Small-ball constants, and exceptional flat points of SPDEs*

Davar Khoshnevisan
University of Utah

Kunwoo Kim
POSTECH

Carl Mueller
University of Rochester

In the memory of Giuseppe Da Prato
(July 23, 1936–October 5, 2023)

June 12, 2024

Abstract

We study small-ball probabilities for the stochastic heat equation with multiplicative noise in the moderate-deviations regime. We prove the existence of a small-ball constant and relate it to other known quantities in the literature. These small-ball estimates are known to imply Chung-type laws of the iterated logarithm (LIL) at typical spatial points; these points can be thought of as "points of flat growth." For this result in a similar context in SPDEs see, for example, the recent work of Chen [3]. We establish the existence of a new family of exceptional spatial points where the Chung-type LIL fails.

Keywords: Stochastic partial differential equations, small-ball probabilities, flat points.
AMS 2020 subject classification: Primary: 60H15, Secondary: 60G17, 60F99.

1 Introduction and main results

Let $X = \{X(t)\}_{t\in\mathcal{T}}$ be a real-valued stochastic process with continuous sample functions, where $\mathcal{T}$ is a compact, separable metric space. By a small-ball probability estimate we mean an approximation of $\log \mathrm{P}\{\sup_{t\in\mathcal{T}} |X(t)| \leqslant \varepsilon\}$ that is ideally valid uniformly for all small ε (say $0 < \varepsilon < 1$). We seek to find asymptotic bounds, and the set $\mathcal{T}$ can also depend on the parameter ε. Such results

*Research supported in part by the NSF grant DMS-2245242 and NRF grants 2019R1A5A1028324 and RS-2023-00244382.

were first developed by Chung [4] for the simple walk and for Brownian motion on $\mathbb{R}$, in order to prove so-called Chung-type laws of the iterated logarithm (LIL). More specifically, Chung's work [4] for a 1-dimensional Brownian motion X (set $\mathcal{T} = \mathcal{T}(\varepsilon) = [0\,,\varepsilon]$ with the usual Euclidean distance) implies that, with probability one,

$$\liminf_{\varepsilon\downarrow 0}\left(\frac{\log|\log\varepsilon|}{\varepsilon}\right)^{1/2}\sup_{s\in[0,\varepsilon]}|X(s)| = \frac{\pi}{\sqrt{8}}.$$

The literature on small-ball probabilities and Chung-type LILs has since grown considerably; see the survey paper of Li and Shao [20] for the development of the theory up to earlier 2000s in the context of Gaussian processes. Dereich, Fehringer, Maroussi, and Scheutzow [6], Klartag and Vershynin [13], and Kuelbs and Li [14] discuss various connections between small-ball probability estimates and other parts of mathematics, specifically approximation theory and quantization problems in Banach space theory. Much of the preceding is concerned mainly with the so-called L^∞ theory for Gaussian measures. A recent survey by Nazarov and Petrova [24] describes up-to-date information, particularly for the closely-related L^2-type theory of small-ball estimates for Gaussian measures. Here, we pursue aspects of some L^∞-type problems for stochastic PDEs of a parabolic type.

Let $\mathbb{T} = [-1\,,1] \cong \mathbb{R}/(2\mathbb{Z})$ denote the one-dimensional torus and consider the following parabolic stochastic PDE (or SPDE) on $\mathbb{R}_+ \times \mathbb{T}$:

$$\begin{cases}\partial_t u(t\,,x) = \partial_x^2 u(t\,,x) + \sigma(u(t\,,x))\dot{W}(t\,,x) & \text{for all } t>0,\ x\in\mathbb{T},\\ \text{subject to } u(0\,,x) = u_0(x) & \text{for all } x\in\mathbb{T}.\end{cases} \tag{1.1}$$

where the forcing is comprised of a space-time white noise $\dot{W} = \{\dot{W}(t\,,x)\}_{t\geqslant 0\,,x\in\mathbb{T}}$ on $\mathbb{R}_+ \times \mathbb{T}$ with an interaction term $\sigma : \mathbb{R} \to \mathbb{R}$ that is a non-random and Lipschitz continuous function, and an initial data $u_0 : \mathbb{T} \to \mathbb{R}$ that is non-random and Lipschitz continuous.

Our goal is to continue the recent analyses of Athreya, Joseph, and Mueller [2], Chen [3], and Foondun, Joseph, and Kim [7] to study small-ball probabilities for a nonlinear system, such as (1.1), and discuss how they relate to sample function properties of the solution to the SPDE (1.1).

Athreya et al prove in [2] that, if in addition $0 < \inf\sigma \leqslant \sup\sigma < \infty$ and $u_0 \equiv 0$, and if the Lipschitz constant for σ is sufficiently small, then there exists a number $K > 1$ such that uniformly for all $\varepsilon \in (0\,,1)$ and $T > 0$,

$$K^{-1}\exp\left(-\frac{KT}{\varepsilon^6}\right) \leqslant \mathrm{P}\left\{\sup_{t\in[0,T]}\|u(t)\|_{C(\mathbb{T})} \leqslant \varepsilon\right\} \leqslant K\exp\left(-\frac{T}{K\varepsilon^6}\right), \tag{1.2}$$

where $\|f\|_{C(\mathbb{T})} = \sup_{x\in\mathbb{T}}|f(x)|$, and $u(t)$ denote the function $x \mapsto u(t\,,x)$ for every $t \geqslant 0$. Foondun et al [7] investigate associated small-ball questions wherein the sup norm $\|\cdots\|_{C(\mathbb{T})}$ is replaced by a Hölder norm. And the main result of Chen [3] implies that, in the same setting as was considered by Athreya et al [2], the following Chung-type LILs holds: There exists a non-random number $\kappa_* \in (0\,,\infty)$ such that

$$\liminf_{\varepsilon \downarrow 0} \frac{(\log|\log\varepsilon|)^{1/6}}{\varepsilon} \sup_{t\in[0,\varepsilon^4]} \sup_{x\in[0,\varepsilon^2]} |u(t\,,x)| = \kappa_* \qquad \text{a.s.} \tag{1.3}$$

In the case that σ is constant — and in fact for much more general Gaussian random fields that are strongly locally non deterministic — some of this type of analysis was carried out by Lee and Xiao [16] slightly earlier.[1]

These references show that, under appropriate conditions, one can establish small-ball probability estimates that are sharp, at the logarithmic level, up to a multiplicative constant [2, 7, 16]. Moreover, one can deduce Chung-type LILs for the solution to (1.1) under natural conditions [3, 16]. In this paper, we study (1.1) dynamically as a process $t \mapsto u(t\,,x)$, one value of x at a time, and show that:

(1) The resulting processes have a tight small-ball estimate with a more-or-less explicit small-ball constant; see Theorem 1.1 below. This appears to be a first example of a family of infinite-dimensional Markov processes that have tight, explicit small-ball probability rates, together with identifiable small-ball constants; and

(2) In addition to a more traditional Chung-type LIL (Corollary 1.2), we prove that one can find exceptional points $x \in \mathbb{T}$ at which other Chung-type LILs hold; see Theorem 1.3. This finding illustrates a new phenomenon that seems to be intimately linked to the infinite-dimensional setting, and also requires novel proof ideas that require multiple applications of the Baire category theorem and delicate, and different, subsequencing arguments for establishing upper and lower bounds that ultimately lead to Chung-type LILs.

In order to describe our results, let $F = \{F(t)\}_{t\geqslant 0}$ denote a *fractional Brownian motion of index* $1/4$; see Mandelbrot and VanNess [23]. That is, F is a continuous, centered Gaussian process such that $F(0) = 0$ and

$$\mathrm{E}\left(|F(t) - F(s)|^2\right) = |t-s|^{1/2} \qquad \text{for all } s, t \geqslant 0.$$

We can deduce from the works of Li [18], Li and Linde [19], and Shao [25] that

$$\lambda = -\lim_{\varepsilon\downarrow 0} \varepsilon^4 \log \mathrm{P}\left\{ \sup_{t\in[0,1]} |F(t)| \leqslant \varepsilon \right\} \quad \text{exists and is in } (0\,,\infty). \tag{1.4}$$

The number λ is the so-called *small-ball constant* for F. It should be possible to combine (1.4) and Monte Carlo methods in order to find a reasonable approximation to λ, but the exact numerical value of λ is not known.

[1] Small–ball probability estimates are also available for a particular family of Gaussian processes that solve semilinear hyperbolic SPDEs. They have a different form from the results here and in Athreya, Joseph, and Mueller [2], Chen [3], and Foondun, Joseph, and Kim [7], and require very different methods of analysis; see Martin [22], which is based in part on a celebrated earlier theorem of Talagrand [28] on the small-ball problem for the Brownian sheet.

Theorem 1.1. *In addition to the preceding assumptions, suppose that σ is bounded. Choose and fix an unbounded, non-increasing, deterministic function $\phi : (0\,,1) \to (0\,,\infty)$ that satisfies the following:*

$$\phi(\varepsilon) = O\left(|\log \varepsilon|\right) \quad \text{as } \varepsilon \downarrow 0. \tag{1.5}$$

Then, for every $x \in \mathbb{T}$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\phi(\varepsilon)} \log \mathrm{P}\left\{ \sup_{t\in[0,\varepsilon]} |u(t\,,x) - u_0(x)| \leqslant \left(\frac{\varepsilon}{\phi(\varepsilon)}\right)^{1/4} \right\} = -\frac{2\lambda}{\pi} |\sigma(u_0(x))|^4,$$

where λ is the small-ball constant that was introduced in (1.4).

As far as we know, the first paper on small-ball probabilities was Chung [4], where the object of main interest was the simple walk on $\mathbb{Z}$ and, through that, the Brownian motion on the line. Chung [4] was also the first to notice that small-ball probabilities can be used to yield a matching law of the iterated logarithm (LIL). Thus, it should not come as a surprise that Theorem 1.1 too implies a Chung-type LIL. Though we pause to point out that additional effort is required to show the next corollary, as it is valid under fewer technical hypotheses than is Theorem 1.1.

Corollary 1.2. *Regardless of whether or not σ is bounded,*

$$\liminf_{\varepsilon\downarrow 0} \left(\frac{\log|\log \varepsilon|}{\varepsilon}\right)^{1/4} \sup_{t\in[0,\varepsilon]} |u(t\,,x) - u_0(x)| = \left(\frac{2\lambda}{\pi}\right)^{1/4} |\sigma(u_0(x))|, \tag{1.6}$$

a.s. for every $x \in \mathbb{T}$, where λ was defined in (1.4).

To be sure of the order of the quantifiers, we note that Corollary 1.2 says that for every non-random point $x \in \mathbb{R}$ there exists a P-null set off of which (1.6) holds. We may view such points x as points of [relatively] "flat growth," for example as compared with points where iterated logarithm fluctuations are observed; see [8]. Corollary 1.2 and Fubini's theorem together show that the collection of all points $x \in \mathbb{T}$ that satisfy (1.6) has full Lebesgue/Haar measure. The remainder of our effort is concerned with studying many of the points $x \in \mathbb{T}$ that are exceptional in the sense that they fail to satisfy (1.6). A standard method for finding such points is to appeal to the theory of limsup random fractals [11] and adapt it to the present small-ball setting for SPDEs. For large-ball problems, this adaptation was done in [8], and we feel that similar methods will yield exceptional points $x \in \mathbb{T}$ for which the rate const $\times\, (\varepsilon^{-1} \log|\log \varepsilon|)^{1/4}$ is replaced by rate const $\times\, (\varepsilon^{-1}|\log \varepsilon|)^{1/4}$ for suitable choices of "const." We have not tried to do that here. Instead, we document the existence of a more subtle family of exceptional points $x \in \mathbb{T}$ whose existence requires new proof ideas. In order to present that family we need some notation.

From now on, we will use the symbol $\rightsquigarrow$ to denote subsequential limits. More precisely, whenever $a, a_1, a_2, \dots \in \mathbb{R}$, then we might write "$a_n \rightsquigarrow a$ as $n \to \infty$" as shorthand for "$\liminf_{n\to\infty} |a_n - a| = 0$."

Theorem 1.3. *Choose and fix a non-random, nonnegative, extended real number $\chi \in [0\,,\infty]$. Then, regardless of whether or not σ is bounded, there exists a deterministic sequence $\varepsilon_n \to 0$ such that there a.s. exists a random $x \in \mathbb{T}$ that satisfies*

$$\left(\frac{\log|\log \varepsilon_n|}{\varepsilon_n}\right)^{1/4} \sup_{t\in[0,\varepsilon_n]} |u(t\,,x) - u_0(x)| \rightsquigarrow \chi^{1/4}|\sigma(u_0(x))| \text{ as } n \to \infty. \tag{1.7}$$

If $\chi \in [0\,, 2\lambda/\pi]$, then there in fact a.s. exists a random $x \in \mathbb{T}$ such that

$$\liminf_{\varepsilon\downarrow 0} \left(\frac{\log|\log \varepsilon|}{\varepsilon}\right)^{1/4} \sup_{t\in[0,\varepsilon]} |u(t\,,x) - u_0(x)| = \chi^{1/4}|\sigma(u_0(x))|. \tag{1.8}$$

We pause to insert a few problems that have eluded us.

Open Problem 1. Can (1.7) be upgraded to (1.8) when $\chi > 2\lambda/\pi$? We suspect the answer is "no."

Open Problem 2. An informal comparison with limsup random fractals suggests that if we replaced $\log|\log \varepsilon_n|$ and $\log|\log \varepsilon|$ respectively by $|\log \varepsilon_n|$ and $|\log \varepsilon|$ in the left-hand sides of (1.7) and (1.8), then the set of $x \in \mathbb{T}$ where the left-hand sides of (1.7) and/or (1.8), altered as mentioned, are equal to a given number c always has non-trivial Hausdorff dimension for a continuum of suitable choices of c. Moreover, by suitably adjusting c, we can ensure that those Hausdorff dimensions can take any value in $[0\,,1]$. The presence of $\log\log$ instead of $\log$ would imply that there should be many more points $x \in \mathbb{T}$ where (1.7) and (1.8) holds. Therefore, we conjecture that the set of points $x \in \mathbb{T}$ at which either condition (1.7) or (1.8) holds has full Hausdorff dimension.

Above and throughout, we view $\mathbb{T}$ as the set $[-1\,,1]$ and identify it with the abelian group $\mathbb{R}/(2\mathbb{Z})$ in the customary manner: We use the additive notation for $\mathbb{T}$, and in fact move back and forth from interpreting $\mathbb{T}$ as the real interval $[-1\,,1]$ to the abelian group $\mathbb{R}/(2\mathbb{Z})$. In particular, we write "$x - y$" instead of "$x - y \pmod 2$" or "xy^{-1}" for $x, y \in \mathbb{T}$, and designate 0 (not 1) as the group identity. We also denote by $\mathrm{d}x$ an infinitesimal element of a Haar measure on $\mathbb{T}$ and do not distinguish between the Lebesgue measure on $[-1\,,1]$, normalized to have total mass 2 and the Haar measure on $\mathbb{T}$, similarly normalized.

We frequently set $\log_+(a) = \log(a \vee \exp(\mathrm{e}))$ for all $a \geqslant 0$.

Suppose A is a compact metric space and $g : A \to \mathbb{R}$ is continuous. Then we often write $\|g\|_{C(B)}$ in place of $\sup_{x\in B}|g(x)|$ whenever $B \subseteq A$. When $B = [a\,,b]$ is a subinterval of $\mathbb{R}$ we might write $\|g\|_{C[a,b]}$ in place of $\|g\|_{C([a,b])}$. Throughout, the $L^k(\Omega)$-norm of a random variable $Z \in L^k(\Omega)$ is denoted by $\|Z\|_k := \{\mathrm{E}\left(|Z|^k\right)\}^{1/k}$ for all $1 \leqslant k < \infty$.

Let us conclude the Introduction with an outline of this paper. In Section 2, we investigate small ball probabilities for the constant-coefficient case $\sigma \equiv 1$ in (1.1). In Section 3, we consider the linearization of the nonlinear equation (1.1) and present detailed estimates for the difference between the nonlinear one and its linearization (see Proposition 3.1). We will use these estimates, along with the results from Section 2, to prove Theorem 1.1. Sections 4 and 5 are dedicated to the proofs of Corollary 1.2 and Theorem 1.3 by using Theorem 1.1 and introducing some novel ideas.

2 The linear case

As is commonly done [5,30], we interpret the SPDE (1.1) as the following random integral equation:

$$u(t\,,x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(u(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y), \tag{2.1}$$

for all $t > 0$ and $x \in \mathbb{T}$, where p denotes the heat kernel on $\mathbb{T}$; that is, for all $r > 0$ and $x, y \in \mathbb{T}$,

$$p_r(x\,,y) = \sum_{n=-\infty}^{\infty} G_r(x - y + 2n), \text{ where } G_r(a) = \frac{\exp\{-a^2/(4r)\}}{\sqrt{4\pi r}}, \tag{2.2}$$

for every $a \in \mathbb{R}$. It is well known that in short times, the increment $u(t\,,x)-u_0(x)$ of the solution to (1.1) is very close to a constant multiple of the solution to the following linearized version of (1.1); see [9,12]. Therefore, we reserve the letter Z specifically for the solution to the following SPDE.

$$\begin{aligned} &\partial_t Z(t\,,x) = \partial_x^2 Z(t\,,x) + \dot{W}(t\,,x) && \text{for all } t > 0,\ x \in \mathbb{T},\\ &\text{subject to } Z(0\,,x) = 0 && \text{for all } x \in \mathbb{T}. \end{aligned}$$

According to (2.1), we may write the solution Z as the following Wiener integral process,

$$Z(t\,,x) = \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\, W(\mathrm{d}s\,\mathrm{d}y) \qquad \text{for all } t > 0, \tag{2.3}$$

where the kernel p was defined in (2.2). In this section, we study the specialization of Theorem 1.1 to the Gaussian random field Z, viewed as an approximation for the process u.

Proposition 2.1. *Choose and fix an unbounded, non-increasing, deterministic function $\phi : (0\,,1) \to (0\,,\infty)$ that satisfies (1.5). Then,*

$$\lim_{\varepsilon\to 0^+} [\phi(\varepsilon)]^{-1} \log \mathrm{P}\left\{ \|Z(t)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4} \right\} = -2\lambda/\pi,$$

where λ was defined in (1.4).

As Z is a nice Gaussian random field, we will prove Proposition 2.1 by following a similar route to that taken in [16], and then appeal to the results in [19,21,25] in order to prove the existence of the small-ball constant and then to identify that constant. It should be pointed out that scaling plays a role in the methods of the latter three references. Thus, a certain amount of additional effort is expended in order to overcome the lack of scaling for Z.

2.1 The linear heat equation on free space

So far, $\dot{W}(t\,,x) = \partial^2_{t,x} W(t\,,x)$ where W denotes a space-time Brownian sheet that is indexed by $(t\,,x) \in \mathbb{R}_+ \times [-1\,,1]$. Without loss of generality, and in a standard manner, we can extend the domain of definition of the Brownian sheet W so that it is in fact a space-time Brownian sheet on the full space $\mathbb{R}_+ \times \mathbb{R}$. This canonically extends the domain of the definition of the white noise $\dot{W}$ to all of $\mathbb{R}_+ \times \mathbb{R}$ as well. With this in mind, let us consider the stochastic heat equation,

$$\begin{cases} \partial_t H(t\,,x) = \partial^2_x H(t\,,x) + \dot{W}(t\,,x) & \text{for all } t > 0,\ x \in \mathbb{R}, \\ \text{subject to } H(0\,,x) = 0 & \text{for all } x \in \mathbb{R}. \end{cases} \tag{2.4}$$

The solution to this SPDE is, by virtue of definition and similarly to (2.1), the following Gaussian random field which is defined as a Wiener integral process,

$$H(t\,,x) = \int_{(0,t)\times\mathbb{R}} G_{t-s}(x\,,y)\, W(\mathrm{d}s\,\mathrm{d}y) \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{2.5}$$

where G was defined in (2.2). The following result is a precise small-ball estimate for the process H at a given spatial point, say $x = 0$, in terms of the same small-ball constant λ that was introduced in (1.4).

Proposition 2.2. $\lim_{\varepsilon\downarrow 0} \varepsilon^4 \log \mathrm{P}\{\sup_{t\in[0,1]} |H(t\,,0)| \leqslant \varepsilon\} = -2\lambda/\pi$.

It is well-known that one can decompose $t \mapsto H(t\,,0)$ as a constant multiple of a fractional Brownian motion F with index $1/4$ plus a continuous Gaussian random field T that is independent of Z and has C^∞ sample functions away from $t = 0$; see [17] (Lemma 2.3 below). One can expect the small-ball probability of the rougher process F to dominate that of the smoother Gaussian process T. Therefore, it remains to make this assertion rigorous. This effort is complicated by the fact that, near $t = 0$, the random field T is not smooth; in fact, T and F are equally smooth locally near $t = 0$. The crux of the argument hinges on estimating how quickly T begins to "look like a C^∞ process," together with a suitable quantitative way to interpret the quoted sentence. This effort will be summarized in Proposition 2.4 below. Proposition 2.2 is proved subsequently in §2.3.

We begin by studying an auxiliary process T.

2.2 An auxiliary Gaussian process

Let V denote a one-parameter white noise on $\mathbb{R}$; that is, V is the weak derivative of a two-sided Brownian motion indexed by $\mathbb{R}$. Consider the centered Gaussian process $T = \{T(t)\}_{t\geqslant 0}$ that is defined by $T(0) = 0$ and

$$T(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1 - \mathrm{e}^{-tz^2/2}}{z} \right) V(\mathrm{d}z) \qquad \text{for } t > 0. \tag{2.6}$$

We shall assume throughout that V and the noise W in (2.4) are independent. Let us recall the following structural decomposition of $H(t)$ in terms of the process T and a fractional Brownian motion of index $1/4$.

Lemma 2.3 (Lei and Nualart [17])**.** *The centered Gaussian process T is continuous. Moreover, its restriction to $[\eta\,,\infty)$ is almost surely C^∞ for every $\eta > 0$. Finally,*

$$F(t) = \frac{H(t\,,0) + T(t)}{(2/\pi)^{1/4}} \qquad [t \geqslant 0]$$

defines a standard fractional Brownian motion of index $1/4$.

The final part of Lemma 2.3 is proved via a direct computation of the covariance function of F. With this aim in mind, let d denote the usual canonical distance that is associated to the Gaussian process T; that is,

$$d(s\,,t) = \|T(t) - T(s)\|_2 \qquad \text{for all } s,t \geqslant 0. \tag{2.7}$$

The regularity assertions of Lemma 2.3 were proved by showing that:

$$\begin{aligned} &\textbf{(a) } d(s\,,t) \lesssim |t-s|^{1/4} \text{ uniformly for all } s,t \geqslant 0; \text{ and} \\ &\textbf{(b) } d(s\,,t) \leqslant C_\eta |t-s| \text{ whenever } s,t \geqslant \eta, \end{aligned} \tag{2.8}$$

where C_η is a number that depends on η but not $(s\,,t)$. The main result of this section is the following lower bound on the small-ball probability of T. Note that, in addition to the assertions in Lemma 2.3, parts (a) and (b) of (2.8) show that while F is smooth away from the origin, it scales roughly as fractional Brownian motion of index $1/4$ near the origin. Nevertheless, the following shows that the small-ball probability of T is significantly larger than that of a fractional Brownian motion with index $1/4$.

Proposition 2.4. *There exists a constant $L > 1$ such that*

$$\mathrm{P}\left\{\|T\|_{C[0,1]} \leqslant r\right\} \geqslant L^{-1}\exp(-L/r) \qquad \textit{for all } r > 0.$$

The proof of Proposition 2.4 requires a few preliminary steps. The first is a careful estimate on the canonical distance that improves (2.8); it is in fact the following interpolation between **(a)** and **(b)** of (2.8).

Lemma 2.5. *There exists a number $c > 0$ such that*

$$d(s\,,t) \leqslant c|t-s|^{1/4}\left[1 \wedge \left(\frac{|t-s|}{s\wedge t}\right)^{3/4}\right] \qquad \textit{for all } s,t > 0.$$

Proof. The definition of the Wiener integral in (2.6) yields the following: For every $t,\varepsilon \geqslant 0$,

$$d(t+\varepsilon\,,t) = \frac{\sqrt{t}}{\pi}\int_0^\infty \left(\frac{1-\mathrm{e}^{-\varepsilon y^2/(2t)}}{y}\right)^2 \mathrm{e}^{-y^2}\,\mathrm{d}y.$$

Since $1-\exp(-c)\leqslant 1\wedge c$ for all $c\geqslant 0$, this yields

$$[d(t+\varepsilon\,,t)]^2\leqslant\frac{\varepsilon^2}{4\pi t^{3/2}}\int_0^{\sqrt{2t/\varepsilon}}y^2\mathrm{e}^{-y^2}\,\mathrm{d}y+\frac{\sqrt{t}}{\pi}\int_{\sqrt{2t/\varepsilon}}^\infty\frac{\mathrm{e}^{-y^2}}{y^2}\,\mathrm{d}y=J_1+J_2,$$

notation being clear from context. On one hand, uniformly for all $t\geqslant\varepsilon>0$,

$$J_1\leqslant\frac{\varepsilon^2}{4\pi t^{3/2}}\int_0^\infty y^2\mathrm{e}^{-y^2}\,\mathrm{d}y\propto\frac{\varepsilon^2}{t^{3/2}}\qquad\text{and}$$
$$J_2\leqslant\frac{\sqrt{t}}{\pi}\int_{\sqrt{2t/\varepsilon}}^\infty\frac{\mathrm{e}^{-y^2}}{y^2}\,\mathrm{d}y\asymp\frac{\varepsilon^{3/2}}{t}\,\mathrm{e}^{-2t/\varepsilon}\lesssim\frac{\varepsilon^2}{t^{3/2}},$$

where we have appealed to l'Hôpital's rule to estimate J_2, as well as the fact that $A^{1/2}\exp(-A)\lesssim 1$ uniformly for all $A\geqslant 1$. On the other hand, when $\varepsilon\geqslant t$, we have

$$J_1\leqslant\frac{\varepsilon^2}{4\pi t^{3/2}}\int_0^{\sqrt{2t/\varepsilon}}y^2\,\mathrm{d}y\lesssim\sqrt{\varepsilon},\ J_2\leqslant\frac{\sqrt{t}}{\pi}\int_{\sqrt{2t/\varepsilon}}^\infty\frac{\mathrm{e}^{-y^2}}{y^2}\,\mathrm{d}y\lesssim\sqrt{t}\int_{\sqrt{2t/\varepsilon}}^\infty\frac{\mathrm{d}y}{y^2}\lesssim\sqrt{\varepsilon},$$

valid uniformly for all $\varepsilon\geqslant t>0$. The lemma follows from putting together the two cases. □

We plan to use Lemma 2.5 to compute a sharp metric entropy bound for the process T on $[0\,,\varepsilon]$. In order to do that, we will need a good covering method which will turn out to depend on the solution to a nice difference equation. Choose and fix a number $c>0$ and consider the initial-value problem,

$$g'=cg^{3/4}\quad\text{on }(0\,,\infty),\quad\text{subject to }g(0)=0,$$

whose only strictly increasing solution is $g(t)=(ct/4)^4$. The following is an asymptotically analogous result for a discrete version of the preceding ODE.

Proposition 2.6. *Choose and fix some $c>0$ and define $a_1=1$ and $a_{n+1}=a_n+ca_n^{3/4}$ for every $n\in\mathbb{N}$. Then, $a_n\sim(cn/4)^4$ as $n\to\infty$ in $\mathbb{N}$.*

Proof. By induction, $a_{n+1}\geqslant a_n$ for all $n\in\mathbb{N}$. We first show that $\lim_{m\to\infty}a_m=\infty$. Indeed, $a_n\geqslant a_1=1$ for all $n\in\mathbb{N}$ and hence $a_{n+1}-a_n=ca_n^{3/4}\geqslant c$. This proves the sub-optimal result that $a_n\geqslant cn$ for all $n\in\mathbb{N}$, which is nevertheless good enough to ensure that $a_n\to\infty$ as $n\to\infty$.

Now we extend the sequence $\{a_n\}_{n\in\mathbb{N}}$ to a function $f:[1\,,\infty)\to[1\,,\infty)$ by linear interpolation. Specifically, let

$$f(t)=a_{\lfloor t\rfloor}+c(t-\lfloor t\rfloor)a_{\lfloor t\rfloor}^{3/4}\qquad\text{for all }t\geqslant 1,$$

where $\lfloor t\rfloor$ denotes the greatest integer $\leqslant t$. Note that f is differentiable on $(1\,,\infty)\setminus\mathbb{N}$, and

$$f'(s)=ca_{\lfloor s\rfloor}^{3/4}=c[h(s)]^{3/4}[f(s)]^{3/4}\qquad\text{for all }s\in(1\,,\infty)\setminus\mathbb{N},\tag{2.9}$$

where

$$h(t) = \frac{a_{\lfloor t\rfloor}}{f(t)} = \frac{a_{\lfloor t\rfloor}}{a_{\lfloor t\rfloor} + c(t-\lfloor t\rfloor)a_{\lfloor t\rfloor}^{3/4}} \qquad \text{for every } t \geqslant 1.$$

Since $0 \leqslant t - \lfloor t\rfloor \leqslant 1$ and $a_{\lfloor t\rfloor} \to \infty$ as $t \to \infty$, we have $h(t) \to 1$ boundedly as $t \to \infty$. And of course $h(t) \leqslant 1$ for all $t \geqslant 1$. We can write (2.9) as $\mathrm{d}f/f^{3/4} = ch^{3/4}\,\mathrm{d}s$ and integrate from 1 to t [ds] in order to find that

$$f(t) = \left(1 + \frac{c}{4}\int_1^t [h(s)]^{3/4}\,\mathrm{d}s\right)^4,$$

for every $t \in [1\,,\infty) \setminus \mathbb{N}$ and hence every $t \geqslant 1$ by continuity. Since $f(n) = a_n$ for all $n \in \mathbb{N}$ and $h(s) \to 1$ boundedly as $s \to \infty$, this proves the result. $\square$

Recall the Gaussian process T and associated intrinsic metric d respectively from (2.6) and (2.7). Let $\mathcal{N}$ denote the metric entropy of the process $\{T(t)\}_{t\in[0,1]}$. That is, for every $r > 0$, define $\mathcal{N}(r)$ to be the minimum number of open d-balls of radius $r > 0$ needed to cover the closed interval $[0\,,1]$. We shall recall the following result which is stated explicitly in Talagrand [29, Lemma 2.2], whose proof follows from combining the entropy estimates of Talagrand [27, Section 3] together with a deep theorem of Kuelbs and Li [14]. A detailed concrete proof of the following can be found in Section 7 of the lecture notes by Ledoux [15].[2]

Lemma 2.7 (Talagrand [27])**.** *Suppose $\mathcal{N} \leqslant \psi$ on $(0\,,1)$ for a function $\psi : (0\,,1) \to \mathbb{R}_+$ that satisfies $\psi(r) \asymp \psi(r/2)$ uniformly for all $r \in (0\,,1)$. Then there exists $K > 0$ such that $\mathrm{P}\{\|T\|_{C[0,1]} \leqslant \varepsilon\} \geqslant K^{-1}\exp(-K\psi(\varepsilon))$ for all $\varepsilon > 0$.*

Armed with Lemma 2.7, we can present the following.

Proof of Proposition 2.4. It suffices to prove that the asserted inequality of the proposition is valid for all $\varepsilon \in (0\,,1/2)$. With that aim in mind, let us choose and fix a real number $\varepsilon \in (0\,,1)$ [N.B.: not $\varepsilon \in (0\,,1/2)$], and define $a_0 = 0$, $a_1 = 1$. Then define iteratively $a_{j+1} = a_j + ca_j^{3/4}$ for all $j \in \mathbb{N}$, where $c > 0$ was defined in Lemma 2.5. Also define

$$t_j = a_j(2\varepsilon/c)^4 \qquad \text{for } j \in \mathbb{Z}_+.$$

According to Lemma 2.5, $d(t_0\,,t_1) \leqslant 2\varepsilon$, and

$$d(t_j\,,t_{j+1}) \leqslant c|t_{j+1} - t_j|/t_j^{3/4} = 2\varepsilon \qquad \text{for all } j \in \mathbb{N}.$$

In other words, $d(t_j\,,t_{j+1}) \leqslant 2\varepsilon$ for all $j \in \mathbb{Z}_+$. It follows readily from this that $\mathcal{N}(\varepsilon) \leqslant 1 + \max\{j \geqslant 0 : a_j \leqslant (c/(2\varepsilon))^4\}$, uniformly for all $\varepsilon \in (0\,,1)$. Proposition 2.6 assures us that $a_j \gtrsim j^4$ uniformly for all $j \in \mathbb{N}$ large. Therefore, we can see that there exists $C > 0$ such that $\mathcal{N}(\varepsilon) \leqslant C/\varepsilon$ uniformly for every $\varepsilon \in (0\,,1)$. Apply Lemma 2.7 with $\psi(\varepsilon) = C/\varepsilon$ to conclude the proof. $\square$

[2] In fact, the 2-parameter process $(T \ominus T)(t\,,s) = T(t) - T(s)$ satisfies $\mathrm{P}\{\|T \ominus T\|_{C([0,1]^2)} \leqslant \varepsilon\} \geqslant K^{-1}\exp\{-K\psi(\varepsilon)\}$. Lemma 2.7 follows from this formulation since $T(0) = 0$.

2.3 Proof of Proposition 2.2

With the results of the preceding subsections under way, we are ready to verify Proposition 2.2. But first we pause to recall the following specialization of [1].

Lemma 2.8 (Anderson [1]). *If X is a centered Gaussian random variable X with values in $C[0\,,1]$, then* $\mathrm{P}\{\|X+f\|_{C[0,1]} \leqslant r\} \leqslant \mathrm{P}\{\|X\|_{C[0,1]} \leqslant r\}$ *for every* $f \in C[0\,,1]$ *and* $r > 0$.

Define T as was done in (2.6), using a noise V that is independent of W and hence also of the solution H to (2.4). Let F be the corresponding fractional Brownian motion with index $1/4$, as was introduced in Lemma 2.3. Because H and T are independent processes, we first condition on T and then appeal to Anderson's inequality (Lemma 2.8) in order to see that, for all $\varepsilon > 0$,

$$\begin{aligned}\mathrm{P}\left\{\|F\|_{C[0,1]} \leqslant (\pi/2)^{1/4}\varepsilon\right\} &\leqslant \sup_{f\in C[0,1]} \mathrm{P}\left\{\|H(\cdot\,,0)+f\|_{C[0,1]} \leqslant \varepsilon\right\} \\ &= \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,1]} \leqslant \varepsilon\right\}.\end{aligned} \tag{2.10}$$

This yields a lower bound on the small-ball probability for $t \mapsto H(t\,,0)$ in terms of the better-studied small-ball probability for fractional Brownian motion. For a complementary inequality let us choose and fix some number $\rho \in (0\,,1)$ and observe from the independence of H and T that for all $\varepsilon > 0$,

$$\mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,1]} \leqslant \rho\varepsilon\right\}\cdot\mathrm{P}\left\{\|T\|_{C[0,1]} \leqslant (1-\rho)\varepsilon\right\} \leqslant \mathrm{P}\left\{\|F\|_{C[0,1]} \leqslant (\pi/2)^{1/4}\varepsilon\right\}.$$

Apply Proposition 2.4 with $r = (1-\rho)\varepsilon$ in order to find a number $L > 0$ such that for all $\varepsilon > 0$,

$$\mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,1]} \leqslant \rho\varepsilon\right\} \leqslant L\mathrm{P}\left\{\|F\|_{C[0,1]} \leqslant (\pi/2)^{1/4}\varepsilon\right\} \cdot \mathrm{e}^{L/[(1-\rho)\varepsilon]}. \tag{2.11}$$

Relabel ε as $\rho\varepsilon$ in order to see from (2.10) and (2.11) that for all $\varepsilon > 0$,

$$\begin{aligned}\mathrm{P}\left\{\|F\|_{C[0,1]} \leqslant (\pi/2)^{1/4}\varepsilon\right\} &\leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,1]} \leqslant \varepsilon\right\} \\ &\leqslant L\mathrm{P}\left\{\|F\|_{C[0,1]} \leqslant (\pi/2)^{1/4}\varepsilon/\rho\right\}\mathrm{e}^{L\rho/[(1-\rho)\varepsilon]}.\end{aligned}$$

Apply (1.4) to see that, as $\varepsilon \downarrow 0$,

$$-\frac{2\lambda+o(1)}{\pi} \leqslant \varepsilon^4 \log \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,1]} \leqslant \varepsilon\right\} \leqslant -\frac{2\lambda\rho^4+o(1)}{\pi}.$$

Since $\rho \in (0\,,1)$ was arbitrary, we may let ρ tend upward to 1 in order to complete the proof of Proposition 2.2. □

2.4 Proof of Proposition 2.1

We now prove Proposition 2.1. The first step is to establish the analogue of Proposition 2.1 for the more regular process H. The following summarizes that result.

Lemma 2.9. *For every unbounded, non-increasing, deterministic function* $\phi : (0\,,1) \to (0\,,\infty)$,

$$\lim_{\varepsilon\to 0^+} [\phi(\varepsilon)]^{-1} \log \mathrm{P}\left\{ \|H(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4} \right\} = -2\lambda/\pi.$$

Proof. The random field H inherits scaling properties from white noise and the free-space heat operator. In particular,

$$\left\{ \rho^{-1/4} H(\rho t\,, \rho^{1/2} x)\,;\, t \geqslant 0,\ x \in \mathbb{R} \right\} \stackrel{d}{=} \{H(t\,,x)\,;\, t \geqslant 0,\ x \in \mathbb{R}\}, \tag{2.12}$$

for all $\rho > 0$. In particular, for every $\varepsilon > 0$,

$$\mathrm{P}\left\{ \|H(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4} \right\} = \mathrm{P}\left\{ \|H(\cdot\,,0)\|_{C[0,1]} \leqslant [\phi(\varepsilon)]^{-1/4} \right\}.$$

The result follows from the above, Proposition 2.2, and the fact that $\phi(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$. □

In the next step in the proof of Proposition 2.1 we show that $H(t\,,0)$ is very close to $Z(t\,,0)$. Since H and Z are Gaussian, it suffices to measure closeness using the variance.

Lemma 2.10. $\mathrm{E}(|H(t\,,0) - Z(t\,,0)|^2) \leqslant 5t$ *for all* $t \geqslant 0$.

Proof. We can compare (2.3) to (2.5) in order to see that $\mathrm{E}(|H(t\,,0) - Z(t\,,0)|^2) \leqslant 4J_1 + J_2$, where

$$J_1 = \int_0^t \frac{\mathrm{d}s}{4\pi s} \int_{-1}^1 \mathrm{d}y\, \left| \sum_{n=1}^\infty \exp\left(-\frac{(y+2n)^2}{4s} \right) \right|^2,$$
$$J_2 = \int_0^t \mathrm{d}s \int_{|y|>1} \mathrm{d}y\, [G_{t-s}(y)]^2.$$

Both terms can be estimated by direct means. Indeed,

$$J_1 \leqslant \int_0^t \frac{\mathrm{d}s}{\pi s} \int_{-1}^1 \mathrm{d}y\, \left| \sum_{n=1}^\infty \exp\left(-\frac{n^2}{4s} \right) \right|^2 = \frac{2}{\pi} \int_0^t \left| \sum_{n=1}^\infty \exp\left(-\frac{n^2}{4s} \right) \right|^2 \frac{\mathrm{d}s}{s}, \quad \text{and}$$
$$J_2 = \int_0^t \mathrm{d}s \int_{|y|>1} \mathrm{d}y\, [G_s(y)]^2 = \int_0^t \frac{\mathrm{d}s}{\sqrt{8\pi s}} \int_{|z|>\sqrt{2}} \mathrm{d}z\, G_s(z),$$

thanks to the fact that $[G_s(y)]^2 = (4\pi s)^{-1/2} G_s(y\sqrt{2})$ for all $s > 0$ and $y \in \mathbb{R}$, and a change of variables. Since $\sum_{n=1}^\infty \exp\{-n^2/(4s)\} \leqslant \int_0^\infty \exp\{-y^2/(4s)\}\,\mathrm{d}y = \sqrt{\pi s}$, it follows that $J_1 \leqslant 2t$. And a familiar Gaussian tail bound yields

$$\int_{|z|>\sqrt{2}} \mathrm{d}z\, G_s(z) \leqslant \exp\{-1/(2s)\},$$

and hence

$$J_2 \leqslant \int_0^t \exp\left(-\frac{1}{2s}\right)\frac{\mathrm{d}s}{\sqrt{8\pi s}} \leqslant \frac{t}{\sqrt{8\mathrm{e}\pi}} \leqslant t,$$

thanks to the elementary fact that $s^{-1/2}\exp\{-1/(2s)\} \leqslant \mathrm{e}^{-1/2}$ for all $s \geqslant 0$. Combine the bounds for J_1 and J_2 in order to deduce the lemma. □

In the next stage of the proof of Proposition 2.1 we show that the somewhat crude approximation offered by Lemma 2.10 is good enough to yield the closeness of the respective small-ball probabilities of $t \mapsto H(t\,,0)$ and $t \mapsto Z(t\,,0)$. In fact, a little extra effort produces the following much better result.

Lemma 2.11. *Let $\phi : (0\,,\infty) \to \mathbb{R}_+$ be an unbounded, nonincreasing, deterministic function that satisfies the local growth condition (1.5). Then,*

$$\limsup_{\varepsilon\downarrow 0} \sqrt{\varepsilon\phi(\varepsilon)}\log \mathrm{P}\left\{\|H - Z\|_{C([0,\varepsilon]\times\mathbb{T})} \geqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} \leqslant -\frac{1}{10}.$$

Proof. Let $D(t\,,x) = H(t\,,x) - Z(t\,,x)$ for all $t \geqslant 0$. Clearly, D is a continuous and centered Gaussian process with

$$\begin{aligned} \mathrm{E}\left(|D(t\,,x)|^2\right) &\leqslant 5t, \text{ and} \\ \mathrm{E}\left(|D(t\,,x) - D(s\,,y)|^2\right) &\lesssim |t-s|^{1/2} + |x-y|, \end{aligned} \tag{2.13}$$

valid uniformly for all $s,t \in [0\,,1]$ and $x,y \in \mathbb{T}$. The first inequality in (2.13) is from Lemma 2.10 and stationarity, and the second is a well-known fact that is used frequently in the regularity theory of SPDEs [30, pp. 319–320]. Therefore, Dudley's theorem yields a positive number c such that

$$0 \leqslant \mathrm{E}\sup_{s\in[0,\varepsilon]}\sup_{y\in\mathbb{T}} D(s\,,y) \leqslant c\,|\varepsilon\log\varepsilon|^{1/2} \qquad \text{uniformly for all } \varepsilon \in (0\,,1/\mathrm{e}];$$

see for example Ledoux [15]. Moreover, by concentration of measure [15] and (2.13),

$$\begin{aligned} &\mathrm{P}\left\{\sup_{s\in[0,\varepsilon]}\sup_{y\in\mathbb{T}}\left|D(s\,,y) - \mathrm{E}\left[\sup_{s\in[0,\varepsilon]}\sup_{y\in\mathbb{T}} D(s\,,y)\right]\right| \geqslant z\right\} \\ &\leqslant 2\exp\left(-\frac{z^2}{2\sup_{s\in[0,\varepsilon]}\mathrm{Var}[D(s\,,0)]}\right) \leqslant 2\mathrm{e}^{-z^2/(10\varepsilon)} \quad \text{for all } z,\varepsilon > 0. \end{aligned}$$

Thus we see that, for every $z > 0$ and $\varepsilon \in (0\,,1/\mathrm{e}]$,

$$\mathrm{P}\left\{\|H - Z\|_{C([0,\varepsilon]\times\mathbb{T})} \geqslant c\,|\varepsilon\log\varepsilon|^{1/2} + z\right\} \leqslant 2\mathrm{e}^{-z^2/(10\varepsilon)}.$$

This and the moderate-deviations condition (1.5) together imply the lemma. □

We have laid the groundwork and are now prepared for the following conclusion to the results of this section.

Proof of Proposition 2.1. Choose and fix an arbitrary number $\rho \in (0,1)$. In accord with Lemmas 2.9 and 2.11,

$$
\begin{aligned}
&\mathrm{P}\left\{\|Z(\cdot\,,0)\|_{C[0,1]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\}\\
&\leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (1+\rho)\,(\varepsilon/\phi(\varepsilon))^{1/4}\right\} + \mathrm{P}\left\{\|H(\cdot\,,0)-Z(\cdot\,,0)\|_{C[0,\varepsilon]} \geqslant \rho\,(\varepsilon/\phi(\varepsilon))^{1/4}\right\}\\
&\leqslant \exp\left\{-\frac{(2\lambda/\pi)+o(1)}{(1+\rho)^4}\phi(\varepsilon)\right\} + \exp\left\{-\frac{\rho^4+o(1)}{6\sqrt{\varepsilon\phi(\varepsilon)}}\right\} \qquad \text{as } \varepsilon \downarrow 0.
\end{aligned}
$$

Since $\rho \in (0,1)$ can be as close to zero as we want, this and (1.5) together imply that

$$
\limsup_{\varepsilon\downarrow 0}[\phi(\varepsilon)]^{-1}\log \mathrm{P}\left\{\|Z(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} \leqslant -2\lambda/\pi.
$$

Likewise, we appeal to Lemmas 2.9 and 2.11 as follows:

$$
\begin{aligned}
&\exp\left\{-\frac{(2\lambda/\pi)+o(1)}{(1-\rho)^4}\phi(\varepsilon)\right\} = \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (1-\rho)\,(\varepsilon/\phi(\varepsilon))^{1/4}\right\}\\
&\leqslant \mathrm{P}\left\{\|Z(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} + \mathrm{P}\left\{\|H(\cdot\,,0)-Z(t\,,0)\|_{C[0,\varepsilon]} \geqslant \rho\,(\varepsilon/\phi(\varepsilon))^{1/4}\right\}\\
&\leqslant \mathrm{P}\left\{\|Z(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} + \exp\left\{-\frac{\rho^4+o(1)}{6}\sqrt{\frac{\phi(\varepsilon)}{\varepsilon}}\right\} \qquad \text{as } \varepsilon \downarrow 0.
\end{aligned}
$$

Since $\rho \in (0,1)$ can be as close to zero as we want, this and (1.5) together imply that $\liminf_{\varepsilon\downarrow 0}[\phi(\varepsilon)]^{-1}\log \mathrm{P}\{\|Z(\cdot\,,0)\|_{C[0,\varepsilon]} \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\} \geqslant -2\lambda/\pi$, and concludes the proof of the proposition. □

3 Linearization, and proof of Theorem 1.1

Consider the space-time random field $\mathscr{E}$ that is defined by setting, for all $t \geqslant 0$ and $x \in \mathbb{T}$,

$$
\mathscr{E}(t\,,x) = u(t\,,x) - (p_t * u_0)(x) - \sigma(u_0(x))Z(t\,,x).
$$

Thus, the random variable $\mathscr{E}(t\,,x)$ measures the linearization error of the solution to (1.1) at the space-time point $(t\,,x) \in \mathbb{R}_+ \times \mathbb{T}$. It is known that $\mathscr{E}(t\,,x) \approx 0$ when $t \approx 0$; this was done independently and nearly at the same time in [12] and [9]. The method of [12] provided detailed bounds for the moments of $\sup|\mathscr{E}|$ but with suboptimal t-dependent rates, and the method of [9] provided a.s. estimates for $\sup|\mathscr{E}|$, with nearly sharp control of the size of $\sup|\mathscr{E}|$, but only under extra smoothness conditions on σ; specifically σ was assumed to be in C^r for a large enough $r \geqslant 3$. Our next proposition improves both of these results. It yields a rate that is unimproveable to leading order, does not require additional smoothness for σ, and provides quantitative bounds on $\mathrm{P}\{\mathscr{E} \approx 0\}$. More precisely, we have

Proposition 3.1. *If σ is bounded, then for every $\nu > 0$ there exists $a = a(\nu) \in (0\,,1)$ such that*

$$\mathrm{P}\left\{\|\mathscr{E}\|_{C([0,t]\times\mathbb{T})} \geqslant at^{1/2}\log_+(1/t)\right\} \lesssim t^\nu \text{ uniformly for every } t \in (0\,,1).$$

The proof of Proposition 3.1 requires a few preliminary calculations. Before we commence with those, let us quickly deduce the following analogue of a result in [9] but valid with no additional smoothness assumptions on σ and with a slightly tighter error rate at the sharp leading order of $t^{1/2}$.

Corollary 3.2. *Regardless of whether or not σ is bounded, there exists an a.s.-finite random variable V such that $\|\mathscr{E}(t)\|_{C(\mathbb{T})} \leqslant Vt^{1/2}\log_+(1/t)$ uniformly for all $t \in [0\,,1]$.*

Proof. The proof uses a stopping-time argument. Choose and fix a real number $N > 0$, and define u_N the same as u – see (1.1) – but with σ replaced by σ_N

$$\sigma_N(x) = \begin{cases} \sigma(N) & \text{if } x > N,\\ \sigma(x) & \text{if } -N < \sigma(x) \leqslant N,\\ \sigma(-N) & \text{if } x \leqslant -N.\end{cases}$$

That is, $u_N(0) = u_0$, and

$$u_N(t\,,x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma_N(u_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y),$$

for $t > 0$ and $x \in \mathbb{T}$. Define

$$T_N = \inf\left\{t \geqslant 0:\ \|u_N(t)\|_{C(\mathbb{T})} > N\right\} \qquad [\inf\varnothing = \infty].$$

Then, T_N is a stopping time with respect to the filtration $\mathcal{F}$ generated by the noise. Basic properties of the Walsh stochastic integral and the continuity of u and u_N together imply that

$$\mathrm{P}\left\{u_N(t) = u(t) \text{ for all } t < T_N\right\} = 1, \tag{3.1}$$

whence also $T_N = \inf\{t \geqslant 0 : \|u(t)\|_{C(\mathbb{T})} > N\}$ almost surely. Therefore, we apply Proposition 3.1 with $\nu = 1$ in order to see that there exists $a > 0$ such that

$$\begin{aligned}
&\mathrm{P}\left\{\sup_{t\in[0,\varepsilon]}\|\mathscr{E}(t)\|_{C(\mathbb{T})} \geqslant a\varepsilon^{1/2}|\log\varepsilon|\,;T_N > 1\right\}\\
&= \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]}\|u_N(t) - p_t * u_0 - \sigma(u_0)Z(t)\|_{C(\mathbb{T})} \geqslant a\varepsilon^{1/2}|\log\varepsilon|\,;T_N > 1\right\}\\
&\leqslant \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]}\|u_N(t) - p_t * u_0 - \sigma(u_0)Z(t)\|_{C(\mathbb{T})} \geqslant a\varepsilon^{1/2}|\log\varepsilon|\right\} \lesssim \varepsilon,
\end{aligned}$$

uniformly for all $\varepsilon \in (0\,,1)$. Replace ε by $\exp(-n)$ as n ranges over $\mathbb{N}$ and sum over n to deduce from the Borel-Cantelli lemma that

$$\limsup_{n\to\infty} \sup_{t\in[0,\exp(-n)]} \frac{\|\mathscr{E}(t)\|_{C(\mathbb{T})}}{\mathrm{e}^{-n/2}n} \leqslant a \quad \text{a.s. on } \{T_N > 1\}. \tag{3.2}$$

If $\exp(-n-1) \leqslant \varepsilon \leqslant \exp(-n)$ and $n \in \mathbb{N}$, then

$$\sup_{s\in[0,\varepsilon]} \frac{\|\mathscr{E}(s)\|_{C(\mathbb{T})}}{\varepsilon^{1/2}|\log\varepsilon|} \leqslant \sup_{t\in[0,\exp(-n)]} \frac{\|\mathscr{E}(t)\|_{C(\mathbb{T})}}{\mathrm{e}^{-(n-1)/2}n}.$$

Therefore, (3.2) implies that

$$\mathrm{P}\left\{\limsup_{\varepsilon\downarrow 0} \sup_{s\in[0,\varepsilon]} \frac{\|\mathscr{E}(s)\|_{C(\mathbb{T})}}{\varepsilon^{1/2}|\log\varepsilon|} \leqslant \frac{a}{\mathrm{e}^{1/2}}\right\} \geqslant \mathrm{P}\{T_N > 1\} \qquad \text{for all } N > 0.$$

Because (3.1) and the a.s.-continuity of u together imply that $\lim_{N\to\infty} T_N = \infty$ a.s., this proves that

$$\limsup_{\varepsilon\downarrow 0} \sup_{s\in[0,\varepsilon]} \frac{\|\mathscr{E}(s)\|_{C(\mathbb{T})}}{\varepsilon^{1/2}|\log\varepsilon|} \leqslant \frac{a}{\sqrt{e}} \qquad \text{a.s.}$$

In particular, the above limsup is finite almost surely. This is another way to state the corollary. □

Now we begin proof of Proposition 3.1 in earnest. Let us define a metric Δ on space-time $\mathbb{R}_+ \times \mathbb{T}$ by setting

$$\Delta\left((t\,,x)\,,(s\,,y)\right) = |t-s|^{1/4} + |x-y|^{1/2} \quad \text{for all } s,t \geqslant 0 \text{ and } x,y \in \mathbb{T}.$$

It might help to recall that we are using the additive notation for elements of $\mathbb{T}$. In particular, $|x-y|^{1/2}$ is shorthand for $|x-y \pmod 2|^{1/2}$ whenever $x,y \in \mathbb{T}$.

The following is a consequence of the large-deviations result of Sowers [26] and well-known relations between tails of a Gaussian law and its moments. Results of the following type are well known and typically used to prove that the process u is continuous all the way up to and including the boundary of $[-1\,,1]$, keeping in mind also that ± 1 are identified with one another here.

Lemma 3.3 (Sowers [26])**.** *If σ is bounded, then*

$$\|u(t\,,x) - u(s\,,y)\|_k \lesssim \sqrt{k}\,\Delta\big((t\,,x)\,,(s\,,y)\big),$$

uniformly for all $x,y \in \mathbb{T}$, $s,t \geqslant 0$, and $k \geqslant 2$.

Next, we present an exponential tail estimate for the linearization error $\mathscr{E}$, valid when σ is bounded.

Lemma 3.4. *If σ is bounded, then there exists $\gamma > 0$ such that*

$$\sup_{t\in(0,1]} \sup_{x\in\mathbb{T}} \mathrm{E}\exp\left(\gamma\left|\frac{\mathscr{E}(t\,,x)}{\sqrt{t}}\right|\right) < \infty.$$

Proof. Compare (2.1) with (2.3) in order to see that

$$\mathscr{E}(t\,,x)=\int_{(0,t)\times\mathbb{T}}p_{t-s}(x\,,y)\,[\sigma(u(s\,,y))-\sigma(u(0\,,x))]\;W(\mathrm{d}s\,\mathrm{d}y),$$

for all $t>0$ and $x\in\mathbb{T}$. Thanks to the Young's inequality for stochastic convolutions (see Khoshnevisan [10, Proposition 5.2]), we have the following for every real number $k\geqslant 2$, $t>0$, and $x\in\mathbb{T}$:

$$\begin{aligned}\|\mathscr{E}(t\,,x)\|_k^2&\leqslant 4k\int_0^t\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\;[p_{t-s}(x\,,y)]^2\,\|\sigma(u(s\,,y))-\sigma(u(0\,,x))\|_k^2\\&\leqslant 4k[\mathrm{Lip}(\sigma)]^2\int_0^t\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\;[p_{t-s}(x\,,y)]^2\,\|u(s\,,y)-u(0\,,x)\|_k^2\\&\lesssim k^2\int_0^t\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\;\left[p_{t-s}(x\,,y)\Delta((0\,,x)\,,(s\,,y))\right]^2.\end{aligned}\tag{3.3}$$

Thanks to (2.2), $(t\,,a)\mapsto p_r(a)-G_r(a)$ is bounded uniformly on $\mathbb{R}_+\times\mathbb{T}$. In this way, we find that

$$\|\mathscr{E}(t\,,x)\|_k^2\lesssim k^2\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,[G_s(y)]^2\left(\sqrt{t-s}+|y|\right)+k^2\int_0^t\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\left(\sqrt{s}+|y|\right),$$

uniformly for all $t>0$, $x\in\mathbb{T}$, and $k\geqslant 2$. Direct computation yields the bound,

$$\int_0^t\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\left(\sqrt{s}+|y|\right)\propto t^{3/2}+t,\qquad\text{valid uniformly for all } t>0.$$

Similarly, we find that for all $t>0$,

$$\begin{aligned}\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\;[G_s(y)]^2\sqrt{t-s}&=\int_0^t\sqrt{t-s}\,G_{2s}(0)\,\mathrm{d}s\propto t,\quad\text{and}\\\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\;[G_s(y)]^2|y|&=\int_0^t\frac{\mathrm{d}s}{s}\int_{-\infty}^{\infty}\mathrm{d}y\;[G_1(y/\sqrt{s})]^2|y|\\&=\int_0^t\mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}w\;[G_1(w)]^2|w|\propto t,\end{aligned}$$

where the constants of proportionality do not depend on t. It follows from the preceding effort that there exists $B>0$ such that

$$\sup_{t\in(0,1]}\sup_{x\in\mathbb{T}}\mathrm{E}\left(|\mathscr{E}(t\,,x)|^k\right)\leqslant(Bk)^k t^{k/2}\quad\text{uniformly for all } k\geqslant 2.$$

By Jensen's inequality, the preceding in fact holds uniformly for all $t>0$, $x\in\mathbb{T}$, and $k\geqslant 1$. Among other things, this and Stirling's formula together yield a constant $C>0$ such that

$$\sup_{t\in(0,1]}\sup_{x\in\mathbb{T}}\mathrm{E}\left(\left|\frac{\mathscr{E}(t\,,x)}{\sqrt{t}}\right|^k\right)\leqslant C^k k!\quad\text{uniformly for all } k\in\mathbb{Z}_+.$$

Choose and fix an arbitrary $\gamma\in(0\,,C)$ and sum the above inequality over all $k\in\mathbb{Z}_+$ in order to deduce the lemma. □

Remark 3.5. We can make an adjustment to the preceding proof in order to see that the distribution of $\mathscr{E}(t\,,x)$ in fact has sub-Gaussian tails when σ is bounded. However, in order to achieve a sub-Gaussian tail that is valid uniformly in $t \in (0\,,1]$ all the way down to $t = 0$, we need to normalize $\mathscr{E}(t\,,x)$ differently. A more precise statement is this: *There exists* $\gamma' > 0$ *such that*

$$\sup_{t\in(0,1]} \sup_{x\in\mathbb{T}} \mathrm{E}\exp\left(\gamma' \left|\frac{\mathscr{E}(t\,,x)}{t^{1/4}}\right|^2\right) < \infty. \tag{3.4}$$

To prove (3.4) we simply adjust the first line of (3.3) by bounding out the difference of the σ's. In this way we obtain the following, thanks to the semigroup property of the heat kernel and a standard bound on the heat kernel on $\mathbb{T}$ at small times: Uniformly for all $t \in (0\,,1]$, $x \in \mathbb{T}$, and $k \geqslant 2$,

$$\begin{aligned}
\|\mathscr{E}(t\,,x)\|_k^2 &\leqslant 16 \sup_{z\in\mathbb{R}} |\sigma(z)|^2 k \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ [p_{t-s}(x\,,y)]^2 \\
&= 16 \sup_{z\in\mathbb{R}} |\sigma(z)|^2 k \int_0^t p_{2s}(0\,,0)\,\mathrm{d}s \lesssim k\sqrt{t}.
\end{aligned}$$

This inequality yields (3.4). By itself, the rate $t^{1/4}$ renders the bound (3.4) useless since the individual terms that define $\mathscr{E}$ are each of the order $t^{1/4}$ in law when $t \approx 0$. However, the observation has its uses. For example, (3.4) is good enough to ensure that, among other things, $\mathscr{E}(t\,,x)$ has sub-Gaussian probability tails.

The preceding remark can be followed up by our next lemma which describes unconditional sub-Gaussian tails for the distribution of the spatio-temporal increments of $\mathscr{E}$ when σ is bounded.

Lemma 3.6. *If* σ *is bounded, then there exists a number* $\gamma_0 > 0$ *such that*

$$\mathrm{E}\exp\left(\sup_{0<s<t\leqslant 1}\ \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \gamma_0 \left|\frac{\mathscr{E}(t\,,x) - \mathscr{E}(s\,,y)}{\Delta((t\,,x)\,,(s\,,y))\sqrt{\log_+(1/\Delta((t\,,x)\,,(s\,,y)))}}\right|^2\right) < \infty.$$

Proof. Since the random field Z is defined in the same way as the random field u but with $\sigma \equiv 1$, Lemma 3.3 implies that

$$\|Z(t\,,x) - Z(s\,,y)\|_k \lesssim \sqrt{k}\,\Delta((t\,,x)\,,(s\,,y)),$$

uniformly for all $x, y \in \mathbb{T}$, $s, t \geqslant 0$, and $k \geqslant 2$. Therefore, the boundedness and Lipschitz continuity of σ yield the bounds,

$$\begin{aligned}
&\|\sigma(u_0(x))Z(t\,,x) - \sigma(u_0(y))Z(s\,,y)\|_k \\
&\leqslant \|\sigma(u_0(x))Z(t\,,x) - \sigma(u_0(x))Z(s\,,y)\|_k + |\sigma(u_0(x)) - \sigma(u_0(y))|\|Z(s\,,y)\|_k \\
&\lesssim \sqrt{k}\,\Delta((t\,,x)\,,(s\,,y)) + |x-y|\|Z(s\,,y)\|_k,
\end{aligned}$$

valid uniformly for all $k \geqslant 2$, $s > 0$, and $y \in \mathbb{T}$. Since Z is a Gaussian random field, a standard computation yields

$$\|Z(s\,,y)\|_k \lesssim \sqrt{k}\,\|Z(s\,,y)\|_2 \lesssim \sqrt{k}\,s^{1/4},$$

uniformly for all $k \geqslant 2$, $s > 0$, and $y \in \mathbb{T}$. It follows that

$$\|\sigma(u_0(x))Z(t\,,x) - \sigma(u_0(y))Z(s\,,y)\|_k \leqslant \sqrt{k}\Delta((t\,,x)\,,(s\,,y)),$$

uniformly for all $k \geqslant 2$, $s,t \in (0\,,1]$, and $x,y \in \mathbb{T}$. It is well known that, because u_0 is Lipschitz continuous,

$$|(p_t * u_0)(x) - (p_s * u_0)(y)| \lesssim \Delta((t\,,x)\,,(s\,,y)),$$

uniformly for all $s,t \in (0\,,1]$ and $x,y \in \mathbb{T}$.[3] Therefore, the preceding bounds together yield the inequality

$$\|\mathscr{E}(t\,,x) - \mathscr{E}(s\,,y)\|_k \lesssim \sqrt{k}\,\Delta((t\,,x)\,,(s\,,y)),$$

valid uniformly for all $s,t \in (0\,,1]$ and $x,y \in \mathbb{T}$. Now a standard metric entropy argument completes the proof. ☐

We are ready to establish the following, which is a slightly weaker fixed-time version of Proposition 3.1, and paves the way toward proving afterward that proposition in complete generality.

Lemma 3.7. *If σ is bounded, then for every $\nu > 0$, there exists a number $K = K(\nu) > 1$ such that*

$$\mathrm{P}\left\{\|\mathscr{E}(t)\|_{C(\mathbb{T})} \geqslant K\sqrt{t}\,\log_+(1/t)\right\} \lesssim t^\nu \text{ uniformly for all } t \in (0\,,1).$$

Proof. Lemmas 3.4 and 3.6, and Chebyshev's inequality together yield a number $C > 0$ such that

$$\begin{gathered}\sup_{t>0}\sup_{x\in\mathbb{T}} \mathrm{P}\left\{|\mathscr{E}(t\,,x)| \geqslant 2\beta\sqrt{t}\right\} \lesssim \exp\left(-C\beta\right), \quad \text{and}\\ \mathrm{P}\left\{\sup_{\substack{x,y\in\mathbb{T}:\\|x-y|\leqslant\varepsilon}} |\mathscr{E}(t\,,x) - \mathscr{E}(t\,,y)| \geqslant \theta\sqrt{\varepsilon\log(1/\varepsilon)}\right\} \lesssim \exp\left(-C\theta^2\right),\end{gathered} \tag{3.5}$$

uniformly for every $\beta,\theta > 0$. Define

$$\mathbb{T}_n = \cup_{i\in[-n,n-1]\cap\mathbb{Z}}\,\{i/n\} \qquad \text{for all } n \in \mathbb{N},$$

and remember that because of the group topology of the torus, the ends of $\mathbb{T} = [-1\,,1]$ are identified with one another. This shows that every point in $\mathbb{T}$ is

[3] This follows for example, from the fact that we can write $(p_t * u_0)(x) = \mathrm{E}u_0(x + B_t)$ for a Brownian motion $\{B_t\}_{t\geqslant 0}$ on $\mathbb{T}$ with speed 2, so that $|(p_t * u_0)(x) - (p_s * u_0)(y)| \leqslant \|u_0(x+B_t) - u_0(y+B_s)\|_1 \leqslant \|u_0\|_{C^{1/2}(\mathbb{R})}\{\|B_t - B_s\|_1 + |x-y|\}^{1/2}$, by the triangle inequality.

within n^{-1} of some point in $\mathbb{T}_n$. Because the cardinality of $\mathbb{T}_n$ is $\leqslant 4n$ uniformly for all $n \in \mathbb{N}$, we can deduce from (3.5) that

$$\begin{aligned}
&\mathrm{P}\left\{ \|\mathscr{E}(t)\|_{C(\mathbb{T})} \geqslant \frac{2\beta}{C}\sqrt{t} \right\} \\
&\leqslant \mathrm{P}\left\{ \max_{x\in\mathbb{T}_n} |\mathscr{E}(t\,,x)| \geqslant \frac{\beta}{C}\sqrt{t} \right\} + \mathrm{P}\left\{ \sup_{\substack{x,y\in\mathbb{T}:\\ |x-y|\leqslant 1/n}} |\mathscr{E}(t\,,x) - \mathscr{E}(t\,,y)| \geqslant \frac{\beta}{C}\sqrt{t} \right\} \\
&\lesssim n\mathrm{e}^{-\beta/2} + \exp\left(-\frac{\beta^2 tn}{C\,\log_+(n)} \right),
\end{aligned}$$

uniformly for all $t \in (0\,,1)$, $\beta > 0$, and $n \in \mathbb{N}$. We apply the preceding with

$$\beta = \beta(t) = 2\kappa \log_+(1/t) \quad \text{and} \quad n = n(t) = C\,\lfloor 1/t \rfloor\,,$$

where $\kappa > \nu \vee 1$ is a fixed number, in order to deduce the result. □

We are ready to establish Proposition 3.1.

Proof of Proposition 3.1. Lemma 3.6 implies that there exists $C > 0$ such that

$$\mathrm{P}\left\{ \sup_{\substack{s,r\in(0,1]:\\ |s-r|\leqslant\varepsilon}} \|\mathscr{E}(s) - \mathscr{E}(r)\|_{C(\mathbb{T})} \geqslant \theta\,\varepsilon^{1/4}\,[\log(1/\varepsilon)]^{1/2} \right\} \lesssim \mathrm{e}^{-C\theta^2}, \tag{3.6}$$

uniformly for all $\varepsilon \in (0\,,1)$ and $\theta > 0$. Now, let us choose and fix some $t \in (0\,,1)$, define

$$S_{n,t} = \cup_{j\in[0,n]\cap\mathbb{N}}\,\{jt/n\} \qquad \text{for all } n \in \mathbb{N},$$

and observe that every point in $[0\,,t]$ is certainly within $1/n$ of some point in $S_{n,t}$. Because the cardinality of $S_{n,t}$ is $\lesssim n$ uniformly for all $n \in \mathbb{N}$, we can deduce from Lemma 3.7 and eq. (3.6) that for every $\nu > 0$ there exists $K = K(\nu) > 0$ such that

$$\begin{aligned}
\mathrm{P}\left\{ \|\mathscr{E}\|_{C([0,t]\times\mathbb{T})} \geqslant 2K\sqrt{t}\,\log_+(1/t) \right\} &\leqslant \mathrm{P}\left\{ \|\mathscr{E}\|_{C(S_{n,t}\times\mathbb{T})} \geqslant K\sqrt{t}\,\log_+(1/t) \right\} \\
&\quad + \mathrm{P}\left\{ \sup_{\substack{s,r\leqslant t:\\ |s-r|\leqslant 1/n}} \|\mathscr{E}(r) - \mathscr{E}(s)\|_{C(\mathbb{T})} \geqslant K\sqrt{t}\,\log_+(1/t) \right\} \\
&\lesssim nt^{\nu+2} + \exp\left(-CK^2 \frac{\sqrt{n}}{\log_+(n)}\, t|\log_+(1/t)|^2 \right),
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$ and $t \in (0\,,1)$. We may now choose and fix an integer $M = M(\nu) > [2\nu/(CK^2)]^2$, and apply the preceding with $n = M\lfloor t^{-2} \rfloor$ in order to complete the proof. □

Proposition 3.1 forms the bulk of the effort of proving Theorem 1.1. Now that we have proved the proposition, we can conclude the proof of Theorem 1.1, which is the first primary offering of this work.

Proof of Theorem 1.1. We can observe that

$$
\begin{aligned}
&\mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |u(t\,,x) - (p_t * u_0)(x)| \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} \\
&\leqslant \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |\sigma(u_0(x))Z(t\,,x)| \leqslant (\varepsilon/\phi(\varepsilon))^{1/4} + a\varepsilon^{1/2}|\log\varepsilon|\right\} \\
&\qquad\qquad\qquad + \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |\mathscr{E}(t\,,x)| \geqslant a\varepsilon^{1/2}|\log\varepsilon|\right\}.
\end{aligned}
$$

Thanks to (1.5), the first probability on the right-hand side decays at least as rapidly as $\exp\{-(2\lambda\sigma^4(u_0(x))/\pi) + o(1))\phi(\varepsilon)\}$. Therefore, we choose $\nu > 2\lambda|\sigma(u_0(x))|/\pi$ too see that, as long as we pick a large enough (which we will), Lemma 3.7 assures us that, as $\varepsilon \downarrow 0$,

$$
\begin{aligned}
&\mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |u(t\,,x) - (p_t * u_0)(x)| \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} \\
&\lesssim \exp\left\{-\left(\frac{2\lambda[\sigma(u_0(x))]^4 + o(1)}{\pi}\right)\phi(\varepsilon)\right\} + \varepsilon^{\nu} \leqslant \exp\left\{-\left(\frac{2\lambda[\sigma(u_0(x))]^4 + o(1)}{\pi}\right)\phi(\varepsilon)\right\},
\end{aligned}
$$

see (1.5). In order to derive a complementary bound, we write

$$
\begin{aligned}
&\mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |\sigma(u_0(x))Z(t\,,x)| \leqslant (\varepsilon/\phi(\varepsilon))^{1/4} + a\varepsilon^{1/2}|\log\varepsilon|\right\} \\
&\leqslant \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |u(t\,,x) - (p_t * u_0)(x)| \leqslant (\varepsilon/\phi(\varepsilon))^{1/4}\right\} \\
&\qquad\qquad\qquad + \mathrm{P}\left\{\sup_{t\in[0,\varepsilon]} |\mathscr{E}(t\,,x)| \geqslant a\varepsilon^{1/2}|\log\varepsilon|\right\},
\end{aligned}
$$

and proceed in parallel to the previous part. This completes the proof since (1.5) assures that

$$
\sup_{t\in[0,\varepsilon]} \|p_t * u_0 - u_0\|_{C(\mathbb{T})} \leqslant \mathrm{Lip}(u_0)\sqrt{\varepsilon} = o\left((\varepsilon/\phi(\varepsilon))^{1/4}\right),
$$

as $\varepsilon \downarrow 0$. This completes the proof. □

4 Proof of Corollary 1.2

As was mentioned in the Introduction, one might anticipate some version of Corollary 1.2, viewed as a natural byproduct of Theorem 1.1. However, it turns out that the proof of Corollary 1.2 requires the introduction of a few subtle ideas that are not altogether standard. Therefore, we use this section to hash out the details of that argument. Throughout this section, let us define

$$
\psi(t) = \left(\frac{t}{\log|\log_+(1/t)|}\right)^{1/4} \qquad \text{for all } t \geqslant 0,
$$

and recall the Gaussian random field H from (2.4) and (2.5). The following is the main step of the proof of Corollary 1.2.

Proposition 4.1. *For every $x \in \mathbb{R}$,*

$$\liminf_{\varepsilon \downarrow 0} \sup_{t \in [0,\varepsilon]} \frac{|H(t\,,x)|}{\psi(\varepsilon)} = \left(\frac{2\lambda}{\pi}\right)^{1/4} \qquad a.s.$$

Before we prove Proposition 4.1, we pause to quickly verify Corollary 1.2. Then, we concentrate on proving Proposition 4.1, which is the main portion of the work.

Sketch of a conditional proof of Corollary 1.2 given Proposition 4.1. We may apply Lemma 2.11 [with $\phi(t) = \delta^{-4} \log\log_+(1/t)$] to see that for every $\delta > 0$ there exists $K = K(\delta) > 0$ such that

$$\mathrm{P}\left\{ \sup_{t \in [0,\varepsilon]} |H(t\,,x) - Z(t\,,x)| \geqslant \delta\psi(\varepsilon) \right\} \leqslant K \exp\left(-\frac{1}{\sqrt{K\varepsilon \log|\log \varepsilon|}} \right), \quad (4.1)$$

uniformly for all $\varepsilon \in (0\,,\mathrm{e}^{-4})$. Because $\delta > 0$ is arbitrary, the Borel-Cantelli lemma then implies that, with probability one,

$$\sup_{t \in [0,\varepsilon]} \|H(t) - Z(t)\|_{C(\mathbb{T})} = o(\psi(\varepsilon)) \qquad \text{as } \varepsilon \downarrow 0.$$

Because $\varepsilon^{1/2}|\log \varepsilon| \ll \psi(\varepsilon)$ as $\varepsilon \downarrow 0$, the preceding and Corollary 3.2 together yield Corollary 1.2. We leave the remaining details to the interested reader. □

Now we start to prove Proposition 4.1. From here on, let us choose a fixed real number $\alpha > 0$, and define

$$t_n = \exp\left(-n^{1+\alpha}\right) \qquad \text{for every } n \in \mathbb{N}. \quad (4.2)$$

Because $\alpha > 0$, a Taylor expansion yields

$$\frac{t_{n+1}}{t_n} \leqslant \exp\left(-(1+\alpha)n^{\alpha}\right) \qquad \text{uniformly for all } n \in \mathbb{N}. \quad (4.3)$$

Lemma 4.2. *For every $\delta \in (0\,,1)$ there exists $M = M(\delta\,,\alpha)$ such that, uniformly for all $n \in \mathbb{N}$,*

$$\mathrm{P}\left\{ \|H\|_{C([0,t_{n+1}]\times[-1,1])} \geqslant \delta\psi(t_n) \right\} \leqslant M \exp\left(-\frac{\exp\left((1+\alpha)n^{\alpha}/2\right)}{M\sqrt{\log_+(n)}} \right).$$

Proof. It is not hard to see that $\mathrm{Var}(H(t\,,0)) \propto \sqrt{t}$ uniformly for all $t > 0$; this is very well known, but also follows essentially immediately from the scaling property (2.12) of the random field H. It is also well known that, for every fixed $T > 0$, $\|H(t\,,x) - H(s\,,y)\|_2 \lesssim |t-s|^{1/4} + |x-y|^{1/2}$ uniformly for all $x, y \in \mathbb{R}$ and $s, t \in [0\,,T]$. In fact, Lemma 3.3 asserts this in a more general context where σ can be nonlinear. A suitable version of Dudley's metric entropy theorem [15, Theorem 6.1] yields a constant $L > 0$ such that

$$\mathrm{E}\left(\|H\|_{C([0,t]\times[-1,1])}\right) \leqslant L t^{1/4} \sqrt{\log_+(1/t)} \qquad \text{for all } t \in (0\,,1).$$

Now we may apply concentration of measure [15] in order to see that there exists $\ell > 0$ such that

$$\mathrm{P}\left\{\|H\|_{C([0,t_{n+1}]\times[-1,1])} \geqslant L t_{n+1}^{1/4}|\log t_{n+1}|^{1/2} + z\right\} \leqslant 2\exp\left(-\frac{\ell z^2}{\sqrt{t_{n+1}}}\right), \quad (4.4)$$

for all $n \in \mathbb{N} \cap [2\,,\infty)$ and $z > 0$. Thanks to (4.3),

$$\begin{aligned} t_{n+1}^{1/4}|\log t_{n+1}|^{1/2} &= \left(\frac{t_{n+1}}{t_n}\right)^{1/4} |\log t_{n+1}|^{1/2} \left(\log|\log t_n|\right)^{1/4}\psi(t_n) \\ &\leqslant \exp\left(-\frac{(1+\alpha)n^\alpha}{4}\right)(n+1)^{1/2}(\log n)^{1/4}\psi(t_n) \leqslant \frac{\delta}{2L}\psi(t_n), \end{aligned}$$

uniformly for all n large enough, and how large depends only on $(\delta\,,\alpha)$. Therefore, we plug into (4.4) $z = \delta\psi(t_n)/2$, and deduce the asserted inequality of the lemma for large n after a few lines of computation. We may increase the constant M, if it is needed, in order to obtain the lemma for all $n \in \mathbb{N}$. □

Next we adopt a localization idea of Lee and Xiao [16], and define a family $\{H_n\}_{n\in\mathbb{N}}$ of space-time Gaussian random fields by setting

$$H_n(t\,,x) = \int_{[t_{n+1},t)\times\mathbb{R}} G_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y), \quad (4.5)$$

for all $(t\,,x) \in [t_{n+1}\,,t_n] \times \mathbb{R}$. If $n \gg 1$ then $H_n \approx H$. The following is a careful way to say this, and contains also a tight quantitative bound on the approximation error, necessary for small-ball probability estimates that follow.

Lemma 4.3. *For every $\delta \in (0\,,1)$ there exists $M = M(\delta\,,\alpha) > 0$ such that, uniformly for all $n \in \mathbb{N}$,*

$$\mathrm{P}\left\{\|H - H_n\|_{C([t_{n+1},t_n]\times[-1,1])} \geqslant \delta\psi(t_n)\right\} \leqslant M\exp\left(-\frac{\exp\left((1+\alpha)n^\alpha/2\right)}{M\sqrt{\log_+(n)}}\right).$$

Proof. Because $H(t\,,x) - H_n(t\,,x) = \int_{(0,t_{n+1})\times\mathbb{R}} G_{t-s}(y-x)\,W(\mathrm{d}s\,\mathrm{d}y)$ for all $n \in \mathbb{N}$, $t \in [t_{n+1}\,,t_n)$ and $x \in \mathbb{R}$, the Wiener isometry implies that

$$\mathrm{E}\left(|H(t\,,x) - H_n(t\,,x)|^2\right) = \int_0^{t_{n+1}} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [G_{t-s}(y-x)]^2 = \int_{t-t_{n+1}}^{t} G_{2s}(0)\,\mathrm{d}s,$$

owing to the semigroup property of the heat kernel. Since $G_{2s}(0) = (8\pi s)^{-1/2}$, it follows that

$$\mathrm{E}\left(|H(t\,,x) - H_n(t\,,x)|^2\right) \propto \sqrt{t} - \sqrt{t - t_{n+1}} \asymp \frac{t_{n+1}}{\sqrt{t}} \leqslant \sqrt{t_{n+1}},$$

uniformly for all $t \in [t_{n+1}, t_n]$, $n \in \mathbb{N}$, and $x \in \mathbb{R}$. Apply (4.3) to see that

$$\sup_{t\in[t_{n+1},t_n]}\sup_{x\in\mathbb{R}} \mathrm{E}\left(|H(t\,,x) - H_n(t\,,x)|^2\right) \lesssim \sqrt{t_n}\exp\left(-\tfrac{1}{2}(1+\alpha)n^\alpha\right), \quad (4.6)$$

uniformly for every $n \in \mathbb{N}$. Thanks to this and a metric entropy argument [15], we can find constants L_1 and L such that

$$\begin{aligned}&\mathrm{E}\|H - H_n\|_{C([t_{n+1},t_n]\times[-1,1])}\\ &\leqslant L_1 t_n^{1/4}|\log t_n|^{1/2}\exp\left(-\tfrac{1}{4}(1+\alpha)n^\alpha\right) \leqslant L t_n^{1/4} n^{(1+\alpha)/2}\exp\left(-\tfrac{1}{4}(1+\alpha)n^\alpha\right),\end{aligned}$$

uniformly for every $n \in \mathbb{N}$. Therefore, (4.6) and concentration of measure [15] together ensure that there exists a number $K > 0$ such that

$$\begin{aligned}&\mathrm{P}\left\{\|H - H_n\|_{C([t_{n+1},t_n]\times[-1,1])} \geqslant L t_n^{1/4} n^{(1+\alpha)/2}\exp\left(-\tfrac{1}{4}(1+\alpha)n^\alpha\right) + z\right\}\\ &\leqslant 2\exp\left(-\frac{z^2}{2\sup\limits_{t\in[t_{n+1},t_n]}\sup\limits_{x\in\mathbb{R}}\mathrm{E}\left(|H(t\,,x) - H_n(t\,,x)|^2\right)}\right) \leqslant 2\exp\left(-\frac{z^2\mathrm{e}^{n^{1+\alpha}/2}}{K\sqrt{t_n}}\right),\end{aligned}$$

uniformly for all $z > 0$ and $n \in \mathbb{N}$. Let $\eta \in (0\,,\delta)$ be an arbitrary number and apply the above with $z = \eta^{1/4}\psi(t_n)$ and appeal to the fact that $t_n^{1/4} n^{\alpha/2}\exp\{-n^{1+\alpha}/2\} \ll z$ for all n large in order to deduce the assertion of the lemma for all sufficiently large n. We may increase M further, if we need to, in order to see that the lemma's statement is valid for every $n \in \mathbb{N}$. □

We are now able to formulate a restricted small-ball estimate, for H, that we shall need shortly.

Lemma 4.4. *For every $\gamma > 0$,*

$$\lim_{n\to\infty}\frac{1}{\log n}\log \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\} = -\frac{2\lambda(1+\alpha)}{\pi\gamma^4}.$$

Proof. Choose and fix $\gamma > 0$. Since

$$\mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,t_n]} \leqslant \gamma\psi(t_n)\right\} \leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\},$$

and because $\log\log 1/t_n = (1+\alpha)\log n$, Proposition 2.2 and scaling – see (2.12) – together imply that

$$-\frac{2\lambda(1+\alpha)}{\pi\gamma^4} \leqslant \liminf_{n\to\infty}\frac{1}{\log n}\log \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}. \tag{4.7}$$

One can obtain a similar bound in the other direction as follows: Owing to Lemma 4.2, for every $\delta \in (0\,,\gamma)$ there exists $M > 0$ such that uniformly for all $n \in \mathbb{N}$,

$$\begin{aligned}&\mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}\\ &\leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,t_n]} \leqslant (\gamma+\delta)\psi(t_n)\right\} + \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,t_{n+1}]} \geqslant \delta\psi(t_n)\right\}\\ &\leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[0,t_n]} \leqslant (\gamma+\delta)\psi(t_n)\right\} + M\exp\left(-\frac{\exp\left((1+\alpha)n^\alpha/2\right)}{M\sqrt{\log_+(n)}}\right).\end{aligned}$$

This proves that

$$\limsup_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\} \leqslant -\frac{2\lambda(1+\alpha)}{\pi(\gamma+\delta)^4}. \tag{4.8}$$

The quantity on the left-hand side does not depend on $\delta \in (0\,,\gamma)$. We therefore obtain the lemma from (4.7) and (4.8) upon letting $\alpha \downarrow 0$. □

When $n \gg 1$, the small-ball probability bound of Lemma 4.4 for the random field H yields an analogous probability bound for the closely related field H_n, viz.,

Lemma 4.5. *For every $\gamma > 0$,*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|H_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\} = -\frac{2\lambda(1+\alpha)}{\pi\gamma^4}.$$

Proof. Lemma 4.3 ensures that for every $0 < \delta < \gamma$ there exists $M > 0$ such that, uniformly for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{P}\left\{\|H_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}\\
&\quad\leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant (\gamma+\delta)\psi(t_n)\right\}\\
&\qquad\qquad + \mathrm{P}\left\{\|H_n(\cdot\, 0) - H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \geqslant \delta\psi(t_n)\right\}\\
&\quad\leqslant \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant (\gamma+\delta)\psi(t_n)\right\} + M\exp\left(-\frac{\exp\left((1+\alpha)n^\alpha/2\right)}{M\sqrt{\log_+(n)}}\right).
\end{aligned}$$

Therefore, Lemma 4.4 ensures that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|H_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}\\
&\leqslant \lim_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant (\gamma+\delta)\psi(t_n)\right\} = -\frac{2\lambda(1+\alpha)}{\pi(\gamma+\delta)^4}.
\end{aligned}$$

In like manner, we can prove that

$$\begin{aligned}
-\frac{2\lambda(1+\alpha)}{\pi(\gamma-\delta)^4} &= \lim_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|H(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant (\gamma-\delta)\psi(t_n)\right\}\\
&\leqslant \liminf_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|H_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}.
\end{aligned}$$

Let $\delta \downarrow 0$ in order to deduce the lemma from the preceding two displays. □

With the preceding preliminary results under way, we can now present the following.

Proof of Proposition 4.1. By the stationarity of $x \mapsto H(\,\cdot\,,x)$, it suffices to prove that

$$\liminf_{\varepsilon\downarrow 0} \sup_{t\in[0,\varepsilon]} \frac{|H(t\,,0)|}{\psi(\varepsilon)} = \left(\frac{2\lambda}{\pi}\right)^{1/4} \qquad \text{a.s.} \tag{4.9}$$

The basic properties of the Wiener integral ensure that the events

$$\left\{\omega \in \Omega : \; \|H_n(\cdot\,,0)\|_{C[t_{n+1},t_n]}(\omega) \leqslant \gamma\psi(t_n)\right\}, \qquad n = 1, 2, \ldots,$$

are independent for every fixed choice of $\gamma > 0$. Therefore, Lemma 4.5 and a standard appeal to the Borel-Cantelli lemma for independent events together yield

$$\liminf_{n\to\infty} \sup_{t\in[t_{n+1},t_n]} \frac{|H_n(t\,,0)|}{\psi(t_n)} = \left(\frac{2\lambda(1+\alpha)}{\pi}\right)^{1/4} \qquad \text{a.s.}$$

Lemma 4.3 and the Borel-Cantelli lemma together imply that

$$\|H(\cdot\,,0) - H_n(t\,,\cdot)\|_{C([t_{n+1},t_n]\times[-1,1])} = o(\psi(t_n)) \qquad \text{as } n\to\infty \quad \text{a.s.}$$

Therefore, we combine the preceding with Lemma 4.2 in order to deduce the following:

$$\liminf_{\varepsilon\downarrow 0} \sup_{t\in[0,\varepsilon]} \frac{|H(t\,,0)|}{\psi(\varepsilon)} \leqslant \liminf_{n\to\infty} \sup_{t\in[t_{n+1},t_n]} \frac{|H_n(t\,,0)|}{\psi(t_n)} = \left(\frac{2\lambda(1+\alpha)}{\pi}\right)^{1/4} \qquad \text{a.s.}$$

Since the left-most quantity is independent of the sequence $\{t_n\}_{n\in\mathbb{N}}$ – and in particular of α – we let $\alpha \downarrow 0$ to see that

$$\liminf_{\varepsilon\downarrow 0} \sup_{t\in[0,\varepsilon]} \frac{|H(t\,,0)|}{\psi(\varepsilon)} \leqslant \left(\frac{2\lambda}{\pi}\right)^{1/4} \qquad \text{a.s.}$$

This proves half of the assertion of (4.9). The other half follows readily from Proposition 2.2, the scaling property (2.12) of H, and a direct application of the Borel-Cantelli lemma. □

5 Proof of Theorem 1.3

Throughout this section, we choose and fix a real number $\theta > 0$, and define

$$\mathcal{D}(n) = \cup_{\substack{j\in\mathbb{Z}_+:\\ 0\leqslant j\leqslant \theta 2^n}} \left\{j2^{-n}\right\} \qquad \text{for all } n \in \mathbb{Z}_+, \qquad \text{so that } |\mathcal{D}(n)| \sim \theta 2^n \text{ as } n\to\infty.$$

Also here and throughout, we choose and fix a second real number $q > 0$ and define $\mathcal{D}_q(1), \mathcal{D}_q(2), \ldots$ to be the following "slowed down" version of $\mathcal{D}(1), \mathcal{D}(2), \ldots$:

$$\mathcal{D}_q(m) = \mathcal{D}(n) \quad \text{whenever } m \in \mathbb{N} \text{ satisfies } 2^{n/q} \leqslant m < 2^{(n+1)/q}.$$

Note in particular that:

1. $\mathcal{D}_q(m) \subseteq \mathcal{D}_q(m+1)$ for every $m \in \mathbb{Z}_+$;
2. $\cup_{m=0}^{\infty}\mathcal{D}_q(m)$ coincides with the set of all dyadic rationals in $[0\,,\theta]$; and
3. $|\mathcal{D}_q(m)| \asymp m^q$, uniformly for all $m \in \mathbb{N}$.

We will use these properties, sometimes without explicit mention, in the sequel. Finally, we choose and fix $\alpha > 0$ throughout this section, and recall the sequence $\{t_n\}_{n\in\mathbb{N}} = \{t_n(\alpha)\}_{n\in\mathbb{N}}$ from (4.2).

Proposition 5.1. *As $n \to \infty$,*

$$\frac{1}{\log n} \log \mathrm{P}\left\{ \min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H(t\,,x)| \leqslant \gamma\psi(t_n) \right\}$$
$$= \frac{1+o(1)}{\log n} \log \mathrm{P}\left\{ \min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant \gamma\psi(t_n) \right\} \to -\left(\frac{2\lambda(1+\alpha)}{\pi\gamma^4} - q \right),$$

provided that $\gamma > 0$ renders the above limit negative; that is, provided that γ satisfies

$$0 < \gamma < \left(\frac{2\lambda(1+\alpha)}{\pi q} \right)^{1/4}. \tag{5.1}$$

The proof of Proposition 5.1 requires first taking three preliminary steps which, in turn, hinge on the introduction of two new objects. Namely, we define for every $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{R}(n) &= \left[x - \sqrt{t_n |\log t_n|}\,, x + \sqrt{t_n |\log t_n|} \right], \quad \text{and} \\ I_n(t\,,x) &= \int_{[t_{n+1},t)\times\mathcal{R}(n)} G_{t-s}(y-x)\, W(\mathrm{d}s\,\mathrm{d}y), \end{aligned} \tag{5.2}$$

for all $(t\,,x) \in [t_{n+1}\,,t_n] \times \mathbb{R}$. Recall the random fields $\{H_n\}_{n=1}^{\infty}$ from (4.5). Our next result shows that H_n and I_n are close, on a suitable scale, and with high probability. The following constitutes the first step of the proof of Proposition 5.1.

Lemma 5.2. *For every $\delta \in (0\,,1)$ and for every closed interval $J \subset \mathbb{R}$, there exists $M = M(\delta\,,\alpha\,,J) > 0$ such that*

$$\mathrm{P}\left\{ \|H_n - I_n\|_{C([t_{n+1},t_n]\times J)} \geqslant \delta\psi(t_n) \right\} \leqslant M \exp\left(-\frac{\exp\left(n^{1+\alpha}\right)}{M\sqrt{\log_+(n)}} \right),$$

uniformly for all $n \in \mathbb{N}$.

Proof. Without too much loss in generality we consider only the case that $J = [-1\,,1]$. The general case is proved by making simple adjustments to the following.

Thanks to (4.5) and (5.2),

$$\mathrm{E}\left(|H_n(t\,,x) - I_n(t\,,x)|^2\right) = \int_0^{t-t_{n+1}} \mathrm{d}s \int_{y\in\mathbb{R}:|y|>2\sqrt{t_n\log(1/t_n)}} \mathrm{d}y\, [G_s(y)]^2$$
$$\leqslant \int_0^{t_n-t_{n+1}} \mathrm{d}s \int_{y\in\mathbb{R}:|y|>2\sqrt{t_n\log(1/t_n)}} \mathrm{d}y\, [G_s(y)]^2$$
$$\propto \int_0^{t_n-t_{n+1}} \frac{\mathrm{d}s}{\sqrt{s}} \int_{y\in\mathbb{R}:|y|>2\sqrt{t_n\log(1/t_n)}} \mathrm{d}y\; G_s\left(y/\sqrt{2}\right) \lesssim \int_0^{t_n} \exp\left(-\frac{t_n\log(1/t_n)}{s} \right) \frac{\mathrm{d}s}{\sqrt{s}},$$

uniformly for all $t \in [t_{n+1}\,, t_n]$, $x \in \mathbb{R}$, and $n \in \mathbb{N}$, thanks to the well-known fact that $\mathrm{P}\{|X| > r\} \leqslant 2\exp(-r^2/(4s))$ for all $r > 0$ if X has a centered normal distribution with variance $2s$ for some $s > 0$. If $s \leqslant t_n$, then $\exp(-t_n \log(1/t_n)/s) \leqslant t_n$. This yields

$$\sup_{t\in[t_{n+1},t_n]} \sup_{x\in\mathbb{R}} \mathrm{E}\left(|H_n(t\,,x) - I_n(t\,,x)|^2\right) \lesssim t_n^{3/2}, \tag{5.3}$$

valid uniformly for every $n \in \mathbb{N}$. Therefore, a metric entropy argument [15] yields the following: Uniformly for all $n \in \mathbb{N}$,

$$\mathrm{E}\|H_n - I_n\|_{C([t_{n+1},t_n]\times[-1,1])} \lesssim t_n^{3/4}\sqrt{\log(1/t_n)} \lesssim t_n^{1/4}\mathrm{e}^{-n^{1+\alpha}/4},$$

with room to spare. Now we apply concentration of measure [15] in conjunction with (5.3) in order to see that there exist $K_i = K_i(\alpha) > 0$ $[i = 1, 2]$ such that, uniformly for all $n \in \mathbb{N}$ and $z > 0$,

$$\mathrm{P}\left\{\|H_n - I_n\|_{C([t_{n+1},t_n]\times[-1,1])} \geqslant K_1 t_n^{1/4}\mathrm{e}^{-n^{1+\alpha}/4} + z\right\} \leqslant 2\mathrm{e}^{-K_2 z^2/t_n^{3/2}}. \tag{5.4}$$

Choose and fix some $\delta \in (0\,,1)$. For all sufficiently large $n \in \mathbb{N}$,

$$K_2 t_n^{1/4} \exp\left(-\frac{n^{1+\alpha}}{4}\right) \leqslant \frac{\delta}{2}\psi(t_n),$$

and how large depends only on $(\delta\,,\alpha)$. Therefore, we plug $z = \delta\psi(t_n)/2$ into (5.4) in order to conclude the proof. ☐

Our next lemma provides the second step in the proof of Proposition 5.1.

Lemma 5.3. *For every $\gamma > 0$,*

$$\lim_{n\to\infty} (\log n)^{-1} \log \mathrm{P}\left\{\|I_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\} = -2\lambda(1+\alpha)/(\pi\gamma^4).$$

Proof. The proof of Lemma 5.3 follows the same pattern as did the proof of Lemma 4.5, but uses respectively Lemma 4.5 and Lemma 5.2 in place of Lemmas 4.3 and 4.4. We leave the remaining details to the interested reader. ☐

The following probability evaluation is the third, and final, preliminary step in our proof of Proposition 5.1.

Lemma 5.4. *We have*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |I_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\} = -\left(\frac{2\lambda(1+\alpha)}{\pi\gamma^4} - q\right),$$

for every γ that satisfies (5.1).

Proof. If $x_1, x_2, \ldots \in \mathbb{R}$ satisfy the following for all distinct $i, j \in \mathbb{N}$,

$$|x_i - x_j| \geqslant 2\sqrt{t_n |\log t_n|} = 2\exp\left(-\frac{n^{1+\alpha}}{2}\right) n^{(1+\alpha)/2},$$

then $\{I_n(\,\cdot\,, x_i)\}_{i=1}^{\infty}$ are obtained by integrating white noise over disjoint sets. In particular, the above condition on $x_1, x_2, \ldots$ ensures that $\{I_n(\,\cdot\,, x_i)\}_{i=1}^{\infty}$ are i.i.d. random variables. If x, y are two distinct points in $\mathcal{D}_q(n)$, then

$$|x - y| \geqslant 2^{-\lfloor q \log_2 n \rfloor} \asymp n^{-q} \gg 2\exp\left(-\frac{n^{1+\alpha}}{2}\right) n^{(1+\alpha)/2},$$

valid for all n large, where how large depends only on $(\alpha\,, q)$. Thus, we can see that $\{I_n(\,\cdot\,, x)\}_{x \in \mathcal{D}_q(n)}$ is an i.i.d. sequence and hence, for every $\gamma > 0$,

$$\begin{aligned}
&\mathrm{P}\left\{\min\nolimits_{x\in\mathcal{D}_q(n)} \sup\nolimits_{t\in[t_{n+1},t_n]} |I_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\}\\
&\quad = 1 - \left(1 - \mathrm{P}\left\{\sup\nolimits_{t\in[t_{n+1},t_n]} |I_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\}\right)^{|\mathcal{D}_q(n)|}\\
&\quad = 1 - \left(1 - \exp\left\{-\frac{2\lambda(1+\alpha)+o(1)}{\pi\gamma^4}\log n\right\}\right)^{|\mathcal{D}_q(n)|} \qquad \text{[by Lemma 5.3]},
\end{aligned}$$

as $n \to \infty$. Since there exists $C > 0$ since $|\mathcal{D}_q(n)| \geqslant Cn^q$ for all $n \in \mathbb{N}$, condition (5.1) implies that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min\nolimits_{x\in\mathcal{D}_q(n)} \sup\nolimits_{t\in[t_{n+1},t_n]} |I_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\}&\\
\geqslant -\left(\frac{2\lambda(1+\alpha)}{\pi\gamma^4} - q\right).&
\end{aligned} \tag{5.5}$$

Conversely, since $|\mathcal{D}_q(n)| \asymp n^q$ uniformly for $n \in \mathbb{N}$, Boole's inequality and the apparent stationarity of $x \mapsto I_n(\,\cdot\,, x)$ yields the following, valid uniformly for all $n \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{P}\left\{\min\nolimits_{x\in\mathcal{D}_q(n)} \sup\nolimits_{t\in[t_{n+1},t_n]} |I_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\}&\\
\lesssim n^q \mathrm{P}\left\{\|I_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}.&
\end{aligned} \tag{5.6}$$

Therefore, in light of (5.5), it remains to prove that

$$\limsup_{n\to\infty} (\log n)^{-1} \log \mathrm{P}\left\{\|I_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\} \leqslant -\frac{2\lambda(1+\alpha)}{\pi\gamma^4}. \tag{5.7}$$

Let us choose and fix some $\delta \in (0\,,\gamma)$, as close to zero as we wish but fixed, and appeal to Lemma 5.2 in order to find a constant $M = M(\delta\,,\alpha) > 0$ such that

$$\begin{aligned}
&\mathrm{P}\left\{\|I_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\}\\
&\quad \leqslant \mathrm{P}\left\{\|H_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant (\gamma+\delta)\psi(t_n)\right\} + M\exp\left(-\frac{\exp\left(n^{1+\alpha}\right)}{M\sqrt{\log_+(n)}}\right),
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This and Lemma 4.5 together imply that

$$\limsup_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\|I_n(\cdot\,,0)\|_{C[t_{n+1},t_n]} \leqslant \gamma\psi(t_n)\right\} \leqslant -\frac{2\lambda(1+\alpha)}{\pi(\gamma+\delta)^4}.$$

Send $\delta \downarrow 0$ to deduce (5.7) and hence the lemma. □

We are ready to prove Proposition 5.1.

Proof of Proposition 5.1. Choose and fix $0 < \delta < \gamma$, where δ is fixed but small enough to ensure that

$$\gamma + \delta < \left(\frac{2\lambda(1+\alpha)}{\pi q}\right)^{1/4}. \tag{5.8}$$

Lemma 5.2 assures us that there exists $M = M(\delta\,,\alpha) > 0$ such that, uniformly for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\} \qquad (5.9)\\
&\leqslant \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |I_n(t\,,x)| \leqslant (\gamma+\delta)\psi(t_n)\right\} + M\exp\left(-\frac{\exp\left(n^{1+\alpha}\right)}{M\sqrt{\log_+(n)}}\right)\\
&\leqslant \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant (\gamma+2\delta)\psi(t_n)\right\} + 2M\exp\left(-\frac{\exp\left(n^{1+\alpha}\right)}{M\sqrt{\log_+(n)}}\right).
\end{aligned}$$

Therefore, Condition (5.8) and Lemma 5.4 together imply that the quantity in the middle line of (5.9) behaves, as $n \to \infty$, as $n^{-\ell(\delta)+o(1)}$, where

$$\ell(\delta) = q - \frac{2\lambda(1+\alpha)}{\pi(\gamma+\delta)^4}.$$

Consequently,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\} \leqslant -\ell(\delta), \qquad (5.10)\\
&\liminf_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant (\gamma+2\delta)\psi(t_n)\right\} \geqslant -\ell(\delta).
\end{aligned}$$

Send $\delta \downarrow 0$ such that (5.8) holds. The first line of (5.10) yields the following, valid under Condition (5.1) alone:

$$\limsup_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\} \leqslant -\ell(0).$$

And we can set $\gamma' = \gamma + 2\delta$ to deduce from the second line in (5.10) that

$$\liminf_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant \gamma'\psi(t_n)\right\} \geqslant -\ell(\delta),$$

for every pair (γ', δ) that satisfies

$$2\delta < \gamma' < 3\delta + \left(\frac{2\lambda(1+\alpha)}{\pi q}\right)^{1/4}.$$

Once again send $\delta \downarrow 0$ to deduce from the preceding effort the following: For every $\gamma > 0$ that satisfies (5.1),

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathrm{P}\left\{\min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)| \leqslant \gamma\psi(t_n)\right\} = -\ell(0).$$

To complete the proof, we rehash the above argument but replace the role of the ordered pair $(H_n\,, I_n)$ with that of $(H\,, H_n)$ and use Lemma 4.3 instead of Lemma 5.2. We leave the remaining details to the interested reader. □

Recall that $\rightsquigarrow$ denotes subsequential convergence. With that in mind, we have the following which is a stronger form of (1.7) when $\sigma \equiv 1$ and the SPDE is on the line rather than the torus.

Lemma 5.5. *With probability one, the random set*

$$\left\{x \in \mathbb{R}:\ \sup_{t\in[0,t_n]} \frac{|H(t\,,x)|}{\psi(t_n)} \rightsquigarrow \left(\frac{2\lambda(1+\alpha)}{\pi(1+q)}\right)^{1/4} \quad \text{as } n \to \infty\right\}$$

is dense in $\mathbb{R}$.

Proof. Choose and fix two numbers

$$0 < \rho_1 < 1 < \rho_2, \tag{5.11}$$

and define

$$\gamma_i = \left(\frac{2\lambda(1+\alpha)}{\pi(\rho_i + q)}\right)^{1/4} \qquad \text{for } i = 1, 2. \tag{5.12}$$

Note that $\gamma_1 > \gamma_2$.

Consider next the events $E_{1,i}, E_{2,i}, \ldots$ $[i = 1, 2]$, where for every $n \in \mathbb{N}$,

$$E_{n,1} = \left\{\omega \in \Omega:\ \min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)|(\omega) < \gamma_1\psi(t_n)\right\}, \quad \text{and}$$
$$E_{n,2} = \left\{\omega \in \Omega:\ \min_{x\in\mathcal{D}_q(n)} \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)|(\omega) \leqslant \gamma_2\psi(t_n)\right\}.$$

Thanks to (4.5) and basic properties of Wiener integrals, the events $\{E_{n,1}\}_{n=1}^\infty$ are independent (say). Moreover, Proposition 5.1 tells us that for $i = 1, 2$,

$$\mathrm{P}(E_{n,i}) = n^{-\rho_i + o(1)} \qquad \text{as } n \to \infty.$$

Therefore, (5.11) and a standard appeal to the Borel-Cantelli lemma together yield the following:

$$\mathrm{P}\left(\cap_{n=1}^\infty \cup_{l=n}^\infty E_{l,1}\right) = 1 \qquad \text{and} \qquad \mathrm{P}\left(\cap_{n=1}^\infty \cup_{l=n}^\infty E_{l,2}\right) = 0. \tag{5.13}$$

Now consider the random sets defined by

$$A_n\,(\rho_1\,,\rho_2) = \left\{ x\in\mathbb{R}:\ \gamma_2\psi(t_n) < \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)|(\omega) < \gamma_1\psi(t_n) \right\}, \quad (5.14)$$

for all $n\in\mathbb{N}$ and ρ_1,ρ_2 that satisfy (5.11). Then, (5.13) says that, with probability one,

$$A_n(\rho_1\,,\rho_2)\cap\mathcal{D}_q(n)\neq\varnothing \quad \text{for infinitely many } n\in\mathbb{N}.$$

Since $\mathcal{D}_q(n)\subset\mathcal{D}_q(n+1)$, this implies that

$$\cup_{k=n}^{\infty}A_k(\rho_1\,,\rho_2)\cap\cup_{m=1}^{\infty}\mathcal{D}_q(m)\neq\varnothing \quad \text{for infinitely many } n\in\mathbb{N}\text{, almost surely.}$$

And because $\cup_{m=1}^{\infty}\mathcal{D}_q(m)$ coincides with the collection of all dyadic rationals in $[0\,,\theta]$, it follows that

$$\cup_{k=n}^{\infty}A_k(\rho_1\,,\rho_2)\cap[0\,,\theta]\neq\varnothing \quad \text{for infinitely many } n\in\mathbb{N}\text{, almost surely.}$$

Because the random field $x\mapsto H(\,\cdot\,,x)$ is stationary, the above continues to hold if we replace $[0\,,\theta]$ by any non-random, bounded, open interval $J\subset\mathbb{R}$. This implies in turn that, with probability one,

$$\cup_{k=n}^{\infty}A_k(\rho_1\,,\rho_2)\cap J\neq\varnothing \quad \text{i.o., } \forall \text{ bounded open interval } J\subset\mathbb{R} \text{ with rational ends,}$$

where "i.o." denotes "infinitely often," and refers to the occurrence of the event in question for infinitely-many [random] $n\in\mathbb{N}$. A consequence of this is that, with probability one,

$$\cup_{k=n}^{\infty}A_k(\rho_1\,,\rho_2) \text{ is dense in } \mathbb{R} \text{ i.o.}$$

Since the [random] set $\cup_{k=n}^{\infty}A_k(\rho_1\,,\rho_2)$ is open for every $n\in\mathbb{N}$, the Baire category theorem ensures that, with probability one,

$$\cap_{\substack{(\rho_1,\rho_2)\in\mathbb{Q}_+^2:\\ 0<\rho_1<1<\rho_2}} \cap_{n=1}^{\infty}\cup_{k=n}^{\infty}A_k(\rho_1\,,\rho_2) \text{ is dense in } \mathbb{R}. \quad (5.15)$$

Thanks to (5.12), we have proved that with probability one,

$$\left\{ x\in\mathbb{R}:\ \sup_{t\in[t_{n+1},t_n]}\frac{|H_n(t\,,x)|}{\psi(t_n)} \rightsquigarrow C^{1/4} \text{ as } n\to\infty \right\} \text{ is dense in } \mathbb{R},$$

where $C=\frac{2\lambda(1+\alpha)}{\pi(1+q)}$. Therefore, Lemma 4.3, and a standard appeal to the Borel-Cantelli lemma together yield the following a.s. statement:

$$\left\{ x\in\mathbb{R}:\ \sup_{t\in[t_{n+1},t_n]}\frac{|H(t\,,x)|}{\psi(t_n)} \rightsquigarrow C^{1/4} \text{ as } n\to\infty \right\} \text{ is dense in } \mathbb{R}. \quad (5.16)$$

Yet another appeal to the Borel-Cantelli lemma, this time in conjunction with Lemma 4.2, implies that with probability one $\sup_{t\in[0,t_{n+1}]}\sup_{x\in[-1,1]}|H(t\,,x)| = o(\psi(t_n))$ as $n\to\infty$. This and (5.16) together yield the lemma. □

The following verifies a stronger form of (1.8) when $\sigma \equiv 1$ and the SPDE is on $\mathbb{R}$ rather than $\mathbb{T}$.

Lemma 5.6. *With probability one, the random set*

$$\left\{x \in \mathbb{R} : \liminf_{\varepsilon \downarrow 0} \sup_{t \in [0,\varepsilon]} \frac{|H(t\,,x)|}{\psi(\varepsilon)} = \left(\frac{2\lambda}{\pi(1+q)}\right)^{1/4} \quad as\ \varepsilon \to 0\right\} \tag{5.17}$$

is dense in $\mathbb{R}$.

Proof. The proof is similar to that of Lemma 5.5, but requires making a number of subtle changes that we describe next. Perhaps most notably, and in contrast with the proof of Lemma 5.5, we will use different sequences for the upper and the lower bounds on the supremum of $|H|$.

For the upper bound, we follow the proof of Lemma 5.5 and let γ_1 be as was defined in (5.12) where $\rho_1 \in (0\,,1)$, but rather than use the random sets A_n from (5.14), we define new random sets $\tilde{A}_n$ as follows:

$$\tilde{A}_n\,(\rho_1\,,\alpha) := \left\{x \in \mathbb{R} :\ \sup\nolimits_{t \in [0\,,t_n]} |H(t\,,x)|(\omega) < \gamma_1 \psi(t_n)\right\}. \tag{5.18}$$

We are including the parameter α, inherited through the choice of the sequence $\{t_n\}_{n \in \mathbb{N}}$ [see (4.2)], for reasons that will become manifest soon. We follow closely the proof of Lemma 5.5 in order to find that with probability one,

$$\cup_{k=n}^{\infty} \tilde{A}_k(\rho_1\,,\alpha) \quad \text{is dense in } \mathbb{R} \text{ i.o..}$$

Next, we introduce a sequence that is notably distinct from $\{t_n\}_{n=1}^{\infty}$: First, choose and fix two numbers c and ρ_2 that satisfy

$$0 < c < 1 < \rho_2,$$

and define

$$s_n = c^n \quad \text{and} \quad \gamma_2 = \left(\frac{2\lambda}{\pi(\rho_2 + q)}\right)^{1/4}.$$

Consider the event $\tilde{E}_1, \tilde{E}_2, \cdots$ where

$$\tilde{E}_n = \left\{\omega \in \Omega :\ \min\nolimits_{x \in \mathcal{D}_q(n)} \sup\nolimits_{s \in [0\,,s_n]} |H(s\,,x)|(\omega) \leqslant \gamma_2 \psi(s_n)\right\} \quad \text{for every } n \in \mathbb{N}.$$

As was done in (5.6), we may appeal to Lemma 2.9 in order to deduce that

$$\mathrm{P}(\tilde{E}_n) = n^{-\rho_2 + o(1)} \qquad \text{as } n \to \infty.$$

Therefore, the Borel-Cantelli lemma yields

$$\mathrm{P}\left(\cap_{n=1}^{\infty} \cup_{l=n}^{\infty} \tilde{E}_l\right) = 0.$$

Define random sets $B_1(\rho_2), B_2(\rho_2), \ldots \subset \mathbb{R}$ via

$$B_n(\rho_2) := \left\{x \in \mathbb{R} :\ \gamma_2 \psi(s_n) < \sup\nolimits_{t \in [0\,,s_n]} |H(t\,,x)|\right\} \quad \text{for } n \in \mathbb{N}.$$

Then a similar argument as the proof of Lemma 5.5 shows us that, with probability 1,

$$\cup_{l=m}^{\infty} B_l(\rho_2) \quad \text{is dense in } \mathbb{R} \text{ i.o.}$$

If there is a realization $[\omega \in \Omega]$ for which the random open sets $\cup_{k=n}^{\infty} \tilde{A}_k(\rho_1\,, \alpha)$ and $\cup_{l=m}^{\infty} B_l(\rho_2)$ are dense, then, for that very realization, the random set $\{\cup_{k=n}^{\infty} \tilde{A}_k(\rho_1)\} \cap \{\cup_{l=m}^{\infty} B_l(\rho_2)\}$ is dense in $\mathbb{R}$ thanks to the Baire category theorem and the fact that $\cup_{k=n}^{\infty} \tilde{A}_k(\rho_1\,, \alpha)$ and $\cup_{l=m}^{\infty} B_l(\rho_2)$ are open sets for every $n, m \geqslant 1$. Therefore, we may apply the Baire category theorem, in much the same way as we just did, one more time in order to establish that, with probability 1,

$$\cap_{\alpha \in \mathbb{Q}_+} \cap_{\substack{(\rho_1,\rho_2)\in\mathbb{Q}_+^2:\\ 0<\rho_1<1<\rho_2}} \cap_{n=1,m=1}^{\infty} \left\{ \cup_{k=n}^{\infty} \tilde{A}_k(\rho_1\,, \alpha) \right\} \cap \left\{ \cup_{l=m}^{\infty} B_l(\rho_2) \right\} \text{ is dense in } \mathbb{R}.$$

We can deduce (5.17) from the above, once we unpack the preceding. $\square$

Proof of Theorem 1.3. First, let us observe that with probability one, the random set

$$\left\{ x \in \mathbb{T} : \sup_{t\in[0,t_n]} \frac{|Z(t\,,x)|}{\psi(t_n)} \rightsquigarrow \left(\frac{2\lambda(1+\alpha)}{\pi(1+q)} \right)^{1/4} \text{as } n \to \infty \right\} \text{ is dense in } \mathbb{T}. \quad (5.19)$$

Indeed, we may appeal to Lemma 2.11

$$\overline{\lim_{\varepsilon\downarrow 0}} \sqrt{\frac{\varepsilon}{\log|\log\varepsilon|}} \log \mathrm{P} \left\{ \|H - Z\|_{C([0,\varepsilon]\times\mathbb{T})} \geqslant \left(\frac{\delta\varepsilon}{\log|\log\varepsilon|} \right)^{1/4} \right\} \leqslant -\frac{\sqrt{\delta}}{10}.$$

In turn, this inequality and a standard application of the Borel-Cantelli lemma together imply that $\|H - Z\|_{C([0,\varepsilon]\times\mathbb{T})} = o(\psi(\varepsilon))$ a.s. as $\varepsilon \downarrow 0$. Therefore, Lemma 5.5 implies (5.19).

Set

$$\chi = \frac{2\lambda(1+\alpha)}{\pi(1+q)},$$

and observe that χ can take any value in $(0\,,\infty)$. This is because the numbers $q > 0$ and $\alpha > 0$ [see (4.2)] can be chosen otherwise arbitrarily.

Next we use Corollary 3.2 in order to deduce from (5.19) that

$$\left\{ x \in \mathbb{T} : \sup_{t\in[0,t_n]} |u(t\,,x) - (p_t * u_0)(x)|/\psi(t_n) \rightsquigarrow \chi^{1/4}|\sigma(u_0(x))| \text{ as } n \to \infty \right\}$$

is dense in $\mathbb{T}$. Because u_0 is Lipschitz continuous, this implies that almost surely,

$$\left\{ x \in \mathbb{T} : \sup_{t\in[0,t_n]} |u(t\,,x) - u_0(x)|/\psi(t_n) \rightsquigarrow \chi^{1/4}|\sigma(u_0(x))| \text{ as } n \to \infty \right\}$$

is dense in $\mathbb{T}$. This proves (1.7) of Theorem 1.3 in the case that $\chi \in (0\,,\infty)$. And when $\chi \in (0\,, 2\lambda/\pi)$, the very same argument works to prove (1.8), except we appeal to Lemma 5.6 in place of Lemma 5.5 everywhere and make adjustments for the change accordingly.

For the proof of (1.8), the case $\chi = 2\lambda/\pi$ is covered already by Corollary 1.2. For the proof of (1.7), the cases where $\chi = 0$ and $\chi = \infty$ remain to be verified; all else has been proved so far. The remaining two cases are handled analogously by making adjustments to the preceding arguments. Therefore, we will describe the changes for the proof of (1.7) in the case that $\chi = 0$ and leave the requisite argument for the remaining case [(1.7) when $\chi = \infty$] to the interested reader.

Choose and fix some $\rho_1 \in (0\,,1)$ and define, in analogy with (5.14),

$$\bar{A}_n(q) = \left\{ x \in \mathbb{R} :\, \sup_{t\in[t_{n+1},t_n]} |H_n(t\,,x)|(\omega) < \gamma_1 \psi(t_n) \right\},$$

where now we are emphasizing the dependence of $\bar{A}_n$ on q and not ρ_1. Thanks to (5.13), another category argument yields the following adaptation of (5.15):

$$\cap_{\substack{q>0\\ q\in\mathbb{Q}}} \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \bar{A}_k(q) \neq \varnothing \quad \text{is dense in } \mathbb{R}.$$

Therefore, we obtain, using the same argument as before, the following adaptation of (5.16):

$$\left\{ x \in \mathbb{R} :\, \sup_{t\in[t_{n+1},t_n]} |H(t\,,x)|/\psi(t_n) \rightsquigarrow 0 \text{ as } n \to \infty \right\} \text{ is dense in } \mathbb{R} \text{ a.s.},$$

and hence

$$\left\{ x \in \mathbb{R} :\, \sup_{t\in[0,t_n]} |H(t\,,x)|/\psi(t_n) \rightsquigarrow 0 \text{ as } n \to \infty \right\} \text{ is dense in } \mathbb{R} \text{ a.s.},$$

thanks to the same argument that was used at the very end of the proof of Lemma 5.5. We now go through the proof of Theorem 1.3 line by line, making only very small changes to adapt the argument, in order to formally justify setting $q = \infty$ in order to finish the proof of the case where $\chi = 0$. This completes our presentation. □

References

[1] Anderson, T. W. The integral of a a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** (1955) 170–176. MR0069229

[2] Athreya, Siva, Joseph, Mathew, and Mueller, Carl. Small ball probabilities and a support theorem for the stochastic heat equation. *Ann. Probab.* **49***(2)* (2021) 2548–2572. MR4317712

[3] Chen, Jiaming. Chung's Law of the Iterated Logarithm for a Class of Stochastic Heat Equations. *Electr. Comm. Probab.* **28** (2023) 1–7. MR4651160

[4] Chung, Kai Lai. On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64** (1948) 205–233. MR0026274

[5] Da Prato, Giuseppe and Zabczyk, Jerzy. *Stochastic Equations in Infinite Dimensions.* First Ed. 1992, Second Ed. 2014. Encyclopedia Math. Appl., 152. Cambridge University Press, Cambridge. MR3236753

[6] Dereich, S., Fehringer, F., Maroussi, A., and Scheutzow, M. On the link between small ball probabilities and the quantization problem for Gaussian measures on Banach spaces. *J. Theoret. Probab.* **16***(1)* (2003) 249–265. MR1956830

[7] Foondun, Mohammud, Joseph, Mathew, and Kim, Kunwoo. Small ball probability estimates for the Hölder semi-norm of the stochastic heat equation. *Probab. Theory Related Fields* **185***(1-2)* (2023) 553–613. MR4528976

[8] Huang, Jingyu and Khoshnevisan, Davar. On the multifractal local behavior of parabolic stochastic PDEs. *Electron. Commun. Probab.* **22** (2017) Paper No. 49, 11. MR3710805

[9] Hairer, Martin and Pardoux, Étienne. A Wong-Zakai theorem for stochastic PDEs. *J. Math. Soc. Japan* **67***(4)* (2015) 1551–1604. MR3417505

[10] Khoshnevisan, Davar. *Analysis of Stochastic Partial Differential Equations.* CBMS Reg. Conf. Ser. Math., 119. American Math. Soc. Providence, RI, 2014, viii+116 pp.

[11] Khoshnevisan, Davar, Peres, Yuval, and Shi, Zhan. Limsup random fractals. *Electr. J. Probab.* **5***(5)* (2000) 24 pp. MR1743726

[12] Khoshnevisan, Davar, Swanson, Jason, Xiao, Yimin, and Zhang, Liang. Weak existence of a solution to a differential equation driven by a very rough fBm. ArXiv e-prints (2013), arXiv:1309.3613.

[13] Klartag, B. and Vershynin R. Small ball probability and Dvoretzky's theorem. *Isreal J. Math.* **157** (2007) 193–207. MR2342445

[14] Kuelbs, James and Li, Wenbo V. Metric entropy and the small ball problem for Gaussian measures. *J. Funct. Anal.* **116***(1)* (1993) 133–157. MR1237989

[15] Ledoux, Michel. *Isoperimetry and Gaussian Analysis.* In: Lectures on Probability Theory and Statistics (Saint-Flour, 1994), 165–294. Lecture Notes in Math., 1648 Springer-Verlag, Berlin, 1996. MR1600888

[16] Lee, Cheuk-Yin and Xiao, Yimin. Chung-type law of the iterated logarithm and exact moduli of continuity for a class of anisotropic Gaussian random fields. *Bernoulli* **29***(1)* (2023) 523–550. MR4497257

[17] Lei, Pedro and Nualart, David. A decomposition of the bifractional Brownian motion and some applications. *Statist. Probab. Lett.* **79***(9)* (2009) 619–624

[18] Li, Wenbo V. A Gaussian correlation inequality and its applications to small ball probabilities. *Electr. Comm. Probab.* **4** (1999) 111–118. MR1741737

[19] Li, Wenbo V. and Linde, Werner. Approximation, metric entropy and small ball estimates for Gaussian measures. *Ann. Probab.* **27***(3)* (1999) 1556–1578. MR1733160

[20] Li, Wenbo V. and Shao, Qi-Man. Gaussian processes: inequalities, small ball probabilities and applications. In: *Stochastic Processes: Theory and Methods*, 533–597. Handbook of Statistics, 19. North Holland Publishing Co., Amsterdam, 2001. MR1861734

[21] Li, Wenbo V. and Shao, Qi-Man. A note on the Gaussian correlation conjecture. In: *High-Dimensional Probability, II (Seattle, WA, 1999)*, 163–171. Progr. Probab. 47. Birkhäuser Boston, Boston, MA, 2000 MR1857321

[22] Martin, A. Small ball asymptotics for the stochastic wave equation. *J. Theoret. Probab.* **17***(3)* (2004) 693–703. MR2091556

[23] Madelbrot, Benoit B. and Van Ness, John W. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** (1968) 422–437. MR0242239

[24] Nazarov, Alexander and Petrova, Yulia. L_2-small ball asymptotics for Gaussian random functions: a survey. *Probab. Surv.* **20** (2023) 608–663. MR460534

[25] Shao, Qi-Man. A Gaussian correlation inequality and its applications to the existence of small ball constant. *Stochastic Process. Appl.* **107***(2)* (2003) 269–287. MR1999791

[26] Sowers, Richard B. Large deviations for a reaction-diffusion equation with non-Gaussian perturbations. *Ann. Probab.* **20***(1)* (1992) 504–537. MR1143433

[27] Talagrand, M. New Gaussian estimates for enlarged balls. *Geometric Funct. Anal.* **3** (1993) 502–520. 1233864

[28] Talagrand, M. The small ball problem for the Brownian sheet. *Ann. Probab.* **22***(3)* (1994) 767–775. MR1334170

[29] Talagrand, M. Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Prob.* **23***(2)* (1995) 767–775.

[30] Walsh, John B. *An Introduction to Stochastic Partial Differential Equations.* In: École d'été probabilités de Saint-Flour, XIV–1984, 265-439. Lecture Notes in Math, 1180, Springer-Verlag, Berlin, 1986. MR0876085

Davar Khoshnevisan. Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, USA, davar@math.utah.edu

Kunwoo Kim. Department of Mathematics, Pohang University of Science and Technology (POSTECH), Pohang, Gyeongbuk, Korea 37673, kunwoo@postech.ac.kr

Carl Mueller. Department of Mathematics, University of Rochester, Rochester, NY 14627, USA,
carl.e.mueller@rochester.edu