An Analysis of Recursively Defined Continuous Functions

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This paper is the final outcome of a joint summer research project in which we canvassed mathematical competitions from around the world (William Lowell Putnam Competition, International Mathematical Olympiad, regional and national mathematical contests etc.), looking for real analysis problems. The reason for that was the intention of the second author to prepare for the Putnam contest. Our motivation for the selection process of the problems is that varying nations have different mathematical cultures; for instance, in Romania, the level of high-school analysis can compare to some extent with the one of introductory real analysis taught in American universities (e.g. the upper-division course Math 104 "Introduction to Analysis" taught by the first author at University of California, Berkeley).

To our surprise we were able to identify a common theme persisting throughout the past twenty years. This theme is represented by 21 problems, involving a continuous function verifying a recursive based formula. Our approach is as follows: first we present the problems, then we formulate the main theoretical results and finally we apply them to solve the problems. The findings of this article could prove useful to future participants of mathematical competitions as there was a distinct trend towards administering problems fitting the criterions presented in this article.

Problems

1.(IMO[1] 1983) [1] Find $f:(0,\infty)\longrightarrow(0,\infty)$ *continuous function satisfying:*

$$f(x\cdot f(y)) = f(x)\cdot y, \quad \forall\, x,y\in(0,\infty)$$
$$\lim_{x\to\infty} f(x) = 0.$$

2.(ROM[2] 1983) [1] Let $a,b\in(0,\frac{1}{2})$ *and* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *a continuous function satisfying*

$$f(f(x)) = a\cdot f(x) + bx, \quad \forall\, x\in\mathbb{R}$$

Prove that $f(0)=0$.

3.(ROM 1984) [1] Let $a\neq 0,\pm 1$. *Find all the functions* $f:\mathbb{R}\longrightarrow\mathbb{R}$, *differentiable at* 0, *verifying*

$$f(f(x)) = ax, \quad \forall\, x\in\mathbb{R}$$

4.(ROM 1985) [1] Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ *be a continuous function such that*

$$f(x^2) = \frac{f(x)}{x}, \quad \forall\, x\neq 0$$

Prove that $f\equiv 0$.

5.(ROM 1985) [1] Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ *be a continuous function at 0 such that*

$$f(ax) = b\cdot f(x) + c, \quad \forall\, x\in\mathbb{R}$$

where $0\leq a\leq 1$, $1<b$ *and* $c\in\mathbb{R}$. *Prove that* f *is constant.*

6.(ROM 1985) [1] Find $f:\mathbb{R}\longrightarrow\mathbb{R}$ *a continuous function at 0 such that*

$$f(x) - 2f(tx) + f(t^2x) = x^2, \quad \forall\, x\in\mathbb{R}$$

where $t\in(0,1)$ *is a fixed number.*

7.(ROM 1986) [1] Find $f:\mathbb{R}\longrightarrow\mathbb{R}$ *a continuous function at 0 such that* $f(0)=1986$ *and*

$$f(x) - f(ax) = x^3 + x^2, \quad \forall\, x\in\mathbb{R}$$

[1] International Mathematical Olympiad

[2] Romanian Mathematical Contest

where $a \in (0,1)$ *is a fixed number.*
8.(ROM 1987,1998) [1] Find $f : (0,\infty) \longrightarrow (0,\infty)$ *continuous function satisfying:*

$$f(x \cdot f(y)) = f(\tfrac{x}{y}) \cdot y^2, \quad \forall\, x, y \in (0,\infty)$$
$$\lim_{x\to\infty} f(x) = 0$$

9.(ROM 1987) [1] Find all the continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$*, verifying*

$$f(x) = f(2x+1), \quad \forall\, x \in \mathbb{R}$$

10.(ROM 1989) [1] Find all the continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$*, verifying*

$$f(x) = f(\sqrt[3]{x^2+2x+12}), \quad \forall\, x \in \mathbb{R}$$

11.(ROM 1991) [1] Find $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a continuous function at 0 such that*

$$f(tx) - f(x) = tx, \quad \forall\, x \in \mathbb{R}$$

where $t \in (0,1)$ *is a fixed number.*
12.(ROM 1991) [1] Find $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a continuous function at 0 such that* $f(0) = 1$ *and*

$$3f(x) - 5f(ax) + 2f(a^2x) = x, \quad \forall\, x \in \mathbb{R}$$

where $a \in (0,1)$ *is a fixed number.*
13.(ROM 1994) [3] Find $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a function differentiable at* $x = 6$ *such that*

$$\frac{f(x)}{2} + 3 = f(\frac{x}{2} + 3), \quad \forall\, x \in \mathbb{R}$$

14.(WLP[3] 1996) [5] Let $c > 0$*. Find all the continuous functions* $f : \mathbb{R} \longrightarrow \mathbb{R}$*, verifying*

$$f(x) = f(x^2 + c), \quad \forall\, x \in \mathbb{R}$$

[3]William Lowell Putnam Competition

15.(ROM 1998) [2] Find all the continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$*, verifying*

$$f(x) = f(x^2 + \frac{1}{4}), \quad \forall\, x \in \mathbb{R}$$

16.(ROM 2000) [4] Let $k \geq 2$ *be an integer and* $f : [0, \infty) \longrightarrow \mathbb{R}$ *a continuous function such that*

$$f(x) = f(x^k), \quad \forall\, x \in [0, \infty)$$

Prove that f *is constant.*

17.(ROM 2000) [4] Find $f : (0, \infty) \longrightarrow (0, \infty)$ *continuous function satisfying:*

$$f(x \cdot f(y)) = f(\tfrac{x}{y}), \quad \forall\, x, y \in (0, \infty)$$
$$\lim_{x \to \infty} f(x) = 0$$

18.(WLP 2000) [5] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function such that*

$$f(2x^2 - 1) = 2x \cdot f(x), \quad \forall\, x \in \mathbb{R}$$

Prove that $f \equiv 0$ *on* $[-1, 1]$*.*

19.(ROM 2000) [4] Let$f : \mathbb{R} \longrightarrow \mathbb{R}_+$ *a continuous function at 0 such that*

$$f(2x) \cdot f(3x) = 32^x, \quad \forall\, x \in \mathbb{R}$$

Prove that $f(x) = 2^x$, $\forall\, x \in \mathbb{R}$*.*

20.(ROM 2000) [4] Let $k > 1$ *be an integer. Find* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a continuous function at 0 such that*

$$k \cdot f(kx) = f(x) + kx, \quad \forall\, x \in \mathbb{R}$$

21.(WLP 2001) [1] Let $a, b \in (0, \frac{1}{2})$ *and* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *a continuous function satisfying*

$$f(f(x)) = a \cdot f(x) + bx, \quad \forall\, x \in \mathbb{R}$$

Prove that there exists $c \in \mathbb{R}$ *such that* $f(x) = c \cdot x$, $\forall\, x \in \mathbb{R}$*.*

Theoretical results

Proposition 1. *Let $D \subseteq \mathbb{R}$ and $h : D \longrightarrow \mathbb{R}$, $g : D \longrightarrow D$ two functions such that h is continuous at a point $L \in D$ and*

$$h(x) \;=\; h(g(x)), \quad \forall\, x \in D$$

If $\forall\, x \in D$, $\lim_n g^n(x)$[4] $= L$, then h is necessarily constant on D.
Proof. Fix $x \in D$. Then, using the hypothesis, we obtain

$$h(x) = h(g(x)) = h(g(g(x))) = \cdots = h(g^n(x))$$

This implies, due to the continuity of h at L,

$$h(x) = \lim_n h(g^n(x)) = h(\lim_n g^n(x)) = h(L)$$

Since x was chosen arbitrarily, it follows that:

$$h(x) \;=\; h(L), \quad \forall\, x \in D$$

Therefore, h is constant on D.
Proposition 2. *i) If $(x_n)_n$ is a sequence defined by $x_{n+1} = ax_n + b$, $\forall n \geq 0$, then:*

$$\begin{aligned} x_n \;&=\; x_0 + nb, \quad a = 1 \\ x_n \;&=\; a^n \cdot (x_0 - \tfrac{b}{1-a}) + \tfrac{b}{1-a}, \quad a \neq 1 \end{aligned}$$

ii)If $(x_n)_n$ is a sequence defined by $x_{n+1} = ax_n + bx_{n-1}$, $\forall n \geq 1$, then:

$$\begin{aligned} x_n \;&=\; (c_1 + nc_2) \cdot \lambda^n, \quad a^2 + 4 \cdot b = 0 \\ x_n \;&=\; c_1\lambda_1^n + c_2\lambda_2^n, \quad a^2 + 4 \cdot b \neq 0 \end{aligned}$$

where c_1, c_2 are constants and λ, λ_1, λ_2 are solutions for the equation $z^2 - -az - b = 0$.
Proof. We will prove both results in the non-trivial cases:

[4] $g^1(x) = g(x), g^2(x) = g(g(x))$ etc.

i) $a \neq 1$. We can write the equation in the form

$$x_{n+1} - \frac{b}{1-a} \;=\; a \cdot (x_n - \frac{b}{1-a}), \quad \forall n \geq 0$$

which clearly gives the desired formula.

ii) $a^2 + 4 \cdot b \neq 0$. Using the fact that $\lambda_1 + \lambda_2 = a$ and $\lambda_1 \lambda_2 = -b$, we can infer that

$$x_{n+1} - \lambda_1 x_n \;=\; \lambda_2 \cdot (x_n - \lambda_1 x_{n-1}) \quad \forall n \geq 1$$

which again, in a few steps, implies the result.

Solutions

Problems 1, 8, 17: Taking $y = 1$ in the main equations we obtain in all the cases $f(x \cdot f(1)) = f(x)$, which implies also:

$$f(x) \;=\; f(x \cdot f(1)) = f(\frac{x}{f(1)})$$

Taking now in PROPOSITION 1 either $g(x) = x \cdot f(1)$ or $g(x) = x \cdot \frac{1}{f(1)}$ depending on whether $f(1) > 1$ or $f(1) < 1$, we obtain, using $\lim_{x \to \infty} f(x) = 0$, that $f \equiv 0$, contradiction. Therefore $f(1) = 1$. Plugging in right now $x = y$ for *Problem 1*, $x = y^2$ for *Problem 8* and $x = y$ for *Problem 17* we obtain, respectively:

$$\begin{aligned} f(x \cdot f(x)) \;&=\; x \cdot f(x), \quad \forall\, x \in (0, \infty) \\ f(x^2 \cdot f(x)) \;&=\; x^2 \cdot f(x), \quad \forall\, x \in (0, \infty) \\ f(x \cdot f(x)) \;&=\; 1, \quad \forall\, x \in (0, \infty) \end{aligned}$$

Denoting $x \cdot f(x) = a$ for *Problems 1, 17* and $x^2 \cdot f(x) = a$ for *Problem 8*, by induction, we can infer that:

$$\begin{aligned} f(a^{2^n}) \;&=\; a^{2^n}, \quad \forall\, n \in \mathbb{Z} \\ f(a^{2^n}) \;&=\; a^{2^n}, \quad \forall\, n \in \mathbb{Z} \\ f(a^{2^n}) \;&=\; 1, \quad \forall\, n \in \mathbb{Z} \end{aligned}$$

These relations, combined once again with the hypothesis $\lim_{x\to\infty} f(x) = 0$, imply that $a = 1$ and so the solutions for the three problems are respectively:

$$\begin{aligned} f(x) &= \tfrac{1}{x}, \quad \forall\, x \in (0,\infty) \\ f(x) &= \tfrac{1}{x^2}, \quad \forall\, x \in (0,\infty) \\ f(x) &= \tfrac{1}{x}, \quad \forall\, x \in (0,\infty) \end{aligned}$$

Problems 2, 21: The two solutions for $z^2 - az - b = 0$ satisfy

$$-1 < \lambda_1 = \frac{a - \sqrt{a^2+4b}}{2} < 0 < \lambda_2 = \frac{a + \sqrt{a^2+4b}}{2} < 1, \quad |\lambda_1| < \lambda_2$$

Let us fix $x \in \mathbb{R}$. According to PROPOSITION 2, applied for $x_n = f^n(x)$, we obtain

$$f^n(x) = c_1(x) \cdot \lambda_1^n + c_2(x) \cdot \lambda_2^n, \quad \forall\, n \in \mathbb{N}$$

where the two constants depend on x. This implies $\lim_n f^n(x) = 0$, $\forall\, x \in \mathbb{R}$, which prompts

$$0 = \lim_n f^{n+1}(x) = f(\lim_n f^n(x)) = f(0)$$

Due to the equation, we also notice that f is one-to-one and so, being continuous, it must be strictly increasing or strictly decreasing. Again, using the equation, we deduce that f cannot be bounded as $x \to \pm\infty$ and so, f must be onto and hence invertible. This implies also:

$$f^n(x) = c_1(x) \cdot \lambda_1^n + c_2(x) \cdot \lambda_2^n, \quad \forall n \in \mathbb{Z}$$

Now suppose f is increasing and that $c_1(x) \neq 0$. Then for n sufficiently large, $c_1(x) \cdot \lambda_1^{-n}$ dominates $c_2(x) \cdot \lambda_2^{-n}$, so $(f^{-n}(x))_n$ alternates in sign but increases in absolute value as n increases. For suitable N we will have $f^{-N-2}(x) >$ $> f^{-N}(x) > 0$, but $f^{-N-1}(x) < f^{-N+1}(x) < 0$. In other words $A > B$, but $f(A) < f(B)$, contradicting the fact that f is increasing. So we must have $c_1(x) = 0$, $\forall x \in \mathbb{R}$. Hence

$$f(x) = \lambda_2 x, \quad \forall x \in \mathbb{R}$$

Similarly it is treated the case when f is decreasing.

Problem 3: Taking $x = f(y)$ in the main equation, we deduce

$$f(ay) = f^3(y) = a \cdot f(y)$$

which implies $f(0) = 0$ and

$$\frac{f(x)}{x} = \frac{f(ax)}{ax}, \quad \forall\, x \neq 0$$

We can apply PROPOSITION 1 either for $h(x) = \frac{f(x)}{x}$ and $g(x) = ax$ or $h(x) = \frac{f(x)}{x}$ and $g(x) = \frac{x}{a}$, depending on whether $|a| \leq 1$ or $|a| > 1$. Due to the differentiability of f at 0, we obtain:

$$\frac{f(x)}{x} = f'(0), \quad \forall\, x \neq 0$$

which implies that either

$$f(x) = \sqrt{a} \cdot x, \quad \forall\, x \in \mathbb{R}$$

or

$$f(x) = -\sqrt{a} \cdot x, \quad \forall\, x \in \mathbb{R}$$

Problem 4: We can write the equation satisfied by f in the form

$$x^2 \cdot f(x^2) = x \cdot f(x), \quad \forall\, x \in \mathbb{R}$$

For $|x| < 1$ we apply PROPOSITION 1 with $h(x) = x \cdot f(x)$ and $g(x) = x^2$, to obtain

$$x \cdot f(x) = 0, \quad \forall\, x \in [-1, 1]$$

For $|x| > 1$, we take advantage of the fact that f is odd and so it is enough to prove the claim for $x > 1$. In this case we apply the same proposition, but with $h(x) = x \cdot f(x)$ and $g(x) = \sqrt{x}$, to obtain

$$x \cdot f(x) = f(1), \quad \forall\, x > 1$$

Continuity of f then ends the problem.

Problem 5: We apply the first part of the PROPOSITION 2 for $x_n = f(a^n x)$ to deduce

$$f(a^n x) \;=\; b^n \cdot (f(x) - \frac{c}{1-b}) + \frac{c}{1-b}, \quad \forall\, n \in \mathbb{N}$$

Using now the continuity of f, we conclude that $f \equiv \frac{c}{1-b}$.
Problem 6: In this instance we use the second part of PROPOSITION 2 for $x_n \;=\; f(t^n x) - (\frac{t^n x}{1-t^2})^2$, which verifies

$$x_{n+1} - 2x_n + x_{n-1} \;=\; 0, \quad \forall\, n \in \mathbb{N}$$

This implies, using the continuity of f at 0, that:

$$f(x) \;=\; (\frac{x}{1-t^2})^2 + C, \quad \forall x \in \mathbb{R}$$

where $C \in \mathbb{R}$ is a constant.
Problem 7: Consider $h(x) = f(x) + \frac{x^3}{a^3-1} + \frac{x^2}{a^2-1}$. Then our equation can be written as

$$h(x) \;=\; h(ax), \quad \forall\, x \in \mathbb{R}$$

Applying now PROPOSITION 1 for h and $g(x) = ax$, we obtain that:

$$h(x) \;=\; h(0), \quad \forall\, x \in \mathbb{R}$$

Therefore $f(x) = -\frac{x^3}{a^3-1} - \frac{x^2}{a^2-1} + 1986$, $\forall\, x \in \mathbb{R}$.
Problem 9: If we try to apply PROPOSITION 1 for $g(x) \;=\; 2x + 1$, the sequence $(g^n(x))_n$ will be divergent for $\forall\, x \neq 1$. The way out of this is to write the equation in the form

$$f(y) \;=\; f(\frac{y-1}{2}), \quad \forall\, y \in \mathbb{R}$$

due to $g^{-1}(y) = \frac{y-1}{2}$, the inverse of the function g. Direct application of the same proposition yields:

$$f(y) \;=\; f(-1), \quad \forall\, y \in \mathbb{R}$$

Problem 10: A standard sequence analysis shows that, for $g(x) = \sqrt[3]{x^2 + 2x + 12}$, we have

$$\lim_n g^n(x) \;=\; 3, \quad \forall\, x \in \mathbb{R}$$

and so $f(x) = f(3)$, $\forall\, x \in \mathbb{R}$.

Problem 11: As in *Probelm 7* we write the main equation in the form

$$f(tx) - \frac{t^2 x}{t-1} \;=\; f(x) - \frac{tx}{t-1}, \quad \forall\, x \in \mathbb{R}$$

to conclude that:

$$f(x) \;=\; \frac{tx}{t-1} + C, \quad \forall\, x \in \mathbb{R}$$

where $C \in \mathbb{R}$ is a constant.

Problem 12: We pick $x_n = f(a^n x) - \frac{a^n x}{2a^2 - 5a + 3}$ which verifies

$$2x_{n+1} - 5x_n + 3x_{n-1} \;=\; 0, \quad \forall\, n \in \mathbb{N}$$

This implies, by Proposition 2, that:

$$x_n \;=\; (3x_0 - 2x_1) + (2x_1 - 2x_0)(\frac{3}{2})^n, \quad \forall\, n \in \mathbb{N}$$

Using the continuity of the function at 0, it follows that $x_1 \;=\; x_0$ and so, using the previous relation, we obtain also that

$$x_n \;=\; x_0, \quad \forall\, n \in \mathbb{N}$$

Using once again the continuity we conclude:

$$f(x) \;=\; \frac{x}{2a^2 - 5a + 3} + 1, \quad \forall\, x \in \mathbb{R}$$

Problem 13: Fix $x \in \mathbb{R}$ and define the sequence

$$x_0 = x,\; x_{n+1} = \frac{x_n}{2} + 3, \quad \forall\, n \in \mathbb{N}$$

Applying the first part of Proposition 2 for this sequence, it follows that:

$$x_n \;=\; \frac{1}{2^n} \cdot (x - 6) + 6, \quad \forall\, n \in \mathbb{N}$$

and $\lim_n x_n = 6$. It can be seen immediately that $f(6) = 6$ and also

$$f(x_{n+1}) = \frac{f(x_n)}{2} + 3, \quad \forall\, n \in \mathbb{N}$$

So using the same result as above, we infer that:

$$f(x_n) \;=\; \frac{1}{2^n} \cdot (f(x) - 6) + 6, \quad \forall\, n \in \mathbb{N}$$

Combining the formulae for x_n and $f(x_n)$, we write

$$\frac{f(x_n) - f(6)}{x_n - 6} \;=\; \frac{f(x) - f(6)}{x - 6}, \quad \forall\, n \in \mathbb{N}$$

which, due to $\lim_n x_n = 6$ and the differentiability of f at 6, implies

$$f(x) \;=\; \lambda \cdot (x - 6) + 6, \quad \forall\, x \in \mathbb{R}$$

where $\lambda \in \mathbb{R}$ is a fixed constant.

Problems 14, 15: First, we remark that f is even and so we can restrict our analysis to $[0, \infty)$. We split the discussion into three cases:

$0 < c < \frac{1}{4}$: In this instance the equation $x^2 - x + c = 0$ has two real roots

$$x_1 \;=\; \frac{1 - \sqrt{1 - 4c}}{2} \;<\; x_2 \;=\; \frac{1 + \sqrt{1 - 4c}}{2}$$

For $0 \leq x < x_2$, we apply Proposition 1 for $g(x) = x^2 + c$ to obtain:

$$f(x) \;=\; f(x_1), \quad \forall\, x \in [0, x_2)$$

For $x_2 < x$, $\lim_n g^n(x) = \infty$ and, as in *Problem 9*, we will use the inverse $g^{-1}(y) = \sqrt{y - c}$ to deduce

$$f(y) \;=\; f(\lim_n g^{-n}(y)) \;=\; f(x_2), \quad \forall\, y \in (x_2, +\infty)$$

This shows that the only solutions in this case are the constant functions.

$c = \frac{1}{4}$: As before, for $0 \leq x \leq \frac{1}{2}$ we apply Proposition 1 for $g(x) = x^2 + \frac{1}{4}$, while for $\frac{1}{2} < x$ we use $g^{-1}(y) = \sqrt{y - \frac{1}{4}}$ to deduce

$$f(x) \;=\; f(\frac{1}{2}), \quad \forall\, x \in [0, \infty)$$

$c > \frac{1}{4}$: In this case none of the above approaches works. If we define $x_0 = 0$ and $x_{n+1} = x_n^2 + c$, then f is completely defined by prescribing its values on $[x_0, x_1) = [0, c)$, because $f([x_n, x_{n+1})) = f([x_0, x_1))$, $\forall\, n \in \mathbb{N}$.
Problem 16: We use the same technique as in *Problem 4*. For $0 \le x < 1$ we apply PROPOSITION 1 with $g(x) = x^k$ to infer

$$f(x) \;=\; f(0), \quad \forall\, x \in [0,1)$$

while for $x > 1$ our choice will be $g^{-1}(x) = x^{\frac{1}{k}}$, which shows:

$$f(x) \;=\; f(1), \quad \forall\, x \in [1, +\infty)$$

These facts prove that f is constant.
Problem 18: For $x \in [-1, 1]$, there exists a $\theta \in [0, \pi]$ such that $x = \cos\theta$. Substituting $x = \cos\frac{\theta}{2}$ in the main equation we obtain

$$\frac{f(\cos\theta)}{\sin\theta} \;=\; \frac{f(\cos\frac{\theta}{2})}{\sin\frac{\theta}{2}}, \quad \forall\, \theta \in (0, \pi)$$

Here we apply PROPOSITION 1 with $h(x) = \frac{f(\cos x)}{\sin x}$ and $g(x) = \frac{x}{2}$ to deduce:

$$\frac{f(\cos\theta)}{\sin\theta} \;=\; \lim_{x\to 1} \frac{f(x)}{\sqrt{1-x^2}} \;=\; 0, \quad \forall\, \theta \in (0, \pi)$$

due to the fact f is differentiable at 1 and $f'(1) = 0$ (these can be deduced easily from the equation). Therefore $f \equiv 0$ on $[-1, 1]$.
Problem 19: Consider $h(x) = \frac{f(x)}{2^x}$. Then, the main equation can be rewritten as

$$h(2x) \cdot h(3x) \;=\; 1, \quad \forall x \in \mathbb{R}$$

This implies:

$$h(x) \;=\; h(\frac{4}{9}\,x), \quad \forall x \in \mathbb{R}$$

We apply PROPOSITION 1 for h and $g(x) = \frac{4}{9}\,x$ to infer that

$$h(x) \;=\; h(0), \quad \forall x \in \mathbb{R}$$

Using the hypothesis on values of f, it follows that:

$$h(x) = 1, \quad \forall x \in \mathbb{R}$$

which concludes the problem.

Problem 20: We take a similar approach to the ones in *Problems 7,11* to transform the recurrence formula into:

$$f(kx) - \frac{k^2x}{k^2-1} = \frac{1}{k} \cdot (f(x) - \frac{kx}{k^2-1}), \quad \forall\, x \in \mathbb{R}$$

Applying now PROPOSITION 2 for $x_n = f(k^n x) - \frac{k^{n+1}x}{k^2-1}$, we obtain

$$f(k^n x) - \frac{k^{n+1}x}{k^2-1} = \frac{1}{k^n} \cdot (f(x) - \frac{kx}{k^2-1}), \quad \forall\, x \in \mathbb{R}\,, \forall\, n \in \mathbb{Z}$$

Taking now $n \to -\infty$, we conclude that

$$f(x) = \frac{kx}{k^2-1}, \forall\, x \in \mathbb{R}$$

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