THE RADIUS OF A SELF-REPELLING STAR POLYMER

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Abstract. We study the effective radius of weakly self-avoiding star polymers in one, two, and three dimensions. Our model includes N Brownian motions up to time T, started at the origin and subject to exponential penalization based on the amount of time they spend close to each other, or close to themselves. The effective radius measures the typical distance from the origin. Our main result gives estimates for the effective radius where in two and three dimensions we impose the restriction that $T \leq N$. One of the highlights of our results is that in two dimensions, we find that the radius is proportional to $T^{3/4}$, up to logarithmic corrections. Our result may shed light on the well-known conjecture that for a single self-avoiding random walk in two dimensions, the end-to-end distance up to time T is roughly $T^{3/4}$.

1. Introduction

Random polymer models have caught the imagination of many mathematicians. Polymers are all around us, and their behavior presents attractive mathematical challenges, many of which still defy solution. See Doi and Edwards [7] for a wide-ranging treatment from the physical point of view, and Madras and Slade [14], den Hollander [6], Giacomin [10], and Bauerschmidt et. al. [2] for rigorous mathematical results. Van der Hofstad and König [17] discuss the one-dimensional case.

In continuous time, we can view a polymer as a Brownian motion with penalization for self-intersections. Here the time parameter represents distance along the polymer. For $T > 0$, let $(B_t)_{t\in[0,T]}$ be a standard Brownian motion in $\mathbb{R}^d$, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, P_T)$. For a probability measure P, we write E^P for the corresponding expectation. Since Brownian motion does not have self-intersections in high dimensions, we study close approaches instead. For any $r > 0$ let $\mathbf{B}_r(x) \subset \mathbb{R}^d$ be the open ball of radius r centered at

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$x \in \mathbb{R}^d$, and define

$$L_T(x) = L_{d,T}(x) := \int_0^T \mathbf{1}_{\mathbf{B}_1(x)}(B_t)dt$$

for $T > 0$. For $\beta > 0$, the typical penalization term is

$$\mathcal{E}_T = \mathcal{E}_{d,\beta,T} := \exp\left(-\beta \int_{\mathbb{R}^d} L_T(x)^2 dx\right).$$

Then we define the penalized measure as

$$Q_T(A) = Q_{d,\beta,T}(A) := \frac{1}{Z_T} E^{P_T}\left[\mathbf{1}_A \mathcal{E}_T\right]$$
$$Z_T = Z_{d,\beta,T} = E^{P_T}\left[\mathcal{E}_T\right]$$

for $A \in \mathcal{F}$.

With these definitions, we call our process weakly self-avoiding Brownian motion.

Note that all of the above quantities depend implicitly on d and all but $P, \mathbf{B}, (B_t), L$ depend on β as well. For simplicity of notation we suppress these dependencies, and we will use similar simplified notation throughout the paper. Furthermore, C will stand for a constant which could change from line to line, and may also depend on d.

One of the most important problems about weakly self-avoiding Brownian motion is to study the radius of the polymer, often defined as the standard deviation of the end-to-end distance,

$$R_T = R_{d,\beta,T} = \left(E^{Q_T}\left[B_T^2\right]\right)^{1/2}.$$

A well-known conjecture from physics states that there exists a scaling exponent ν_d not depending on β such that, in some unspecified sense,

$$R_{d,\beta,T} \approx T^{\nu_d}$$

as $T \to \infty$. All that is rigorously known is that $\nu_1 = 1$ (Bolthausen [4], Greven and den Hollander [11]) and that $\nu_d = 1/2$ for $d \geq 5$ (Hara and Slade [12]). It is believed that $\nu_2 = 3/4$, and there are connections to $\mathrm{SLE}_{8/3}$ (see Lawler, Schramm, and Werner [13]). This conjecture has received enormous attention, and Duminil-Copin and Smirnov [8], page 9, write "The derivation of these exponents seems to be one of the most challenging problems in probability." In [2] Section 1.5.2, we learn that "Almost nothing is known rigorously about ν in dimensions 2, 3, 4. It is an open problem to show that the mean-square displacement grows at least as rapidly as simple random walk, and grows more slowly than ballistically". In our language, this means that for $d \in \{2, 3, 4\}$ it has not been proved that $\nu_d \geq 1/2$ or $\nu_d < 1$. One of the highlights of the present work is that we do obtain the exponent $3/4$ in $d = 2$, see Theorem 1.1. Of course this does not settle the above conjecture.

In the real world, many polymers are branched and do not consist of a single strand. van der Hofstad and König [17], (Section 3.1 pages 16 - 18) give a short discussion of branched polymers taking values in $\mathbb{R}$, and present a conjecture for the growth of the radius. As far as we know, the conjecture is still open. Slade and van der Hofstad [15] use the lace expansion to study the radius for branched polymers in high dimensions, and show that they behave as if there were no self-avoidance.

Since self-avoiding polymers present difficult challenges in low dimensions, we focus on the case of star polymers, which are not too different from random walks. Star polymers are polymers joined at the point $t = 0$, and there is an extensive physics literature about them. See the seminal paper of Daoud and Cotton [5], and for more recent work see [1] and [16].

We now give a brief overview of the results from [5] which are relevant to this paper, and ask the reader to keep in mind that these results come from mathematically non-rigorous arguments.

First, the authors formulate a notion of radius relevant to star polymers. Then, for a given value of T, they discuss 3 regions:

(1) The swollen region far from the origin, where pairs of paths rarely overlap.
(2) The unswollen region closer to the origin, where many paths overlap.
(3) The core, which is even closer to the origin.

Our results deal with regions (2) and (3). One of the principal results of [5] is equation (19), which states that for very long chains, or for high temperatures,

$$R \approx N^{3/5} v^{1/5} f^{1/5} \ell. \tag{1.1}$$

To aid the comparison with our results, we give a dictionary for the notation in the above equation.

Quantity	Our Notation	From [5]
Number of branches	N	f
Length along the polymer	T	N
Self-avoidance parameter	β	v
Length of each polymer element	not included	ℓ

Translating to our notation, (1.1) means, for $d = 3$, that

$$R_T \approx \beta^{1/5} N^{1/5} T^{3/5} \tag{1.2}$$

for large values of T. Again, this is a nonrigorous physical result. Our main result, Theorem 1.1, gives a range of values for R_T in $d = 3$, that includes (1.2). The physical reasoning in Daoud and Cotton's paper yields the following conjecture for the two dimensional model,

$$R \approx \beta^{1/4} N^{1/4} T^{3/4}. \tag{1.3}$$

We refer to Bishop et al. [3] who studied the two dimensional star polymer model. In Theorem 1.1 we verify (1.3) up to a logarithmic constant.

We will now give rigorous definitions, and redefine the notation P_T, Q_T, $L_T(\cdot)$, $\mathcal{E}_T$, R_T used earlier. Assume that for each $T > 0$ we have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P_T)$, and on this space, for $d \geq 1$, we have a collection $(B_t^{(k)})_{t\in[0,T];\, k\in\{1,\dots,N\}}$ of independent adapted $\mathbb{R}^d$-valued standard Brownian motions started at the origin. Without loss of generality, we can assume that $\Omega = (C[0,T])^N$ is canonical path space for the Brownian motions.

We define a weakly self-avoiding model as follows. For $N \in \mathbb{N} = \{1, 2, \dots\}$, $T > 0$, $x \in \mathbb{R}^d$, and $\mathbf{B}_1(x) \subset \mathbb{R}^d$ as before, consider the occupation measure

$$L_T(x) = L_{T,d,N}(x) := \sum_{k=1}^{N} \left|\{t \in [0,T] : B_t^{(k)} \in \mathbf{B}_1(x)\}\right| \tag{1.4}$$

where $|S|$ denotes the Lebesgue measure of a Borel set $S \subset \mathbb{R}$. Our penalization factor is defined as

$$\mathcal{E}_T := \mathcal{E}_{d,\beta,N,T} = \exp\left(-\beta \int_{\mathbb{R}^d} L_T(x)^2 dx\right).$$

We define a probability $Q_T = Q_{T,d,N,\beta}$ and a normalizing factor $Z_T = Z_{T,d,N,\beta}$ as

$$Q_T(S) := \frac{1}{Z_T} E^{P_T}\big[\mathbf{1}_S \mathcal{E}_T\big], \qquad Z_T := E^{P_T}\big[\mathcal{E}_T\big]. \tag{1.5}$$

for $S \in \mathcal{F}_T$.

For any set of real numbers $A = \{a_1, ..., a_N\}$ we denote by $\text{med}(A)$ the median of the set. We define the radius of the star polymer as follows,

$$R_T = R_{d,\beta,N,T} := \text{med}\left(\left\{\sup_{t\in[0,T]} |B_t^{(k)}| \,:\, k = 1, ..., N\right\}\right). \tag{1.6}$$

Our goal is to study the behavior of R_T under the measure Q_T.

Note that we have chosen a different definition of radius in this case. However, the original definition of radius in terms of end-to-end distance may not be the most physically relevant. Indeed, the median distance from the center of mass may be an easier quantity to measure experimentally. In the scientific literature, the radius of gyration is usually defined as the square root of the mean square distance to the center of mass, see Fixman [9].

In order to to study our radius R_T, we introduce the following events

$$\begin{aligned} A^{(<)}_{T,r} &= \{R_T \le r\}, \\ A^{(>)}_{T,r} &= \{R_T \ge r\}. \end{aligned} \tag{1.7}$$

We will show that for appropriate functions $r_1(T), r_2(T)$,

$$\begin{aligned} \lim_{T\to\infty} Q_T\big(A^{(<)}_{T,r_1(T)}\big) &= 0, \\ \lim_{T\to\infty} Q_T\big(A^{(>)}_{T,r_2(T)}\big) &= 0. \end{aligned}$$

Our main result is stated in the following theorem.

Theorem 1.1. *There exists positive constants C_d, c_d and C not depending on N, T and β such that,*

(i) *for $d = 1$, for all $\beta, N \ge 1$, and $T \ge c_1^{-1}$,*

$$Q_T\left(c_1\beta^{\frac{1}{3}}N^{\frac{1}{3}}T \le R_T \le C_1\beta^{\frac{1}{3}}N^{\frac{1}{3}}T\right) \ge 1 - \exp\left(-C\beta^{\frac{2}{3}}N^{\frac{5}{3}}T\right),$$

(ii) *for $d = 2$, for all $\beta \ge 1$, $N \ge (c_2^{-4/3} \vee 1)$ and $c_2^{-4/3} \le T \le N$,*

$$\begin{aligned} Q_T\left(c_2\beta^{\frac{1}{4}}N^{\frac{1}{4}}T^{\frac{3}{4}}(\log(\beta T))^{-\frac{1}{2}} \le R_T \le C_2\beta^{\frac{1}{4}}N^{\frac{1}{4}}T^{\frac{3}{4}}(\log(\beta T))^{\frac{1}{2}}\right) \\ \ge 1 - \exp\left(-C\beta^{\frac{1}{2}}N^{\frac{3}{2}}T^{\frac{1}{2}}\log(\beta T)\right), \end{aligned}$$

(iii) *for $d = 3$, for all $\beta \ge 1$, $N \ge (c_3^{-2} \vee 1)$ and $c_3^{-2} \le T \le N$,*

$$\begin{aligned} Q_T\left(c_3\beta^{\frac{1}{6}}N^{\frac{1}{6}}T^{\frac{1}{2}}(\log(\beta T))^{-\frac{1}{3}} \le R_T \le C_3\beta^{\frac{1}{4}}N^{\frac{1}{4}}T^{\frac{3}{4}}(\log(\beta T))^{\frac{1}{2}}\right) \\ \ge 1 - \exp\left(-C\beta^{\frac{1}{2}}N^{\frac{3}{2}}T^{\frac{1}{2}}\log(\beta T)\right). \end{aligned}$$

Note that the upper and lower bounds in part (iii) of include the physical result given in (1.2).

The proof of Theorem 1.1 is given in Section 3.

2. STRATEGY

To analyze (1.5), suppose we are given functions $r_i : (0,\infty) \to (0,\infty)$ with $i \in \{1,2\}$. We define

$$
\begin{aligned}
q_T^{(<)} &= Q_T(A^{(<)}_{T,r_1(T)}) = E^{P_T}\Big[\mathbf{1}_{A^{(<)}_{T,r_1(T)}}\mathcal{E}_T\Big], \\
q_T^{(>)} &= Q_T(A^{(>)}_{T,r_2(T)}) = E^{P_T}\Big[\mathbf{1}_{A^{(>)}_{T,r_2(T)}}\mathcal{E}_T\Big].
\end{aligned}
\tag{2.1}
$$

Note that $q_T^{(<)}, q_T^{(>)}$ implicitly depend on r_1, r_2 and also d, N, β. Then

$$
q_T^{(<)} \le \sup_{\omega \in A^{(<)}_{T,r_1(T)}} \mathcal{E}_T(\omega), \tag{2.2}
$$

$$
q_T^{(>)} \le E^{P_T}\Big[\mathbf{1}_{A^{(>)}_{T,r_2(T)}}\Big] = P_T\big(A^{(>)}_{T,r_2(T)}\big). \tag{2.3}
$$

We first consider (2.2). Now (1.6) shows that on $A^{(<)}_{T,r_1(T)}$, at least $[N/2]$ Brownian motions satisfy $\sup_{t\in[0,T]}|B_t^{(k)}| \le r_1(T)$. Here for any $x \in \mathbb{R}$, $[x]$ is the greatest integer less than or equal to x. On $A^{(<)}_{T,r_1(T)}$, let $\{k_1, \ldots, k_{[N/2]}\}$ be the first $[N/2]$ indices of the Brownian motions satisfying this condition, and define $L_T^{(\mathrm{med})}$ be the total occupation measure of these Brownian motions.

On $A^{(<)}_{T,r_1(T)}$ we have that $L_T^{(\mathrm{med})}(\cdot)$ is supported on $\mathbf{B}_{r_1(T)+1}(0)$ and $\int_{\mathbb{R}^d} L_T^{(\mathrm{med})}(x)dx = [N/2]T$. The Cauchy-Schwarz inequality shows that among nonnegative functions f supported on $\mathbf{B}_{r_1(T)+1}(0)$, such that $\int_{\mathbb{R}^d} f(x)dx = [N/2]T$, the minimum of $\int_{\mathbb{R}^d} f(x)^2dx$ is achieved when f equals a constant K on $\mathbf{B}_{r_1(T)+1}(0)$, and in that case

$$
K = \frac{[N/2]T}{V_d \cdot (r_1(T)+1)^d},
$$

where V_d is the volume the unit d-dimensional ball.

So, on $A^{(<)}_{T,r_1}$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} L_T(y)^2 dy &\ge K^2 |\mathbf{B}_{r_1(T)+1}(0)| \\
&\ge \left(\frac{[N/2]T}{V_d \cdot (r_1(T)+1)^d}\right)^2 V_d \cdot (r_1(T)+1)^d \\
&= C\frac{N^2T^2}{(r_1(T)+1)^d}.
\end{aligned}
$$

Then by the definition of $\mathcal{E}_T$, we have

$$q_T^{(<)} \le \sup_{\omega \in A^{(<)}_{T,r_1(T)}} \mathcal{E}_T(\omega) \le \exp\left(-\beta C \frac{N^2T^2}{(r_1(T)+1)^d}\right). \tag{2.4}$$

Now we turn to (2.3). In order to bound $q_T^{(>)}$, we use the following large deviations result.

Lemma 2.1. *Given $p \in (0,1)$, let $(X_i)_{i\ge1}$ be an infinite sequence of Bernoulli random variables with $P(X_i = 1) = p$ and $P(X_i = 0) = q := 1-p$. Define $S_n := \sum_{i=1}^n X_i$. For $\alpha \in (p,1)$ we have*

$$P(S_n > \alpha n) \le \left(\frac{(1-\alpha)p}{\alpha q}\right)^{\alpha n} \left[q + p\frac{\alpha q}{(1-\alpha)p}\right]^n.$$

Proof of Lemma 2.1. Using the moment generating function of the binomial distribution and choosing $t > 0$, for $\alpha \in (p,1)$ we get

$$\begin{aligned} P\left(S_n > \alpha n\right) &= P\big(\exp\left(tS_n\right) > \exp\left(\alpha t n\right)\big) \\ &\le \frac{E\left[\exp\left(tS_n\right)\right]}{\exp\left(\alpha t n\right)} \\ &= \exp\left(-\alpha t n\right)\left[q + pe^t\right]^n \\ &= \left(e^{-\alpha t}\left[q+pe^t\right]\right)^n. \end{aligned} \tag{2.5}$$

Let $f(t) = e^{-\alpha t}[q + pe^t]$ and note that f is differentiable on $\mathbb{R}$. We compute

$$\begin{aligned} f'(t) &= -\alpha e^{-\alpha t}[q+pe^t] + e^{-\alpha t}pe^t \\ &= e^{-\alpha t}[-\alpha q - \alpha pe^t + pe^t] \\ &= e^{-\alpha t}[-\alpha q + (1-\alpha)pe^t] \end{aligned}$$

We find that $f(t) \to \infty$ as $t \to \infty$. Since $\alpha > p$, we have

$$e^{\alpha t} f'(0) = -\alpha q + (1-\alpha)p = p - \alpha(p+q) = p - \alpha < 0. \tag{2.6}$$

Thus f achieves its minimum over $[0,\infty)$ in $(0,\infty)$. Let t^* be the infimum of those $t \in (0,\infty)$ for which the minimum is achieved.

Solving $f'(t^*) = 0$, we find $-\alpha q + (1-\alpha)pe^{t^*} = 0$ and so

$$t^* = \log \frac{\alpha q}{(1-\alpha)p}.$$

Combining (2.5) and (2.6) and substituting $t = t^*$ gives

$$P(S_n > \alpha n) \le \left(\frac{(1-\alpha)p}{\alpha q}\right)^{\alpha n} \left[q + p\frac{\alpha q}{(1-\alpha)p}\right]^n.$$

This proves Lemma 2.1. $\square$

Assuming that $p_N \in (0,1/2)$ and $\alpha = 1/2$, we conclude

$$
\begin{aligned}
q_T^{(>)} &= P(S_N > N/2) \leq \left(\frac{p_N}{q_N}\right)^{N/2} [2q_N]^N \\
&= 2^N [pq]^{N/2} \leq (4p_N)^{N/2}
\end{aligned}
\tag{2.7}
$$

We expect that q_N is close to 1 for large T, so not much is lost in the final step above.

The probability that a single Brownian motion exits $\mathbf{B}_{r_2(T)}(0)$ by time T is bounded by

$$
\begin{aligned}
p_N &= P_T\Big(\sup_{t\in[0,T]} |B_t^{(k)}| > r_2(T)\Big) \\
&\leq C\frac{1}{r_2(T)} \exp\left(-\frac{r_2(T)^2}{2T}\right),
\end{aligned}
\tag{2.8}
$$

by standard Brownian estimates.

Continuing, we use (2.7) and (2.8) to get

$$
\begin{aligned}
q_T^{(>)} &\leq (4p_N)^{N/2} \\
&\leq \left(\frac{C}{r_2(T)}\right)^{N/2} \exp\left(-\frac{Nr_2(T)^2}{2T}\right).
\end{aligned}
$$

Now assuming that $r_2(T)^2/T > C_0 > 0$ we can absorb $(C/r_2(T))^{N/2}$ into the exponential to get

$$
q_T^{(>)} \leq \exp\left(-\frac{CNr_2(T)^2}{T}\right),
\tag{2.9}
$$

for some constant $C > 0$.

From (1.6) it follows that on $A_{T,r_2}^{(>)}$, at least $[N/2]$ of the Brownian paths exit from $\mathbf{B}_{r_2(T)}(0)$ within time $[0,T]$.

Our next argument is only heuristic, but it allows us to guess a formula for R_T. Recall that Q_T in (1.5) involves the ratio of P_T to Z_T. We think of R_T as the critical value of $r_1(T)$ and $r_2(T)$ for which both $q_T^{(<)}, q_T^{(>)}$ are close to Z_T. We believe that if $r_1(T) \approx r_2(T) \approx R_T$, then $q_T^{(<)}, q_T^{(>)}$ will be close. Thus we set $r_1(T) \approx r_2(T)$ and equalize the powers appearing in (2.4) and (2.9). Following this path, we conclude that (ignoring constants)

$$
\frac{Nr_2^2(T)}{T} \approx \beta\frac{N^2T^2}{r_1(T)^d}
$$

hence setting $r_1(T) \approx r_2(T) \approx R_T$, we get the guess

$$
R_T \approx \beta^{\frac{1}{d+2}} N^{\frac{1}{d+2}} T^{\frac{3}{d+2}}.
$$

This leads us to guess that

$$Z_{N,T} \approx \beta^{\frac{2}{d+2}} N^{\frac{d+4}{d+2}} T^{\frac{4-d}{d+2}}. \tag{2.10}$$

We describe our results regarding $Z_{N,T}$ in the following theorem.

Theorem 2.2. *Let $Z_{N,T}$ be defined as in (1.5). Then there exists a constant $C_{2.2} > 0$ not depending on N, T and β such that,*

(i) *for $d = 1$, for all $\beta, T \geq 1$,*

$$\log Z_{N,T} \geq -C_{2.2} \beta^{\frac{2}{3}} N^{\frac{5}{3}} T,$$

(ii) *for $d = 2, 3$, for all $\beta \geq 1$ and $1 \leq T \leq N$,*

$$\log Z_{N,T} \geq -C_{2.2} \beta^{\frac{1}{2}} N^{\frac{3}{2}} T^{\frac{1}{2}} \log(\beta T),$$

The proof of Theorem 2.2 is given in Section 3.

3. Proof of Theorems 1.1 and 2.2

Proof of Theorem 2.2. In order to derive the bounds Theorem 2.2 we use a change of measure that will impose a radial drift $\lambda_k(t)$ on each Brownian motion $B^{(k)}$, of magnitude

$$|\lambda_k(t)| = \kappa(\alpha + 1)t^{\alpha}, \tag{3.1}$$

for some $\kappa > 0$ and $\alpha < 0$ to be determined. Now we describe the directions $(\theta_k)_{k\in\{1,\dots,N\}}$ of the drifts, where each θ_k is a point on the unit sphere $\mathbf{S}_d \subset \mathbb{R}^d$. Let $\theta := (\theta_k)_{k\in\{1,\dots,N\}}$ be an ensemble of i.i.d. random points chosen according to the uniform probability measure on $\mathbf{S}_d$. Specifically, given T, N we define $(\theta_k)_{k\in\{1,\dots,N\}}$ over a probability space $(\Omega_\theta, \mathcal{F}_\theta, P_\theta)$ and form the product space

$$(\overline{\Omega}, \mathcal{G}, \mathbb{P}) = (\Omega_\theta \times \Omega, \mathcal{F}_\theta \times \mathcal{F}, P_\theta \times P_T).$$

Sometimes we write $\mathbb{P}_{T,N}$ to emphasize the dependence on T. Note that there is also an implicit dependence on N. Finally, we denote $\lambda := (\lambda_k)_{k\in\{1,\dots,N\}}$.

Roughly speaking, we want the drift to be stronger than the standard deviation $t^{1/2}$ of Brownian motion. From (3.1) it follows that the accumulated drift up to time t is given by

$$\kappa(\alpha + 1) \int_0^t s^{\alpha} ds = \kappa t^{\alpha+1}, \tag{3.2}$$

so we require that $\alpha > -1/2$.

Given θ and N, T, we denote the measure on canonical path space $(C[0,T])^N$ which is induced by the drifts λ as P_T^λ. The Radon-Nikodym derivative with respect to P_T in (1.5) is given by

$$\frac{dP_T^\lambda}{dP_T} = \exp\left(\sum_{k=1}^N \left\{\int_0^T \lambda_k(t)\cdot dB_t^{(k)} - \frac{1}{2}\int_0^T |\lambda_k(t)|^2 dt\right\}\right).$$

We can also define the corresponding product probability $\mathbb{P}_T^\lambda$ as before, using P_T^λ instead of P_T.

We can express Z_T in (1.5) in terms of $\mathcal{E}_T$ and the Radon-Nikodym derivative as follows

$$Z_T = E^{\mathbb{P}_T^\lambda}\left[\mathcal{E}_T\left(\frac{dP_T^\lambda}{dP_T}\right)^{-1}\right].$$

We can use Jensen's inequality on $\log Z_T$, since the logarithm function is concave to get

$$\begin{aligned}\log Z_T &\geq E^{\mathbb{P}_T^\lambda}\left[\log \mathcal{E}_T - \log\left(\frac{dP_T^\lambda}{dP_T}\right)\right] \\ &\geq -\beta E^{\mathbb{P}_T^\lambda}\left[\int_{\mathbb{R}^d} L_T(y)^2 dy\right] - E^{\mathbb{P}_T^\lambda}\left[\log\left(\frac{dP_T^\lambda}{dP_T}\right)\right] \\ &=: -\beta I_1(\beta,N,T) - I_2(\beta,N,T).\end{aligned} \tag{3.3}$$

Using (3.1) we can easily compute

$$\begin{aligned}I_2(\beta,N,T) &= E^{\mathbb{P}_T^\lambda}\left[\sum_{k=1}^N \left\{\int_0^T \lambda_k(t)\cdot dB_t^{(k)} - \frac{1}{2}\int_0^T |\lambda_k(t)|^2 dt\right\}\right] \\ &= -\frac{N}{2}\kappa^2(\alpha+1)^2 E\left[\int_0^T t^{2\alpha} dt\right] \\ &= -\frac{\kappa^2(\alpha+1)^2}{2} N \cdot \frac{T^{2\alpha+1}}{2\alpha+1}.\end{aligned} \tag{3.4}$$

We can compare $I_2(\beta,N,T)$ in (3.4) with (2.10) in order to determine the constants in the drift. Neglecting multiplicative constants it follows that we must have

$$\kappa^2 N \cdot \frac{T^{2\alpha+1}}{2\alpha+1} = \beta^{\frac{2}{d+2}} N^{\frac{d+4}{d+2}} T^{\frac{4-d}{d+2}}.$$

This will lead to the following choice of drift parameters:

$$\kappa = \beta^{\frac{1}{d+2}} N^{\frac{1}{d+2}}, \quad \alpha = -\frac{d-1}{d+2}, \quad d = 1,2,3. \tag{3.5}$$

Remark 3.1. *While it is tempting to use the drift in (3.5) for $d = 1, 2, 3$, we find that in the case of $d = 3$ it is suboptimal. This choice of drift for $d = 3$ yields the following bound on $Z_{N,T}$.*

$$\log Z_{N,T} \geq -C_{2.2}\beta^{\frac{3}{5}} N^{\frac{8}{5}} T^{\frac{4}{5}} \log(\beta T). \tag{3.6}$$

Since (3.6) is suboptimal we do not prove it, but the interested reader can verify it by modifying the arguments in Sections 4, 5, and 6.

It is better to use the same drift parameter for $d = 3$ as for $d = 2$, namely $\kappa = \beta^{\frac{1}{4}} N^{\frac{1}{4}}$, $\alpha = -1/4$. This choice gives the bound in Theorem 2.2(ii), that is for $d = 3$,

$$\log Z_{N,T} \geq -C_{2.2}\beta^{\frac{1}{2}} N^{\frac{3}{2}} T^{\frac{1}{2}} \log(\beta T),$$

which is clearly better than (3.6) for $\beta, N, T \geq 1$. The reason for that is that the bound on $Z_{N,T}$ depends on a comparison between the two components on the right hand side of (3.3): $I_1(\beta, N, T)$ which involves the occupation measure L_T, and on $I_2(\beta, N, T)$ which contains the Radon-Nikodym derivative and is computed in (3.4). The bound on $I_1(\beta, N, T)$ in Proposition 3.2 below, is obtained by bounding the terms in right-hand side of (4.2): $J_1(\beta, N, T)$ which depends on self-intersection occupation measure of each branch of the polymer and $J_2(\beta, N, T)$ which involves cross-intersection occupation measure of pairs of branches. The bound on $J_1(\beta, N, T)$, which is derived on Section 6, is the more restrictive one and eventually determines the result of Proposition 3.2. In particular in the proof of Lemma 6.1 (see (6.7)) we show that this bound should be similar to $d = 2, 3$. Hence choosing similar drifts for the cases of $d = 2, 3$ leads to an optimal bound for our method.

For the reminder of the paper, we therefore fix

$$\kappa = \begin{cases} \beta^{\frac{1}{3}} N^{\frac{1}{3}}, & d = 1, \\ \beta^{\frac{1}{4}} N^{\frac{1}{4}}, & d = 2, 3, \end{cases} \qquad \alpha = \begin{cases} 0, & d = 1, \\ -\frac{1}{4}, & d = 2, 3. \end{cases} \tag{3.7}$$

The rest of this paper is dedicated to the estimation of $I_1(\beta, N, T)$. This is essentially given in the following proposition which together with (3.3) and (3.4) concludes the proof of Theorem 2.2.

Proposition 3.2. *There exists a constant $C_{3.2} > 0$ not depending on N, T and β such that,*

- **(i)** *for $d = 1$, for all $\beta, T \geq 1$,*
$$I_1(\beta, N, T) \leq C_{3.2}\beta^{-\frac{1}{3}} N^{\frac{5}{3}} T.$$
- **(ii)** *for $d = 2, 3$, for all $\beta \geq 1$ and $1 \leq T \leq N$,*
$$I_1(\beta, N, T) \leq C_{3.2}\beta^{-\frac{1}{2}} N^{\frac{3}{2}} T^{\frac{1}{2}} \log(\beta T).$$

The proof of Proposition 3.2 for $d = 2, 3$ is postponed to Section 4 and the case of $d = 1$, which is much simpler, is postponed to Section 7. □

Proof of Theorem 1.1. Note that Theorem 2.2 provides a lower bound on $Z_{N,T}$ of the form

$$\log Z_{N,T} \geq -f(\beta, N, T), \tag{3.8}$$

for some positive function f of (β, N, T).

From (1.5), (1.7), (2.1), (2.4) and (3.8) we have

$$\begin{aligned} \log Q_T\big(A^{(<)}_{T,r_1(T)}\big) &= \log q^{(<)}_T - \log Z_{N,T} \\ &\leq -\beta C \frac{N^2T^2}{(1+r_1(T))^d} + f(\beta, N, T), \end{aligned} \tag{3.9}$$

hence by choosing $r_1(T)$ as in the upper bound in the statement of Theorem 1.1, with c_d small enough and β, N, T as in the hypothesis, we ensure that $r_1(T) \geq 1$ and therefore,

$$\begin{aligned} -\beta C \frac{N^2T^2}{(1+r_1(T))^d} + f(\beta, N, T) &\leq -\beta \widetilde{C} \frac{N^2T^2}{(1 \vee r_1(T))^d} + f(\beta, N, T) \\ &\leq -\beta \hat{C} \frac{N^2T^2}{r_1(T)^d} + f(\beta, N, T) \\ &< -cf(\beta, N, T), \end{aligned}$$

for some constant $c > 0$ and we get the lower bound on R_T.

Similarly from (1.5), (1.7), (2.1), (2.9) and (3.8) we have

$$\begin{aligned} \log Q_T\big(A^{(>)}_{T,r_2(T)}\big) &= \log q^{(>)}_T - \log Z_{N,T} \\ &\leq -\frac{CNr_2(T)^2}{T} + f(\beta, N, T), \end{aligned} \tag{3.10}$$

hence by choosing $r_2(T)$ as in the statement of Theorem 1.1 we ensure that

$$-\frac{CNr_2(T)^2}{T} + f(\beta, N, T) < -c' f(\beta, N, T),$$

for some constant $c' > 0$ and we get the upper bound on R_T. Note for $d = 1$ that $r_1(T)$ and $r_2(T)$ agree up to a constant; for $d = 2$, $r_1(T)$ and $r_2(T)$ agree up to logarithmic terms; while in $d = 3$ there is a gap between them. □

4. Proof of Proposition 3.2 for $d = 2, 3$

This section is dedicated to the bound on $I_1(\beta, N, T)$ in Proposition 3.2 for $d = 2, 3$.

Proof of Proposition 3.2 for $d = 2, 3$. Recall that L_T was defined in (1.4). From (3.3) and the statement of Proposition 3.2, we see that we need the bound for $d = 2, 3$,

$$E^{\mathbb{P}_T^\lambda}\left[\int_{\mathbb{R}^d} L_T(y)^2 dy\right] \leq \beta^{-\frac{1}{2}} N^{\frac{3}{2}} T^{\frac{1}{2}}. \tag{4.1}$$

for α and κ as in (3.7).

Note that

$$\begin{aligned}\int_{\mathbb{R}^d} L_T(y)^2 dy &= \int_{\mathbb{R}^d}\left(\sum_{k=1}^N \int_0^T \mathbf{1}_{\mathbf{B}_1(y)}(B_t^{(k)})dt\right)^2 dy\\ &= \sum_{k,\ell=1}^N \int_0^T\int_0^T \left(\int_{\mathbb{R}^d} \mathbf{1}_{\mathbf{B}_1(y)}(B_t^{(k)})\mathbf{1}_{\mathbf{B}_1(y)}(B_s^{(\ell)})dy\right) dtds.\end{aligned}$$

In fact, for $a, b \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \mathbf{1}_{\mathbf{B}_1(y)}(a)\mathbf{1}_{\mathbf{B}_1(y)}(b)dy \leq \mathbf{1}_{\mathbf{B}_2(0)}(b-a)|\mathbf{B}_1(0)| = C_d \mathbf{1}_{\mathbf{B}_2(0)}(b-a).$$

Thus, if $f_{t,s}^{(k,\ell,\alpha)}(z)$ is the probability density function of $B_t^{(k)} - B_s^{(\ell)}$ under $\mathbb{P}_T^\lambda$, then using Fubini's Theorem we get,

$$\begin{aligned}\int_{\mathbb{R}^d} E^{\mathbb{P}_T^\lambda}\left[L_T(y)^2\right] dy &\leq C_d \sum_{k,\ell=1}^N \iint_{[0,T]^2}\int_{\mathbf{B}_2(0)} f_{t,s}^{(k,\ell,\alpha)}(z)dzdsdt\\ &= C_d \sum_{k=1}^N \iint_{[0,T]^2}\int_{\mathbf{B}_2(0)} f_{t,s}^{(k,k,\alpha)}(z)dzdsdt\\ &\quad + C_d \sum_{k\neq\ell} \iint_{[0,T]^2}\int_{\mathbf{B}_2(0)} f_{t,s}^{(k,\ell,\alpha)}(z)dzdsdt\\ &=: J_1(\beta, N, T) + J_2(\beta, N, T).\end{aligned} \tag{4.2}$$

Note that $J_1(\beta, N, T)$ represents the represents the sum of mean-squared self-intersection occupation measure of each branch of the star polymer. $J_2(\beta, N, T)$ represents the sum over all pairs of branches of their mean-squared cross-intersection occupation measure. The following proposition gives some essential bounds on each of these terms.

Proposition 4.1. *There exists a constant $C > 0$ not depending on N, T and β such that for all $1 \leq T \leq N$ and $\beta \geq 1$,*

$$J_i(\beta, N, T) \leq C\beta^{-\frac{1}{2}} N^{\frac{3}{2}} T^{\frac{1}{2}} \log(\beta T), \quad i = 1, 2.$$

The proof of Proposition 4.1 is long and involved. In Section 5 we prove the bound on cross-intersection occupation measure, which involves two different Brownian motions $B^{(k_i)}$ for $i = 1, 2$ and $k_1 \neq k_2$, . The bound on self-intersection occupation measure is derived in Section 6. From (4.1), (4.2) and Proposition 4.1 we get the bounds in Proposition 3.2. □

5. Cross-intersections occupation measure for $d = 2, 3$

In this section we derive the bounds on $J_2(\beta, N, T)$ from Proposition 4.1 for $d = 2, 3$. We recall that $f_{t,s}^{(k,\ell,\alpha)}(z)$ is the probability density function of $B_t^{(k)} - B_s^{(\ell)}$ under $\mathbb{P}_T^{\lambda}$. Our next task is to estimate $f_{t,s}^{(k,\ell,\alpha)}(z)$. Since f only depends on the angle θ between the two drifts of $B^{(k)}$ and $B^{(\ell)}$, from now on we write $f_{t,s}^{\theta,\alpha}$ for our probability density. Note that θ is uniformly distributed on $[0, \pi]$.

Instead of working with P_T^{λ}, we will work with P_T and consider processes $B_t^{(k)} + D_t^{(k)}, B_s^{(\ell)} + D_s^{\ell}$, where $D^{(k)}, D^{(\ell)}$ are the respective drift processes. First note that

$$B_t^{(k)} - B_s^{(\ell)} \sim \mathcal{N}(0, t + s),$$

where $\mathcal{N}(\cdot, \cdot)$ stands for the d-dimensional normal distribution.

Recall that the drift magnitudes are given by (3.2), namely

$$|D_t^{(k)}| = \kappa t^{\alpha+1}, \qquad |D_s^{(\ell)}| = \kappa s^{\alpha+1}. \tag{5.1}$$

For the remainder of the proof we assume that $\theta \in [0, \pi]$. First we get a lower bound on $|z - D_t^{(k)} + D_s^{(\ell)}|^2$. Let $a = D_t^{(k)} - D_s^{(\ell)}$. Then

$$|a|^2 = |(a - z) + z|^2 \leq 2|z - a|^2 + 2|z|^2 \leq 2|z - a|^2 + 8,$$

since $|z| \leq 2$ in the domain of integration of $J_2(\beta, N, T)$ (see (4.2)). Now we use the law of cosines to deduce

$$\begin{aligned} |D_t^{(k)} - D_s^{(\ell)}|^2 &= |D_t^{(k)}|^2 + |D_s^{(\ell)}|^2 - 2|D_t^{(k)}| \cdot |D_s^{(\ell)}| \cos\theta \\ &= \left(|D_t^{(k)}| - |D_s^{(\ell)}|\right)^2 + 2|D_t^{(k)}| \cdot |D_s^{(\ell)}| \left[1 - \cos\theta\right]. \end{aligned} \tag{5.2}$$

Elementary geometry shows that for some constant $C > 0$ and for all $\theta \in [0, \pi]$,

$$1 - \cos\theta \geq C\theta^2 \tag{5.3}$$

and so

$$|D_t^{(k)} - D_s^{(\ell)}|^2 \geq \left(|D_t^{(k)}| - |D_s^{(\ell)}|\right)^2 + C|D_t^{(k)}| \cdot |D_s^{(\ell)}|\theta^2. \tag{5.4}$$

We also recall the following integral bound. For any $K > 0$ we have

$$\int_0^\infty \exp\left(-K\theta^2\right) d\theta = \frac{\sqrt{\pi}}{2\sqrt{K}}. \tag{5.5}$$

Remark 5.1. *The bound in (5.5) will be used in Lemma 5.2 (see (5.11)) and in Lemma 5.4 (see (5.25)) in order to bound $J_2(\beta, N, T)$ for two different cases. Note that the aforementioned bounds can be improved for $d = 3$ by using the following bound for any $K > 0$,*

$$\int_0^\infty \exp\left(-K\theta^2\right) \theta d\theta = \frac{1}{2K},$$

for the expression for $J_2(\beta, N, T)$ in (5.7) which has an extra factor of θ (see explanation after (5.7)). As mentioned in Remark 3.1 this will not have any affect on the result of Theorem 2.2. Therefore, for the sake of simplicity we provide one proof for the bounds $J_2(\beta, N, T)$ from Proposition 4.1 for $d = 2, 3$ using (5.5).

We derive a preliminary bound for $J_2(\beta, N, T)$ in (4.2) where we distinguish between the following cases.

Case 1: $d = 2$**.** Note that $J_2(\beta, N, T)$ is bounded by

$$\begin{aligned} J_2(\beta, N, T) \leq CN^2 \int_0^\pi \iint_{0 \leq s \leq t \leq T} \frac{1}{t+s} \\ \times \int_{\mathbf{B}_2(0)} \exp\left(-\frac{\left|z - D_t^{(k)} + D_s^{(\ell)}\right|^2}{2(t+s)}\right) dz ds dt d\theta. \end{aligned} \tag{5.6}$$

Case 2: $d = 3$**.** By the same argument as in Case 1 we get,

$$\begin{aligned} J_2(\beta, N, T) \leq CN^2 \int_0^\pi \iint_{0 \leq s \leq t \leq T} \frac{1}{(t+s)^{3/2}} \\ \times \int_{\mathbf{B}_2(0)} \exp\left(-\frac{\left|z - D_t^{(k)} + D_s^{(\ell)}\right|^2}{2(t+s)}\right) dz ds dt d\theta. \end{aligned} \tag{5.7}$$

Note that in the second integral above, using polar coordinates would give $\theta d\theta$. The interested reader can check that using $d\theta$ gives the same end result. Of course we can bound θ by π and so replace $\theta d\theta$ by $d\theta$.

Let V be the subspace generated by the first two coordinates of $\mathbb{R}^3$, so for $z = (z_1, z_2, z_3)$ we denote by $z_V = (z_1, z_2)$. In fact we can find a coordinate system in which $D_t^{(k)}$ is parallel to z_1, so the above integral is bounded by

(5.8)

$$\begin{aligned} &J_2(\beta,N,T)\\ &\le CN^2\int_0^{\pi}\iint_{0\le s\le t\le T}\frac{1}{(t+s)}\int_{\mathbf{B}_2^V(0)}\exp\left(-\frac{\left|z_V-D_t^{(k)}+D_s^{(\ell)}\right|^2}{2(t+s)}\right)dz_V\\ &\qquad\qquad\times\frac{1}{(t+s)^{1/2}}\int_{-\infty}^{\infty}\exp\left(-\frac{z_3^2}{2(t+s)}\right)dz_3dsdtd\theta\\ &\le CN^2\int_0^{\pi}\iint_{0\le s\le t\le T}\frac{1}{(t+s)}\\ &\qquad\qquad\times\int_{\mathbf{B}_2^V(0)}\exp\left(-\frac{\left|z_V-D_t^{(k)}+D_s^{(\ell)}\right|^2}{2(t+s)}\right)dz_Vdsdtd\theta, \end{aligned}$$

where $\mathbf{B}_2^V(0)$ is the projection of $\mathbf{B}_2(0)$ to V. From (5.6) and (5.8) it follows that we need to bound the same integral for the cases where $d=2,3$ but $D_t^{(k)}, D_s^{(\ell)}$ will change according to the dimension as implied by (3.7) and (5.1). In the following we therefore work on to the combined cases $d=2,3$.

In order to do that we distinguish between a few cases which depend on the range of (t,s) in the above integral.

Define

$$\begin{aligned} &\mathbf{R}_1=\{(s,t):0\le s\le t\le T, t\ge 4\},\\ (5.9)\qquad &\mathbf{R}_2=\left\{(s,t):0\le s\le t\le T,\ t<4,\ |D_t^{(0)}-D_s^{(\ell)}|^2\le 8\right\},\\ &\mathbf{R}_3=\left\{(s,t):0\le s\le t\le T,\ t<4,\ |D_t^{(0)}-D_s^{(\ell)}|^2> 8\right\}. \end{aligned}$$

Moreover for $i=1,2,3$, and $d=2,3$ define:

$$\begin{aligned} \overline{J}_d(\beta,N,T,\mathbf{R}_i)&=CN^2\int_0^{\pi}\iint_{\mathbf{R}_i}\frac{1}{t+s}\\ &\quad\times\int_{\mathbf{B}_2(0)}\exp\left(-\frac{\left|z-D_t^{(0)}+D_s^{(\ell)}\right|^2}{2(t+s)}\right)dzdsdtd\theta. \end{aligned}\tag{5.10}$$

We now present a sequence of technical lemmas that will helps us to bound $J_2(\beta,N,T)$. The proof of Proposition 4.1 for $J_2(\beta,N,T)$ is given in the end of this section.

Lemma 5.2. *There exists a constant $C_{5.2}>0$ not depending on N,T and β such that the following bound holds for $d=2,3$:*

$$\overline{J}_d(\beta,N,T,\mathbf{R}_1)\le C_{5.2}\beta^{-1/2}N^{3/2}T^{1/2},\quad \textit{for all } \beta,T\ge 1.$$

Proof. Using (5.4) we get

$$|z - D_t^{(k)} + D_s^{(\ell)}|^2 \geq \left(|D_t^{(k)}| - |D_s^{(\ell)}|\right)^2 + C|D_t^{(k)}| \cdot |D_s^{(\ell)}|\theta^2 - 4,$$

since $|z| \leq 2$ in the domain of integration of $\overline{J}_d(\beta, N, T, \mathbf{R}_i)$ (see (5.10)).

Note that since on $\mathbf{R}_1$ we have $s+t > 4$, it follows that $4/(s+t) \leq 1$ and by integrating over z we get

$$\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_1) \\
&= N^2 \int_0^\pi \iint_{(s,t)\in\mathbf{R}_1} \frac{1}{t+s} \int_{\mathbf{B}_2(0)} \exp\left(-\frac{|z - D_t^{(k)} + D_s^{(\ell)}|^2}{2(t+s)}\right) dz ds dt d\theta \\
&\leq CN^2 \int_0^\pi \iint_{(s,t)\in\mathbf{R}_1} \frac{1}{t+s} \\
&\qquad \times \exp\left(-\frac{\left(|D_t^{(k)}| - |D_s^{(\ell)}|\right)^2 + C|D_t^{(k)}| \cdot |D_s^{(\ell)}|\theta^2}{2(t+s)} + 1\right) ds dt d\theta.
\end{aligned}$$

Next we factor the above exponential, incorporate e^1 into the constant, and apply our elementary integral (5.5) to obtain

$$\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_1) \\
&\leq CN^2 \iint_{(s,t)\in\mathbf{R}_1} \frac{1}{t+s} \exp\left(-\frac{\left(|D_t^{(k)}| - |D_s^{(\ell)}|\right)^2}{2(t+s)}\right) \\
&\qquad \times \left[\int_0^\pi \exp\left(-\frac{C|D_t^{(k)}| \cdot |D_s^{(\ell)}|\theta^2}{2(t+s)}\right) d\theta\right] ds dt \\
&= CN^2 \iint_{(s,t)\in\mathbf{R}_1} \frac{1}{\sqrt{(t+s)|D_t^{(k)}| \cdot |D_s^{(\ell)}|}} \\
&\qquad \times \exp\left(-\frac{\left(|D_t^{(k)}| - |D_s^{(\ell)}|\right)^2}{2(t+s)}\right) ds dt.
\end{aligned} \tag{5.11}$$

Then we use the fact that $0 \leq s \leq t$, and so $|D_s^{(\ell)}| \leq |D_t^{(k)}|$. From (3.7) and (5.1) we have $|D_t^{(k)}| = \beta^{1/4} N^{1/4} t^{3/4}$, so we get

$$\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_1) \\
&\leq \tilde{C}N^2 \iint_{(s,t)\in\mathbf{R}_1} t^{-\frac{1}{2}}\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}t^{-\frac{3}{8}}s^{-\frac{3}{8}} \\
&\qquad\qquad\qquad \times \exp\left(-C\frac{\beta^{\frac{1}{2}}N^{\frac{1}{2}}[t^{\frac{3}{4}} - s^{\frac{3}{4}}]^2}{t}\right) dsdt \\
&\leq \tilde{C}\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \int_0^T \int_0^t t^{-\frac{7}{8}}s^{-\frac{3}{8}} \\
&\qquad\qquad\qquad \times \exp\left(-C\beta^{\frac{1}{2}}N^{\frac{1}{2}}t^{-1}[t^{\frac{3}{4}} - s^{\frac{3}{4}}]^2\right) dsdt.
\end{aligned}$$

Now by making a change variables to $u = s/T$ and $v = t/T$ it follows that,

(5.12)

$$\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_1) \\
&= \tilde{C}\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \int_0^1 \int_0^v (Tv)^{-\frac{7}{8}}(Tu)^{-\frac{3}{8}} \\
&\qquad \times \exp\left(-C\beta^{\frac{1}{2}}N^{\frac{1}{2}}(Tv)^{-1}[(Tv)^{\frac{3}{4}} - (Tu)^{\frac{3}{4}}]^2\right) d(Tu)d(Tv) \\
&= \tilde{C}\beta^{-\frac{1}{4}}N^{\frac{7}{4}}T^{\frac{3}{4}} \int_0^1 \int_0^v v^{-\frac{7}{8}}u^{-\frac{3}{8}} \exp\left(-C\beta^{\frac{1}{4}}N^{\frac{1}{4}}T^{\frac{1}{2}}v^{-1}\left[v^{\frac{3}{4}} - u^{\frac{3}{4}}\right]^2\right) dudv.
\end{aligned}$$

We will show that the following bound holds for all $K \geq 1$,

$$\int_0^1 \int_0^v v^{-\frac{7}{8}}u^{-\frac{3}{8}} \exp\left(-CKv^{-1}[v^{\frac{3}{4}} - u^{\frac{3}{4}}]^2\right) dudv \leq K^{-1/2}. \tag{5.13}$$

We choose $K = \beta^{1/2}N^{1/2}T^{1/2}$ as in (5.12) so the bound in (5.13) will give us

$$\overline{J}_d(\beta, N, T, \mathbf{R}_1) \leq C\beta^{-1}N,$$

for $\beta, T \geq 1$, and this will complete the proof for $d = 2$ and $d = 3$.

By the mean value theorem, there exists $r \in (u, v)$ such that

$$v^{\frac{3}{4}} - u^{\frac{3}{4}} = cr^{-\frac{1}{4}}(v - u) \geq cv^{-\frac{1}{4}}(v - u).$$

Hence following (5.12), it is enough to bound

$$\begin{aligned}
&\int_0^1 \int_0^v v^{-\frac{7}{8}}u^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}[v - u]^2\right) dudv \\
&= \int_0^1 \int_0^v v^{-\frac{7}{8}}(v - w)^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dwdv.
\end{aligned} \tag{5.14}$$

We examine the inner integral in (5.14), and write

$$\begin{aligned}
&\int_0^v v^{-\frac{7}{8}}(v-w)^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dw \\
&= \int_0^{v/2} v^{-\frac{7}{8}}(v-w)^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dw \\
&\quad + \int_{v/2}^{v} v^{-\frac{7}{8}}(v-w)^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dw \\
&=: H_1(v) + H_2(v).
\end{aligned} \tag{5.15}$$

First dealing H_1, we have

$$\begin{aligned}
H_1(v) &= \int_0^{v/2} (v-w)^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dw \\
&\le Cv^{-\frac{3}{8}} \int_0^{\infty} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dw \\
&\le Cv^{-\frac{3}{8}} K^{-1/2} v^{\frac{3}{4}} \\
&\le Cv^{\frac{3}{8}} K^{-1/2},
\end{aligned}$$

using (5.5). Integrating over v as well, we get

$$\begin{aligned}
\int_0^1 v^{-\frac{7}{8}} H_1(v) dv &= CK^{-1/2} \int_0^1 v^{-\frac{7}{8}} v^{\frac{3}{8}} dv \\
&= CK^{-1/2}.
\end{aligned} \tag{5.16}$$

Turning to H_2, we have

$$\begin{aligned}
H_2(v) &= \int_{v/2}^{v} (v-w)^{-\frac{3}{8}} \exp\left(-Kv^{-\frac{3}{2}}w^2\right) dw \\
&\le \exp\left(-CKv^{-\frac{3}{2}}v^2\right) \int_{v/2}^{v} (v-w)^{-\frac{3}{8}} dw \\
&\le Cv^{\frac{5}{8}} \exp\left(-CKv^{\frac{1}{2}}\right).
\end{aligned}$$

Integrating over v as well, we get

$$\begin{aligned}
\int_0^1 v^{-\frac{7}{8}} H_2(v)dv &\leq \int_0^1 v^{-\frac{7}{8}} v^{\frac{5}{8}} \exp\left(-CKv^{\frac{1}{2}}\right) dv \\
&= \int_0^1 v^{-\frac{1}{4}} \exp\left(-CKv^{\frac{1}{2}}\right) dv \\
&= K^{-\frac{3}{2}} \int_0^1 (K^2 v)^{-\frac{1}{4}} \\
&\qquad \times \exp\left(-C(K^2 v)^{\frac{1}{2}}\right) d(K^2 v) \\
&= K^{-\frac{3}{2}} \int_0^1 r^{-\frac{1}{4}} \exp\left(-Cr^{\frac{1}{2}}\right) dr \\
&\leq K^{-3/2}, \quad \text{for all } K \geq 1.
\end{aligned} \tag{5.17}$$

Note that we have used

$$\int_0^1 x^{-\eta/2} e^{-x^{\eta}} dx < \infty \quad \text{for all } \eta \in (0,2).$$

From (5.14), (5.15), (5.16) and (5.17) we get (5.13) and Lemma 5.2 follows. □

The following technical lemma gives us some essential bounds on $\overline{J}_d(\beta, N, T, \mathbf{R}_2)$ which was defined in (5.10).

Lemma 5.3. *There exists a constant $C_{5.3} > 0$ not depending on N, T and β such that the following bound hold for $d = 2, 3$:*

$$\overline{J}_d(\beta, N, T, \mathbf{R}_2) \leq C_{5.3} \beta^{-1/2} N^{3/2}, \quad \textit{for all } \beta \geq 1.$$

Proof. Note that on $\mathbf{R}_2$ since $|z|^2 < 4$ we we expect $|z - D_t^{(0)} - D_s^{(\ell)}|^2$ to be of the same order of $s+t$. In this case we get that the integral over z in the right-hand side of (5.10) is approximately 1, that is,

$$\int_{\mathbf{B}_2(0)} \frac{1}{s+t} \exp\left(-C \frac{|z - D_t^{(0)} + D_s^{(\ell(\theta))}|^2}{2(s+t)}\right) dz \approx 1.$$

Hence we will absorb this integral as a multiplicative factor in our bounds on $\overline{J}_d(\beta, N, T, \mathbf{R}_2)$ as follows,

(5.18)

$$\begin{aligned}\overline{J}_d(\beta, N, T, \mathbf{R}_2) &= CN^2 \int_0^{\pi} \iint_{\mathbf{R}_4} \frac{1}{t+s} \\ &\qquad \times \int_{\mathbf{B}_2(0)} \exp\left(-\frac{\left|z - D_t^{(0)} + D_s^{(\ell)}\right|^2}{2(t+s)} \right) dz ds dt d\theta \\ &\leq CN^2 \int_0^{\pi} \iint_{\mathbf{R}_4} ds dt d\theta.\end{aligned}$$

From (3.7) and (5.1) we get

$$\begin{aligned}|D_t^{(0)} - D_s^{(\ell(\theta))}|^2 &= \beta^{\frac{1}{2}} N^{\frac{1}{2}} \left(t^{\frac{3}{4}} - s^{\frac{3}{4}} \cos\theta \right)^2 \\ &\quad + \beta^{\frac{1}{2}} N^{\frac{1}{2}} \left(s^{\frac{3}{4}} \sin\theta \right)^2.\end{aligned} \tag{5.19}$$

Since on $\mathbf{R}_2$ we have $|D_t^{(0)} - D_s^{(\ell(\theta))}|^2 \leq 8$, we get that both terms on the right side of (5.19) must be bounded by 8. Thus we have $\beta^{1/2} N^{1/2} \left(s^{3/4} \sin\theta \right)^2 \leq 8$ and so using $\sin\theta \leq C\theta$ on $\theta \in [0, \pi]$ we have

$$\theta \leq C s^{-\frac{3}{4}} \beta^{-\frac{1}{4}} N^{-\frac{1}{4}}. \tag{5.20}$$

From (5.19) we also have

$$\beta^{\frac{1}{2}} N^{\frac{1}{2}} \left(t^{\frac{3}{4}} - s^{\frac{3}{4}} \cos\theta \right)^2 \leq 8.$$

Therefore if $t > 4\beta^{-\frac{1}{4}} N^{-\frac{1}{4}}$ we get

$$(t^{\frac{3}{4}} - 4N^{-\frac{1}{4}} \beta^{-\frac{1}{4}})^{\frac{4}{3}} < s \leq t.$$

Together with (5.18) and (5.20) we get

$$
\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_2) \\
&\le CN^2 \int_{\theta \le Cs^{-\frac{3}{4}}\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}} \iint_{\mathbf{R}_2} dsdtd\theta \\
&\le C\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \iint_{\mathbf{R}_2} s^{-\frac{3}{4}} dsdt \\
&\le C\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \int_{\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}}^{4} \int_{(t^{\frac{3}{4}}-4N^{-\frac{1}{4}}\beta^{-\frac{1}{4}})^{\frac{4}{3}}}^{t} s^{-\frac{3}{4}} dsdt \\
&\quad + C\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \int_0^{\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}} \int_0^4 s^{-\frac{3}{4}} dsdt \\
&\le C\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \int_{\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}}^{4} \big(t^{\frac{1}{4}} - (t^{\frac{3}{4}} - 4N^{-\frac{1}{4}}\beta^{-\frac{1}{4}})^{\frac{1}{3}}\big) dt \\
&\quad + C\beta^{-\frac{1}{4}}N^{\frac{7}{4}} \int_0^{\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}} dt \\
&\le C\beta^{-\frac{1}{2}}N^{\frac{3}{2}} \left(1 + \int_{\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}}^{4} t^{-\frac{1}{2}} dt\right) \\
&\le C\beta^{-\frac{1}{2}}N^{\frac{3}{2}}.
\end{aligned} \tag{5.21}
$$

where we have used,

$$t^{\frac{1}{2}} - (t^{\frac{3}{4}} - 4N^{-\frac{1}{4}}\beta^{-\frac{1}{4}})^{\frac{1}{3}} \le Ct^{-\frac{1}{2}}N^{-\frac{1}{4}}\beta^{-\frac{1}{4}}.$$

This completes the proof of Lemma 5.3. □

The following lemma gives us some essential bounds on $\overline{J}_d(\beta, N, T, \mathbf{R}_3)$ which was defined in (5.10).

Lemma 5.4. *There exists a constant $C > 0$ not depending on N, T and β such that the following bound holds for $d = 2, 3$:*

$$\overline{J}_d(\beta, N, T, \mathbf{R}_3) \le C_{5.4}\beta^{-1/2}N^{3/2}, \quad \textit{for all } \beta \ge 1.$$

Proof. Recalling that $|z| < 2$, we have

$$|z - D_t^{(k)} + D_s^{(\ell)}|^2 \ge |D_t^{(k)} + D_s^{(\ell)}|^2 - 4^2 > \frac{1}{2}|D_t^{(k)} + D_s^{(\ell)}|^2 \tag{5.22}$$

Using (3.7), (5.1) and (5.4), for $\ell = \ell(\theta)$ we get

$$
\begin{aligned}
|D_t^{(0)} + D_s^{(\ell)}|^2 &\ge \big(|D_t| - |D_s|\big)^2 + 2|D_t| \cdot |D_s|\theta^2 \\
&\ge \beta^{1/2}N^{1/2}\left[\big(t^{3/4} - s^{3/4}\big)^2 + 2t^{3/4}s^{3/4}\theta^2\right].
\end{aligned} \tag{5.23}
$$

From (5.22) and (5.23) and we can abound right-hand side of (5.10) as follows,

$$
\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_3) \\
&\leq CN^2 \iint_{\mathbf{R}_3} \frac{1}{t+s} \exp\left(-\frac{\frac{1}{2}\beta^{\frac{1}{2}} N^{\frac{1}{2}} \left[\left(t^{\frac{3}{4}} - s^{\frac{3}{4}} \right)^2 \right]}{2(t+s)} \right) \\
&\quad \times \int_0^{\pi} \exp\left(-\frac{\beta^{\frac{1}{2}} N^{\frac{1}{2}} t^{\frac{3}{4}} s^{\frac{3}{4}} \theta^2}{2(t+s)} \right) d\theta ds dt.
\end{aligned} \tag{5.24}
$$

Next, we integrate inner integral on the right-hand side of (5.24) over θ using $s+t \leq 2t$ and (5.5) to get

$$
\begin{aligned}
\int_0^{\pi} \exp\left(-\frac{\beta^{\frac{1}{2}} N^{\frac{1}{2}} s^{\frac{3}{4}} t^{\frac{3}{4}} \theta^2}{2t} \right) d\theta &\leq C\beta^{-\frac{1}{4}} N^{-\frac{1}{4}} s^{-\frac{3}{8}} t^{\frac{1}{8}} \\
&\leq C\beta^{-\frac{1}{4}} N^{-\frac{1}{4}} t^{-\frac{1}{4}}.
\end{aligned} \tag{5.25}
$$

Using the mean value theorem, we see that there exists $r = r(s,t)$ such that

$$
t^{3/4} - s^{3/4} = Cr^{-\frac{1}{4}}(t-s) \geq Ct^{-\frac{1}{4}}(t-s). \tag{5.26}
$$

Plugging in (5.25) into (5.24) and integrating over s, t and using (5.26), again using our basic exponential integral estimate (5.5), and carrying over the results of (5.25), we get

(5.27)

$$
\begin{aligned}
&\overline{J}_d(\beta, N, T, \mathbf{R}_3) \\
&\leq C \int_0^4 \int_0^t \left(\beta^{-\frac{1}{4}} N^{-\frac{1}{4}} t^{-\frac{1}{4}}\right) \frac{1}{t} \exp\left(-\frac{C\beta^{\frac{1}{2}} N^{\frac{1}{2}} \left(t^{\frac{3}{4}} - s^{\frac{3}{4}}\right)^2}{4t}\right) dsdt \\
&\leq C\beta^{-1} N^{-\frac{1}{4}} \int_0^1 t^{-\frac{5}{4}} \int_0^t \exp\left(-\frac{\beta^{\frac{1}{2}} N^{\frac{1}{2}} \left(t^{\frac{3}{4}} - s^{\frac{3}{4}}\right)^2}{4t}\right) dsdt \\
&\leq C\beta^{-1} N^{-\frac{1}{4}} \int_0^4 t^{-\frac{5}{4}} \int_0^t \exp\left(-\frac{\beta^{\frac{1}{2}} N^{\frac{1}{2}} t^{-\frac{1}{2}} (t-s)^2}{4t}\right) dsdt \\
&\leq C\beta^{-1} N^{-\frac{1}{4}} \int_0^4 t^{-\frac{5}{4}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{4}\beta^{\frac{1}{2}} N^{\frac{1}{2}} t^{-\frac{3}{2}} (t-s)^2\right) dsdt \\
&\leq C\beta^{-\frac{1}{4}} N^{-\frac{1}{4}} \int_0^4 t^{-\frac{5}{4}} \beta^{-\frac{1}{4}} N^{-\frac{1}{4}} t^{\frac{3}{4}} dt \\
&\leq C\beta^{-\frac{1}{2}} N^{-\frac{1}{2}} \int_0^1 t^{-\frac{1}{2}} dt \\
&\leq C\beta^{-\frac{1}{2}} N^{-\frac{1}{2}}.
\end{aligned}
$$

Multiplying by N^2, we get $\beta^{-1/2}N^{3/2}$ as required. This finishes the proof of Lemma 5.4.

□

Proof of Proposition 4.1 for $J_2(\beta, N, T)$. The bounds on $J_2(\beta, N, T)$ for $d = 2, 3$ follow directly from Lemmas 5.2–5.4. □

6. Self-intersection occupation measure for $d = 2, 3$

In this section we derive the bounds on $J_1(\beta, N, T)$ from Proposition 4.1 for $d = 2, 3$. Recall that in this case we are dealing with the occupation measure terms related to a single branch which is intersects with itself, therefore there are N such contributions.

We recall that $f_{t,s}^{(k,k,\alpha)}$ is the probability density function of $B_t^{(k)} - B_s^{(k)}$ under P_T^λ. Instead of working with P_T^λ, we will work with P_T and consider processes $B_t^{(k)} + D_t^{(k)}$, where $D^{(k)}, D^{(\ell)}$ are the respective drift processes. First note that

$$B_t^{(k)} - B_s^{(k)} \sim \mathcal{N}(0, t-s),$$

where as before $\mathcal{N}(\cdot, \cdot)$ stands for the d-dimensional normal distribution.

Recall that the drifts magnitudes are given by (5.1). Without loss of generality, we can assume that $s < t$ and that $D^{(k)}$ points in the $\mathbf{e}_1$ direction, where $\mathbf{e}_1$ is the first coordinate vector.

For simplicity we define

$$\zeta = \beta^{\frac{1}{4}} N^{\frac{1}{4}}, \tag{6.1}$$

then using (5.1) and (3.7) we get

$$\begin{aligned} f_{t,s}^{(k,k,\alpha)}(z) &= \frac{1}{(2\pi(t-s))^{-d/2}} \exp\left(-\frac{|z-(\zeta t^{\frac{3}{4}} - \zeta s^{\frac{3}{4}})\mathbf{e}_1|^2}{2(t-s)}\right) \\ &= (2\pi(t-s))^{-d/2} \exp\left(-\frac{(z_1 - \zeta t^{\frac{3}{4}} + \zeta s^{\frac{3}{4}})^2}{2(t-s)}\right) \\ &\quad \times \prod_{k=2}^{d} \exp\left(-\frac{z_k^2}{2(t-s)}\right). \end{aligned} \tag{6.2}$$

For convenience we again split the area of integration in $J_1(\beta, N, T)$, $\{(s,t) : 0 \le s \le t \le T\}$ to the following subregions.

$$\begin{aligned} \hat{\mathbf{R}}_1 &= \{(s,t) \in [0,T]^2 : t^{\frac{3}{4}} - s^{\frac{3}{4}} > C_{(6.3)} \zeta^{-1}\}, \\ \hat{\mathbf{R}}_2 &= \{(t,s) \in [0,T]^2 : t^{\frac{3}{4}} - s^{\frac{3}{4}} < C_{(6.3)} \zeta^{-1}\}, \end{aligned} \tag{6.3}$$

where $C_{(6.3)} > 0$ is a constant to be specified later.

Define

$$\hat{J}(\beta, N, T, \hat{\mathbf{R}}_i) = N \iint_{\hat{\mathbf{R}}_i} \int_{\mathbf{B}_2(0)} f_{t,s}^{(1,1,\alpha)}(z) dz ds dt. \tag{6.4}$$

Since the self-occupation measure is similar for all branches of the polymers it follows from (4.2),

$$J_1(\beta, N, T) \le C \sum_{i=1}^{2} \hat{J}(\beta, N, T, \hat{\mathbf{R}}_i). \tag{6.5}$$

We introduce a few technical lemmas that will help us to bound $J_1(\beta, N, T)$.

Lemma 6.1. *There exist positive constants $C_{(6.3)}$ and $C_{6.1}$ not depending on N, T and β such that the following bound holds for $d = 2, 3$:*

$$\hat{J}(\beta, N, T, \hat{\mathbf{R}}_1) \le C_{6.1} \beta^{-\frac{1}{2}} N^{\frac{3}{2}} T^{\frac{1}{2}} \log(\beta T), \quad \textit{for all } 1 \le T \le N,\ \beta \ge 1.$$

Proof. From (6.3) we can choose $C_{(6.3)}$ in the definition of $\hat{\mathbf{R}}_1$ such that

$$(z_1 - \zeta t^{\frac{3}{4}} + \zeta s^{\frac{3}{4}})^2 \ge \frac{1}{2}(\zeta t^{3/4} - \zeta s^{3/4})^2, \quad \text{for all } t, s \in \hat{\mathbf{R}}_1. \tag{6.6}$$

From (6.2), (6.4) and (6.6) it follows that
(6.7)

$$\begin{aligned}\hat{J}(\beta,N,T,\hat{\mathbf{R}}_1) &\le N\iint_{\hat{\mathbf{R}}_1}\frac{1}{\sqrt{2\pi(t-s)}}\int_{|z_1|<2}e^{-\frac{\left(z_1-\zeta t^{\frac{3}{d+2}}+\zeta s^{\frac{3}{4}}\right)^2}{2(t-s)}}dz_1\\ &\qquad\times\prod_{k=2}^{d}\left(\int_{|z_k|<2}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{z_k^2}{2(t-s)}}dz_k\right)dsdt\\ &\le N\iint_{\hat{\mathbf{R}}_1}\frac{1}{\sqrt{2\pi(t-s)}}\int_{|z_1|<2}e^{-\frac{\left(\zeta t^{\frac{3}{4}}-\zeta s^{\frac{3}{4}}\right)^2}{4(t-s)}}dz_1\\ &\qquad\times\prod_{k=2}^{d}\left(\int_{z_k\in\mathbf{R}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{z_k^2}{2(t-s)}}dz_k\right)dsdt\\ &\le N\iint_{\hat{\mathbf{R}}_1}\frac{C}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{\zeta^2\left(t^{\frac{3}{4}}-s^{\frac{3}{4}}\right)^2}{4(t-s)}\right)dsdt.\end{aligned}$$

Since $0<s<t$, the mean value theorem implies that for some $r\in(s,t)$ we have

$$\begin{aligned}\frac{\left(t^{\frac{3}{4}}-s^{\frac{3}{4}}\right)^2}{t-s} &= \left(\frac{t^{\frac{3}{4}}-s^{\frac{3}{4}}}{t-s}\right)^2(t-s)=\left(r^{-\frac{1}{4}}\right)^2(t-s)\\ &\ge t^{-\frac{1}{2}}(t-s),\end{aligned}$$

where in the last line follows because $r<t$.

We therefore have

$$\hat{J}(\beta,N,T,\hat{\mathbf{R}}_1)\le N\iint_{\hat{\mathbf{R}}_1}\frac{C}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{1}{4}t^{-\frac{1}{2}}(t-s)\right)dsdt.$$

Using the fact that $0\le s<t\le T$ and

$$\int_{\mathbf{B}_2(0)}f_{t,s}^{(k,k,\alpha)}(z)dz\le 1,$$

together with (6.4) we arrive at
(6.8)

$$\begin{aligned}&\hat{J}(\beta,N,T,\hat{\mathbf{R}}_1)\\ &\le N\iint_{\hat{\mathbf{R}}_1}\left(\left(\frac{C}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{1}{4}T^{-\frac{1}{2}}\zeta^2(t-s)\right)\right)\wedge 1\right)dsdt.\end{aligned}$$

Define

$$\gamma_1=\frac{5}{4},\qquad\gamma_2=\frac{3}{4}, \tag{6.9}$$

and

$$\gamma := \frac{1}{4}(N\beta)^{-\frac{1}{2}}T^{\frac{1}{2}}\log(T^{\gamma_1}\beta^{\gamma_2}). \tag{6.10}$$

From (6.1) and (6.10) we get for all $t-s>\gamma$,

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{1}{4}T^{-\frac{1}{2}}\zeta^2(t-s)\right)\\
&\le C\gamma^{-1/2}\exp\left(-\frac{1}{4}T^{-\frac{2d-2}{d+2}}\zeta^2\gamma\right)\\
&\le C\gamma^{-1/2}\exp\left(-\frac{1}{4}T^{-\frac{1}{2}}N^{\frac{1}{2}}\beta^{\frac{1}{2}}\gamma\right)\\
&\le C(N\beta)^{\frac{1}{4}}T^{-\frac{1}{4}}\log(T^{\gamma_1}\beta^{\gamma_2})^{-1/2}(T^{\gamma_1}\beta^{\gamma_2})^{-1}\\
&\le C\beta^{-\frac{1}{2}}N^{\frac{1}{2}}T^{-\frac{3}{2}}(\log(\beta T))^{-1/2},
\end{aligned} \tag{6.11}$$

where we have plugged in the values of γ_i from (6.9) in the last inequality.

From (6.8), (6.11) and since $0<s<t<T$ it follows that

$$\begin{aligned}
&\hat{J}(\beta,N,T,\hat{\mathbf{R}}_1)\\
&\le CN\int_0^T\int_0^{t-\gamma}\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{1}{4}t^{-\frac{1}{2}}(t-s)\right)dsdt\\
&\quad+CN\int_0^T\int_{t-\gamma}^t 1dsdt\\
&\le CN\int_0^T\int_0^T\beta^{-\frac{1}{2}}N^{\frac{1}{2}}T^{-\frac{3}{2}}(\log(\beta T))^{-1/2}dsdt\\
&\quad+CNT\gamma\\
&\le C\beta^{-\frac{1}{2}}N^{\frac{3}{2}}T^{\frac{1}{2}}(\log(\beta T))^{-1/2}\\
&\quad+CNT(N\beta)^{-\frac{1}{2}}T^{\frac{1}{2}}\log(T\beta)\\
&\le C\beta^{-\frac{1}{2}}N^{\frac{3}{2}}T^{\frac{1}{2}}(\log(\beta T))^{-1/2}\\
&\quad+C\beta^{-\frac{1}{2}}N^{\frac{1}{2}}T^{\frac{3}{2}}\log(T\beta).
\end{aligned}$$

By choosing

$$T\le N, \tag{6.12}$$

we get

$$\hat{J}(\beta,N,T,\hat{\mathbf{R}}_1)\le C\beta^{-\frac{1}{2}}N^{\frac{3}{2}}T^{\frac{1}{2}}\log(\beta T),\quad \text{for all } T,\beta\ge 1,$$

This completes the proof of Lemma 6.1. □

Lemma 6.2. *There exists a constant $C_{6.2} > 0$ not depending on N, T and β such that the following bound holds for $d = 2, 3$:*

$$\hat{J}(\beta, N, T, \hat{\mathbf{R}}_2) \leq C_{6.2}\beta^{-2/3}N^{1/3}, \quad \textit{for all } \beta \geq 1.$$

Proof. From (6.1) and (6.3) it follows that on $\hat{\mathbf{R}}_2$ we have

$$\begin{aligned} s^{\frac{3}{d+2}} &\leq t^{\frac{3}{d+2}} \leq C\beta^{-\frac{1}{d+2}}N^{-\frac{1}{d+2}} + s^{\frac{3}{d+2}}, \\ s &\leq C\beta^{-\frac{1}{3}}N^{-\frac{1}{3}}. \end{aligned} \tag{6.13}$$

By integrating over z in the right-hand side of (6.4) and then using (6.13) we get

$$\begin{aligned} \hat{J}(\beta, N, T, \hat{\mathbf{R}}_2) &\leq N|\hat{\mathbf{R}}_2| \\ &\leq N\int_0^{C\beta^{-\frac{1}{3}}N^{-\frac{1}{3}}} \int_0^{(C\beta^{-\frac{1}{4}}N^{-\frac{1}{4}}+s^{\frac{3}{4}})^{\frac{4}{3}}} dtds \\ &\leq N\int_0^{C\beta^{-\frac{1}{3}}N^{-\frac{1}{3}}} (C\beta^{-\frac{1}{4}}N^{-\frac{1}{4}} + s^{\frac{3}{4}})^{\frac{4}{3}}ds \\ &\leq CN\int_0^{C\beta^{-\frac{1}{3}}N^{-\frac{1}{3}}} (\beta^{-\frac{1}{4}}N^{-\frac{1}{4}})^{\frac{4}{3}}ds \\ &\leq C\beta^{-\frac{2}{3}}N^{\frac{1}{3}}. \end{aligned} \tag{6.14}$$

This completes the proof of Lemma 6.2. □

Proof of Proposition 4.1 for $J_1(\beta, N, T)$. The bounds on $J_1(\beta, N, T)$ for $d = 2, 3$ follow directly from Lemmas 6.1 and 6.2 and from (6.5). □

7. Proof of Proposition 3.2 for $d = 1$

Following (3.7) we choose

$$\kappa = \beta^{1/3}N^{1/3}, \quad \alpha = 0, \tag{7.1}$$

for the parameters of drift magnitude given by (5.1). Note that in the one-dimensional case we give all N particles drift in the positive direction of $\mathbb{R}$.

7.1. Cross-intersection occupation measure.

We follow the argument in Section 5, with the change that under $\mathbb{P}_T^{\lambda}$ all drifts go in the positive direction on $\mathbb{R}$. Recall that there are $O(N^2)$ pairs of particles.

We therefore have as in (4.2) as follows,

(7.2)

$$J_2(\beta,N,T) := CN^2\int_0^T\int_0^T\int_{\mathbf{B}_2(0)} f_{t,s}^{(k,\ell,\alpha)}(z)dzdsdt \leq CN^2\int_0^T\int_0^T\left(\left(\frac{1}{(t+s)^{1/2}}\int_{|z_1|<2} e^{-\frac{(z_1-\beta^{1/3}N^{1/3}(t-s))^2}{2(t+s)}}dz_1\right)\wedge 1\right)dsdt.$$

The following lemma give us the essential bound on $J_2(\beta,N,T)$.

Lemma 7.1. *There exists a constant $C_{7.1}>0$ not depending on N,T and β such that*

$$J_2(\beta,N,T)\leq C_{7.1}\beta^{2/3}N^{5/3}T, \quad \textit{for all } \beta,T\geq 1.$$

Proof. From we have (7.2),

(7.3)

$$J_2(\beta,N,T) \leq CN^2\int_0^T\int_0^T\left(\left(\frac{1}{(t+s)^{1/2}}\int_{|z_1|<2} e^{-\frac{(z_1-\beta^{1/3}N^{1/3}(t-s))^2}{2(t+s)}}dz_1\right)\wedge 1\right)dsdt.$$

We fix $C_0>2$ satisfying,

(7.4)

$$(2-\beta^{1/3}N^{1/3}(t-s))^2\geq \frac{1}{2}\beta^{2/3}N^{2/3}(t-s)^2, \quad \text{for all } t-s>C_0\beta^{-\frac{1}{3}}N^{-\frac{1}{3}},$$

and consider the following regions:

$$\begin{aligned}\hat{\mathbf{R}}_1 &= \{(t,s)\in[0,T]^2 : \beta^{1/3}N^{1/3}(t-s)>C_0\}\\ \hat{\mathbf{R}}_2 &= \{(t,s)\in[0,T]^2 : \beta^{1/3}N^{1/3}(t-s)\leq C_0\}.\end{aligned} \tag{7.5}$$

We define

$$J_2(\beta,N,T,\hat{\mathbf{R}}_i) := N^2\iint_{\hat{\mathbf{R}}_i}\int_{\mathbf{B}_2(0)} f_{t,s}^{(1,1,\alpha)}(z)dzdsdt, \quad i=1,2, \tag{7.6}$$

and note that

$$J_2(\beta,N,T)\leq\sum_{i=1}^2 J_i(\beta,N,T,\hat{\mathbf{R}}_i). \tag{7.7}$$

From (7.3), (7.4), (7.5) and (7.6) we get that

(7.8)

$$\begin{aligned}
&J_2(\beta,N,T,\hat{\mathbf{R}}_1)\\
&\le CN^2\int_0^T\int_0^t\left(\frac{1}{(2\pi(t+s))^{1/2}}e^{-\beta^{2/3}N^{2/3}\frac{(t-s)^2}{4(t+s)}}\right)\wedge 1dsdt\\
&\le CN^2\int\int_{t-s>\delta}\left(\frac{1}{(2\pi(t+s))^{1/2}}e^{-\beta^{2/3}N^{2/3}\frac{(t-s)^2}{4(t+s)}}\right)\wedge 1dsdt\\
&\quad+CN^2\int\int_{0<t-s\le\delta}\left(\frac{1}{(2\pi(t+s))^{1/2}}e^{-\beta^{2/3}N^{2/3}\frac{(t-s)^2}{4(t+s)}}\right)\wedge 1dsdt\\
&\le CN^2\int\int_{t-s>\delta}\frac{1}{(2\pi(t+s))^{1/2}}e^{-\beta^{2/3}N^{2/3}\frac{(t-s)^2}{4(t+s)}}dsdt+CN\int_0^T\int_{t-\delta}^t dsdt\\
&=: H_1(\beta,N,T)+H_2(\beta,N,T),
\end{aligned}$$

where we define

$$\delta=\beta^{-1/3}N^{-1/3}. \tag{7.9}$$

First we note that

$$H_2(\beta,N,T)\le CN^2T\delta\le\beta^{-1/3}N^{5/3}T. \tag{7.10}$$

Next we deal with (A). Let us change variables to

$$a=t-s,\qquad\qquad b=t+s$$

and absorb the Jacobian determinant into the constant C. We find

$$H_1(\beta,N,T)\le CN^2\int_0^{2T}\frac{1}{(2\pi b)^{1/2}}\left(\int_\delta^\infty e^{-\beta^{2/3}N^{2/3}\frac{a^2}{4b}}da\right)db. \tag{7.11}$$

Dealing with the inner integral, we use the bound on the integral over the Gaussian density to get,

(7.12)

$$\begin{aligned}
\frac{1}{(2\pi b)^{1/2}}\int_\delta^\infty e^{-\beta^{2/3}N^{2/3}\frac{a^2}{4b}}da &\le \frac{N^{-1/3}\beta^{-1/3}}{(2\pi bN^{-2/3}\beta^{-2/3})^{1/2}}\int_{-\infty}^\infty e^{-\frac{a^2}{4bN^{-2/3}\beta^{-2/3}}}da\\
&\le CN^{-1/3}\beta^{-1/3}.
\end{aligned}$$

Applying (7.12) to the inner integral in (7.11) with we get

$$\begin{aligned}
H_1(\beta,N,T)&\le CN^2\int_0^{2T}CN^{-1/3}\beta^{-1/3}db\\
&\le C\beta^{-1/3}N^{5/3}T.
\end{aligned} \tag{7.13}$$

Plugging-in our estimates from (7.10) and (7.13) to (7.8) we get

$$J_2(\beta,N,T,\hat{\mathbf{R}}_1)\le C\beta^{-1/3}N^{5/3}T. \tag{7.14}$$

From (7.5) and (7.6) it follows that

$$J_2(\beta, N, T, \hat{\mathbf{R}}_2) \leq N^2|\hat{\mathbf{R}}_2| \leq CN^2\beta^{-1/3}N^{-1/3}T \leq C\beta^{-1/3}N^{5/3}T. \tag{7.15}$$

From (7.7), (7.14) and (7.15) we get Lemma 7.1. ☐

7.2. **Self-intersection occupation measure.** Using (5.1) and (7.1) we deduce by following similar lines as in (6.2), (6.3) and (6.5), we need to bound the following integral:

(7.16)

$$J_1(\beta, N, T) := N\int_0^T\int_0^T\int_{\mathbf{B}_2(0)} f_{t,s}^{(1,1,\alpha)}(z)dzdsdt = N\int_0^T\int_0^T \frac{1}{(2\pi(t-s))^{1/2}}\int_{|z|<2} e^{-\frac{(z-\beta^{1/3}N^{1/3}(t-s))^2}{2(t-s)}}dzdsdt.$$

In the following lemma we derive the bound on $J_1(\beta, N, T)$.

Lemma 7.2. *There exists a constant $C_{7.2} > 0$ not depending on N, T and β such that*

$$J_1(\beta, N, T) \leq C_{7.2}\beta^{2/3}N^{2/3}T, \quad \textit{for all } \beta, T \geq 1.$$

Proof. Recalling $\hat{\mathbf{R}}_i$ defined in (7.5) we denote

$$J_1(\beta, N, T, \hat{\mathbf{R}}_i) := N\iint_{\hat{\mathbf{R}}_i}\int_{\mathbf{B}_2(0)} f_{t,s}^{(1,1,\alpha)}(z)dzdsdt, \quad i = 1, 2, \tag{7.17}$$

and note that

$$J_1(\beta, N, T) \leq \sum_{i=1}^{2} J_i(\beta, N, T, \hat{\mathbf{R}}_i).$$

Using the fact that $\int_{\mathbf{B}_2(0)} f_{t,s}^{(1,1,\alpha)}(z)dz \leq 1$ and (7.4), we get for δ as in (7.9),

(7.18)

$$J_1(\beta, N, T, \mathbf{R_1}) \leq CN\int\int_{\hat{\mathbf{R}}_1\cap\{t-s>\delta\}} \frac{1}{2\pi(t-s)}e^{-\frac{\beta^{2/3}N^{2/3}(t-s)^2}{4(t-s)}}dsdt + CN\int\int_{\hat{\mathbf{R}}_1\cap\{t-s\leq\delta\}} dsdt.$$

Note that this integral is similar to (7.8) (with a difference by a factor of N) hence we get the bound

$$J_1(\beta, N, T, \hat{\mathbf{R}}_1) \leq C\beta^{2/3}N^{2/3}T. \tag{7.19}$$

Using again the trivial bound on the integral over $f^{(1,1,\alpha)}$ we get from (7.17) and (7.5) that

$$J_1(\beta, N, T, \hat{\mathbf{R}}_2) \leq C \iint_{\hat{\mathbf{R}}_2} ds dt \leq C\beta^{-1/3} N^{2/3} T, \tag{7.20}$$

where we used a similar bound as in (7.15) with a difference of a factor of N in the last inequality.

From (7.19), (7.20) and (7.17) we get the result of Lemma 7.2. □

Now we are ready to prove Theorem 2.2 for $d = 1$.

Proof of Proposition 3.2 for $d = 1$. The proof follows similar lines to the proof in the case where $d = 2, 3$ only we now use Lemmas 7.1 and 7.2 instead of Proposition 4.1. □

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