dw/2$. This means our new integral is

$$\int \frac{x\, dx}{\sqrt{1-x^2}} = -\int \frac{dw}{2\sqrt{w}} = -\frac{1}{2}\int u^{-1/2}\, dw \qquad = -\frac{1}{2} 2u^{1/2} - C = -\sqrt{1-x^2} - C,$$

and the original integral is

$$\int \arcsin x \, dx = x \arcsin x + \sqrt{1-x^2} + C.$$

4. (20 points)

(a) Find the partial fraction decomposition of

$$\frac{x^2+3x}{x^2-1}.$$

Answer:

The fraction is improper so first use long division to write:

$$\frac{x^3+3x}{x^2-1} = 1 + \frac{3x+1}{x^2-1}.$$

Since the denominator is a difference of squares $x^2-1 = (x-1)(x+1)$ we next seek constants A, B such that:

$$\frac{3x+1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

which is equivalent to solving the linear system:

$$A + B = 3$$
$$A - B = 1$$

Adding these equations gives $2A = 4$ so $A = 2$ and therefore $B = 1$. Thus:

$$\frac{x^3+3x}{x^2-1} = 1 + \frac{2}{x-1} + \frac{1}{x+1}.$$

(b) Write out the form of the partial fraction decomposition of the function

$$\frac{x^3-2}{(x+1)^3(x^2+1)^2(x-1)} = \underline{\hspace{8cm}}$$

Do not determine the numerical values of the coefficients.

Answer:

All the factors are linear except x^2+1, which has discriminant $b^2-4ac = -4 < 0$ (has complex roots $\pm i$) so does not factor over the real numbers. Thus there is a linear factor of multiplicity 3, an irreducible quadratic factor of multiplicity 2 and a linear factor of multiplicity 1. So the partial fraction decomposition will look like:

$$\frac{x^3-2}{(x+1)^3(x^2+1)^2(x-1)} = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3}{(x+1)^3} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2} + \frac{D}{x-1}.$$

(c) Let

$$f(x) = \frac{1}{x-1} + \frac{2x+3}{x^2+1}.$$

Evaluate

$$\int f(x)dx.$$

Answer:

Split the integral:

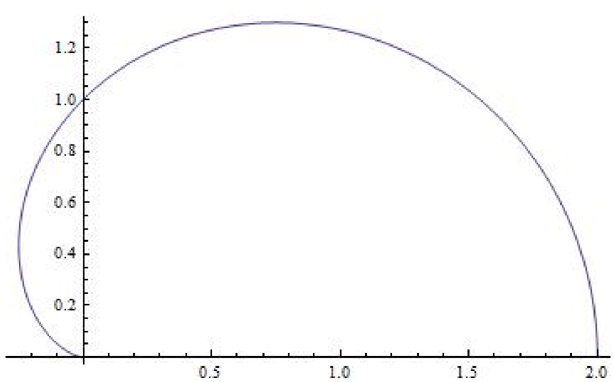
$$\begin{aligned}\int f(x)dx &= \int \frac{1}{x-1}dx + \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx \\ &= \ln|x-1| + \int \frac{2x}{x^2+1}dx + 3\arctan x\end{aligned}$$

Substitute $u = x^2+1$ and hence $du = 2xdx$ to get:

$$\int f(x)dx = \ln|x-1| + \ln|x^2+1| + 3\arctan(x) + C.$$

5. (15 points)

The cardioid is the curve defined in polar coordinates by $r = 1+\cos\theta$. Find the area of the region bounded above by the cardioid and below by the x-axis.



Answer:

Solution: It is easily verified that the region R bounded above by the cardioid and below by the x-axis is given by

$$R = \{(r, \theta) : 0 \leq r \leq 1 + \cos\theta, 0 \leq \theta \leq \pi\}.$$

We use the formula for area inside a polar curve to compute that the area A of the region R is given by

$$\begin{aligned} A &= \frac{1}{2}\int_0^\pi (1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\,d\theta \\ &= \frac{1}{2}\int_0^\pi \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)\,d\theta = \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi \\ &= \frac{3\pi}{4}. \end{aligned}$$

6. (20 points)

Find the arc length of the parametric curve $x(t) = e^t\cos t$, $y(t) = e^t \sin t$ connecting the point $(1, 0)$ to the point $(e^{2\pi}, 0)$.

Answer:

Solution: First observe that the points $(1, 0)$ and $(e^{2\pi}, 0)$ correspond to $t = 0$ and $t = 2\pi$, respectively. It follows that the arc length of this curve is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^{2\pi} \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{e^{2t}\cos^2 t - 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t}\,dt \\ &= \int_0^{2\pi} \sqrt{2e^{2t}(\cos^2 t + \sin^2 t)}\,dt \\ &= \sqrt{2}\int_0^{2\pi} e^t\,dt = \sqrt{2}[e^t]_0^{2\pi} = \sqrt{2}(e^{2\pi} - 1). \end{aligned}$$

Part B

7. (20 points)

(a) Find a power series representation centered at 1 as well as the radius and interval of convergence for the function

$$f(x) = \frac{x-1}{x+2}.$$

Answer:

Write $f(x)$ as the sum $\frac{a}{1-r}$ of a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ [which converges iff $|r| < 1$]

$$f(x) = \frac{x-1}{x+2} = \frac{x-1}{3+(x-1)} = \frac{\frac{1}{3}(x-1)}{1+\left(\frac{x-1}{3}\right)} = \sum_{n=1}^{\infty} \tfrac{1}{3}(x-1)\left(\frac{x-1}{3}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}(x-1)^n$$

This converges if and only if:

$$|r| = \frac{|x-1|}{3} < 1 \iff |x-1| < 3$$

So the radius of convergence is $R = 3$ and the interval of convergence is $(-2, 4)$.

(b) Write the following integral as a power series in $x - 1$. What is the radius of convergence of this power series?

$$\int \frac{x-1}{x+2} dx$$

Answer:

By the integration theorem:

$$\begin{aligned} \int \frac{x-1}{x-2} dx &= \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}(x-1)^n dx \quad \text{for } |x-1| < 3 \\ &= \sum_{n=1}^{\infty} \int \frac{(-1)^{n-1}}{3^n}(x-1)^n dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}(n+1)(x-1)^{n+1} \end{aligned}$$

with the same radius of convergence $R = 3$.

8. (20 points)

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$$

Answer:

First, consider the series

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$$

for absolute convergence. Since $n > (\ln n)^2$ for $n \geq 2$,

$$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n}.$$

We also know that the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test with $p = 1$. Therefore, it follows from the comparison test that the series diverges.

Now, we consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$$

for conditional convergence. It is an alternating series satisfying

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}\ln n} = 0.$$

It is also obvious that $\frac{1}{\sqrt{n}\ln n}$ is a decreasing function of n. So by the Alternating Series test, the original series converges.

Therefore, the series is a conditionally convergent series.

9. (20 points)

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{\sqrt[3]{n}}.$$

Answer:

Solution: We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = |a_{n+1}| \cdot \left|\frac{1}{a_n}\right| = \frac{3^{n+1}|x-2|^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{3^n|x-2|^n}$$
$$= 3 \cdot \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \cdot |x-2| \to 3|x-2|$$

as $n \to \infty$. From

$$3|x-2| < 1 \Leftrightarrow |x-2| < \frac{1}{3},$$

the radius of convergence $R = 1/3$.

Now consider the boundary case $x = 5/3$ or $x = 7/3$. Plugging $x = 5/3$ in original series expression, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}},$$

which converges by the alternating series test.

Plugging $x = 7/3$ in original series expression, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}},$$

which diverges by the p-series test with $p = 1/3 < 1$.

So the interval of convergence is $[5/3, 7/3)$.

10. (20 points) Let

$$f(x) = \frac{x^2}{1+2x}.$$

(a) Find the Taylor series of $f(x)$ centered at $x = 0$.

Answer:

Write

$$\begin{aligned} \frac{x^2}{1+2x} &= \frac{x^2}{1-(-2x)} \\ &= x^2 \sum_{n=0}^{\infty} (-2x)^n \qquad \text{(using geometric series expansion)} \\ &= \sum_{n=0}^{\infty} (-2)^n x^{n+2}. \end{aligned}$$

(b) Find the radius of convergence.

Answer:

One way is to note that $f(x)$ is not defined at $x = -1/2$. This is a distance of $1/2$ away from the center $x = 0$. Thus, the radius of convergence will be $1/2$. Alternatively, you can use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+3}}{(-2)^n x^{n+2}} \right| = |2x|$$

For the series to converge by the Ratio test, we must have $|2x| < 1$, which means $|x| < 1/2$. Thus, again, the radius of convergence is $1/2$.

(c) Compute $f^{(100)}(0)$.

Answer:

For Taylor series centered at $x = a$, the coefficient of x^n is $f^{(n)}(a)/n!$. Thus, we need the coefficient of x^{100}. This happens when $n = 98$. Thus, we have $f^{(100)}(0)/100! = (-2)^{98}$, and so

$$f^{(100)}(0) = 100! \cdot (-2)^{98} = 100! \cdot 2^{98}.$$

11. (20 points)

(a) Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^5}$$

is absolutely convergent, conditionally convergent, or divergent.

Answer:

The series converges by the alternating series test. It converges absolutely by the intgeral test or the p-test.

(b) Estimate the sum of the series with an accuracy of $.01 = 1/100$.

Answer:

The alternating series is

$$1 - \frac{1}{2^5} + \frac{1}{3^5} + \cdots = 1 - \frac{1}{32} + \frac{1}{243} + \cdots$$

Its third terms is less that $.005 = 1/200$, so the sum of the first two terms will give the desired precision. That sum is

$$1 - \frac{1}{32} = \frac{31}{32} = .96875.$$