ON THE RADIUS OF SELF-REPELLENT FRACTIONAL BROWNIAN MOTION

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ABSTRACT. We study the radius of gyration R_T of a self-repellent fractional Brownian motion $\left\{B_t^H\right\}_{0\leq t\leq T}$ taking values in $\mathbb{R}^d$. Our sharpest result is for $d=1$, where we find that with high probability,

$$R_T \asymp T^{\nu}, \quad \text{with } \nu = \frac{2}{3}(1+H).$$

For $d>1$, we provide upper and lower bounds for the exponent ν, but these bounds do not match.

1. INTRODUCTION

Self-avoiding random walks are among the most extensively studied models in statistical physics. A variant, the *self-repellent walk* (also known as the *weakly self-avoiding walk*), provides a weaker version of the self-avoiding walk. This variation adjusts the probability distribution of a simple random walk by imposing penalties on paths with self-intersections. In contrast, a self-avoiding walk is a random walk that strictly prohibits any self-intersections. The *Domb-Joyce model* [1] constitutes a discrete-time version of the self-repellent walk, while the *Edwards model* [2] provides a continuous alternative, known as the *self-repellent Brownian motion*. For an in-depth treatment of self-avoiding walks, readers are directed to the monograph [3], lecture notes [4], and the survey [5] that highlights recent advancements in the field.

This paper primarily focuses on the Edwards model associated with self-repellent *fractional Brownian motions* (fBm's). This particular model provides an apt framework to analyze the properties of polymer molecules in good solvents, as discussed in detail in [6]. Extensive investigation of the self-repellent fBm has been undertaken in [7, 8, 9], contingent on the presence of a square-integrable (self-intersection) local time for the fBm. However, it is known that the fBm with the *Hurst parameter* H might not possess a square-integrable local time if $dH\geq 1$—for instance, if $d=2$ and $H=1/2$; see [10, 11]. As observed in [12], the local time characteristic in the self-repellent Brownian motion is not essential. Instead, one can work with the *occupation measure*. In particular, we consider the occupation measure of balls of radius 1. Thus, we penalize paths that come close to their past positions, rather than passing through exactly the same points. Substituting local time with the occupation measure should theoretically maintain the outcome, which is validated specifically within dimension 1. This insight facilitates the characterization of self-repellent fBm's across all dimensions $d\geq 1$ and for the entire range of the Hurst parameter $H\in(0,1)$.

Let $\left\{B_t^H\right\}_{t\geq 0}$ be a fractional Brownian motion with Hurst index $H\in(0,1)$, taking values in $\mathbb{R}^d$. That is, $B_t^H=\left(B_t^{H,1},\dots,B_t^{H,d}\right)$ where $\left(B_\cdot^{H,i}\right)_{i=1}^d$ are independent one-dimensional

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fractional Brownian motions with Hurst index H. Thus, each $B_{\cdot}^{H,i}$ is a centered Gaussian process on $[0,\infty)$ with covariance

$$\mathbb{E}\left[B_s^{H,i}B_t^{H,i}\right]=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

We define the occupation time as follows:

$$L_T(y)\coloneqq\left|\left\{t\in[0,T]:B_t^H\in\mathbf{B}_1(y)\right\}\right|=\int_0^T\mathbf{1}_{\mathbf{B}_1(y)}\left(B_t^H\right)\mathrm{d}t, \tag{1.1}$$

where $|S|$ denotes the Lebesgue measure of the set S, and $\mathbf{B}_r(y)$ is the open ball in $\mathbb{R}^d$, centered at y, of radius $r>0$. Define

$$\mathcal{E}_T\coloneqq\exp\left(-\beta\int_{\mathbb{R}^d}L_T(z)^2\mathrm{d}z\right) \tag{1.2}$$

and for an event A, let

$$\mathbb{Q}_T(A)\coloneqq\frac{1}{Z_T}\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_A\mathcal{E}_T\right],\qquad Z_T\coloneqq\mathbb{E}^{\mathbb{P}_T}\left[\mathcal{E}_T\right]. \tag{1.3}$$

Then, under probability measure $\mathbb{Q}_T$, $\left\{B_t^H:0\le t\le T\right\}$ is a self-repellent fBm.

In this paper, we will investigate the *radius of gyration* R_T (see [13]) of the self-repellent fractional Brownian motion B_t^H.

$$R_T\coloneqq\left[\frac{1}{T}\int_0^T\left|B_t^H-\overline{B}_T^H\right|^2\mathrm{d}t\right]^{1/2}\quad\text{with}\quad\overline{B}_T^H\coloneqq\frac{1}{T}\int_0^TB_t^H\mathrm{d}t. \tag{1.4}$$

It is worth noting that another customary radius is the *mean square end-to-end distance*, which is often seen in mathematical papers:

$$\left(\mathbb{E}^{\mathbb{Q}_T}\left[\left|B_T^H\right|^2\right]\right)^{1/2},\qquad\text{where }\left|B_T^H\right|\coloneqq\sqrt{\left(B_T^{H,1}\right)^2+\cdots+\left(B_T^{H,d}\right)^2}.$$

Physicists often base their reasoning on universality, namely the belief that changing the details of a model will not affect its large-scale behavior. We believe that the exact definition of the radius is unlikely to change our final result. It is expected that, maybe up to a logarithmic correction,

$$R_T\asymp T^\nu$$

for some exponent ν depending on the dimension d and the Hurst parameter H. It has been conjectured in [8] and [9] that

$$\nu=\frac{2(1+H)}{2+d}. \tag{1.5}$$

In the case of one dimension, the radius exhibits ballistic behavior, $\nu=1$, for the self-repellent random walk, as shown in [14, 15]. For fBm, the corresponding result is as follows:

Theorem 1.1 ($d = 1$)**.** *Let B^H be a one-dimensional fBm with Hurst parameter $H \in (0,1)$, and let $\mathbb{Q}_T$ and R_T be defined as in (1.3) and (1.4), respectively. Then, for any $\beta > 0$, there exist nonrandom constants $T_\beta \geq e$, and C_*, C^*, $C_{1.12} > 0$ such that the following inequality holds whenever $T \geq T_\beta$:*

$$\mathbb{Q}_T\left(C_*\beta^{1/3}T^{\frac{2(1+H)}{3}} \leq R_T \leq C^*\beta^{1/3}T^{\frac{2(1+H)}{3}}\right) \geq 1 - 2\exp\left(-C_{1.12}\beta^{2/3}T^{\frac{2(2-H)}{3}}\right),$$

where the constants C_*, C^* are given in (1.8) and $C_{1.12}$ in (1.12).

In other words, the radius of the one-dimensional self-repellent fBm is completely solved:

$$R_T \asymp T^\nu \quad \text{with} \quad \nu = \frac{2}{3}(1+H)$$

with high probability for large T. This theorem proves the conjectured claim (1.5) for $d = 1$, if we use the radius of gyration. In particular, when $H = 1/2$, $R_T \asymp T$, which coincides with the classical result for the Brownian motion/random walk in [14, 15].

The analogous question in dimensions $d = 2, 3, 4$ is completely open even in the special case of self-repellent random walk/Brownian motion. In dimensions $d \geq 5$, the *lace expansion* was successfully used to show that $\nu = 1/2$ for self-repellent random walk; see [16, 17]. The lace expansion is not expected to work in the fBm case, since it requires the Markov property. We have the following result for all dimensions:

Theorem 1.2 ($d \geq 1$)**.** *Let B^H be a d-dimensional fBm with Hurst parameter $H \in (0,1)$, and let $\mathbb{Q}_T$ and R_T be defined as in (1.3) and (1.4), respectively. Then, for any $\beta > 0$, there exists some nonrandom constant $T_\beta \geq e$ such that the following inequality holds whenever $T \geq T_\beta$:*

$$\mathbb{Q}_T\left(C_*\, \underline{R}_T \leq R_T \leq C^*\, \overline{R}_T\right) \geq 1 - 2\exp\left(-C_{1.12}\gamma_{d,H}(\beta)F_{d,H}(T)\right). \tag{1.6}$$

In (1.6), $\gamma_{d,H}(\beta)$ and $F_{d,H}(T)$ are defined in Table 1 with

$$\beta^{a,b} := \beta^a \mathbf{1}_{\{0<\beta\leq 1\}} + \beta^b \mathbf{1}_{\{\beta>1\}}. \tag{1.7}$$

The constants C_* and C^* are defined as follows:

$$C_* := \left(\frac{C_{1.10}}{2C_{1.12}}\right)^{1/d} \quad \text{and} \quad C^* := \left(\frac{2C_{1.12}}{C_{1.11}}\right)^{1/2}, \tag{1.8}$$

with $C_{1.10}$, $C_{1.11}$, and $C_{1.12}$ being the positive constants appearing in (1.10), (1.11), and (1.12), respectively. The bounds $\underline{R}_T$ and $\overline{R}_T$ in (1.6) are equal to

$$\begin{aligned}\underline{R}_T &= \underline{R}_T(d,H,\beta) := \left(\frac{\beta T^2}{\gamma_{d,H}(\beta)F_{d,H}(T)}\right)^{1/d},\\ \overline{R}_T &= \overline{R}_T(d,H,\beta) := \left(\gamma_{d,H}(\beta)T^{2H}F_{d,H}(T)\right)^{1/2};\end{aligned} \tag{1.9}$$

see Table 2 for their explicit expressions for various cases.

In the one-dimensional case, $dH = H < 1$, and thus

$$\underline{R}_T = \overline{R}_T = \beta^{1/3}T^{\frac{2(1+H)}{3}};$$

see the column $d = 1$ in Table 3b below for some concrete values. Therefore, Theorem 1.1 is a direct corollary of Theorem 1.2.

	$dH<1$		$dH=1$		$dH>1$	
	$d=1$	$d\geq 1$	$H=1/2$ and $d=2$	$H<1/2$	$H\geq 1/2$	$H<1/2$
$\gamma_{d,H}(\beta) \coloneqq$	$\beta^{2/3}$	$\beta^{\frac{2(1-H)}{3-(d+2)H}}$	$\beta^{1/2,\,2/3}$	β	$\beta^{2/3}$	β
$F_{d,H}(T) \coloneqq$	$T^{\frac{2(2-H)}{3}}$	$T^{1+\frac{(1-2H)(1-dH)}{3-(d+2)H}}$	T	$T\log(T)$	$T^{\frac{2(2-H)}{3}}$	T

TABLE 1. Definitions of $\gamma_{d,H}(\beta)$ and $F_{d,H}(T)$, with the case $d=1$ being specially highlighted in the gray background. Recall that the notation $\beta^{1/2,\,2/3}$ is given in (1.7).

	$dH<1$		$dH>1$	
	$d=1$	$d\geq 1$	$H\geq 1/2$	$H<1/2$
$\underline{R}_T$	$\beta^{1/3}T^{\frac{2(1+H)}{3}}$	$\left(\beta^{\frac{1-dH}{3-(d+2)H}}T^{\frac{2-2dH^2}{3-(d+2)H}}\right)^{1/d}$	$\left(\beta^{1/3}T^{2(1+H)/3}\right)^{1/d}$	$T^{1/d}$
$\overline{R}_T$		$\beta^{\frac{1-H}{3-(2+d)H}}T^{\frac{2-(d-1)H-2H^2}{3-(d+2)H}}$	$\beta^{1/3}T^{2(1+H)/3}$	$\beta^{1/2}T^{(1+2H)/2}$

(A) $dH\neq 1$

	$dH=1$	
	$H=1/2$ and $d=2$	$H=1/d$ and $d\geq 3$
$\underline{R}_T$	$\beta^{1/4,1/6}\,T^{1/2}$	$(T/\log(T))^{1/d}$
$\overline{R}_T$	$\beta^{1/4,1/3}\,T$	$\beta^{1/2}T^{(1+2H)/2}\sqrt{\log(T)}$

(B) $dH=1$

TABLE 2. Explicit expressions of $\underline{R}_T$ and $\overline{R}_T$, as defined in (1.9), for various values of (d,H), with the case $d=1$ being specially highlighted in the gray background. Recall that the notation $\beta^{a,b}$ is given in (1.7).

The proof of Theorem 1.2 builds on techniques from Mueller and Neuman [12]. Given $r>0$, define the following events and probabilities:

$$A_{r,T}^{(<)} \coloneqq \{R_T\leq r\}, \qquad q_{r,T}^{(<)} \coloneqq \mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{A_{r,T}^{(<)}}\mathcal{E}_T\right],$$
$$A_{r,T}^{(>)} \coloneqq \{R_T\geq r\}, \qquad q_{r,T}^{(>)} \coloneqq \mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{A_{r,T}^{(>)}}\mathcal{E}_T\right].$$

The following two lemmas, whose proofs are deferred to Sections 2.1 and 2.2, respectively, will be used in the proof of Theorem 1.2.

Lemma 1.3. *For any $T \geq 0$, the following two inequalities hold:*

$$q_{r,T}^{(<)} \leq \exp\left(-C_{1.10}\frac{\beta\, T^2}{r^d}\right), \quad \text{for all } r \geq 1, \tag{1.10}$$

$$q_{r,T}^{(>)} \leq \exp\left(-C_{1.11}\frac{r^2}{T^{2H}}\right), \quad \text{for all } r > 0. \tag{1.11}$$

where

$$C_{1.10} = C_{1.10}(d) := \frac{9\Gamma(1+d/2)}{2^{5+d}\,\pi^{d/2}} \quad \text{and} \quad C_{1.11} = C_{1.11}(d,H) > 0.$$

Lemma 1.4. *There exist constants $C_{1.12} = C_{1.12}(d,H) > 0$ and $T_\beta > e$, such that for all $T \geq T_\beta$, $d \geq 1$, and $H \in (0,1)$, we have*

$$\log Z_T \geq -C_{1.12}\,\gamma_{d,H}(\beta) F_{d,H}(T), \tag{1.12}$$

where $\gamma_{d,H}(\beta)$ and $F_{d,H}(T)$ are defined in Table 1.

The paper is organized as follows. The proofs of Theorem 1.2 and lemmas 1.3 and 1.4 are given in Section 2. We then extend our discussions on our results in Section 3. Finally, in the appendix, we include some known results for fBm's and the corresponding Girsanov theorem.

2. Proof of the main result

We will defer the proofs of the lemmas to the next section, opting to initially establish Theorem 1.2 through their application. Here, let us briefly outline our strategy, which has been successfully employed in Brownian cases in [12, Theorem 1.1].

Let $a < b$ be real numbers, and let $c > 0$. Recall that $\mathbb{Q}_T$ is defined by (1.3). Let X be a random variable. Then, to prove

$$\mathbb{Q}_T\left(a \leq X \leq b\right) \geq 1 - 2\exp(-c), \tag{2.1}$$

it suffices to show that

$$\mathbb{Q}_T(X \leq a) = \frac{\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{X\leq a\}}\mathcal{E}_T\right]}{Z_T} \leq \exp(-c),$$

and

$$\mathbb{Q}_T(X \geq b) = \frac{\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{X\geq b\}}\mathcal{E}_T\right]}{Z_T} \leq \exp(-c).$$

Additionally, the above two inequalities are ensured by the next three inequalities:

$$Z_T \geq \exp(-c), \tag{2.2}$$

$$\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{X\leq a\}}\mathcal{E}_T\right] \leq \exp(-2c), \tag{2.3}$$

and

$$\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\{X\geq b\}}\mathcal{E}_T\right] \leq \exp(-2c). \tag{2.4}$$

Following this idea, we are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. By the definition of $F_{d,H}$ in Table 1, it is clear that the function $T^2/F_{d,H}(T)$ is monotone increasing on $[e,\infty)$ with $\lim_{T\to\infty} T^2/F_{d,H}(T)=\infty$ for any $d\geq 1$ and $H\in(0,1)$. Hence, when T is large enough, we can ensure that

$$r_* := \left(\frac{C_{1.10}\beta T^2}{2C_{1.12}\gamma_{d,H}(\beta)F_{d,H}(T)}\right)^{1/d} \geq 1.$$

Plugging the above r_* to (1.10) shows that

$$\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{R_T\leq r_*}\right] = q^{(<)}_{r_*,T} \leq \exp\left(-C_{1.10}\frac{\beta T^2}{r_*^d}\right) = \exp\big(-2C_{1.12}\gamma_{d,H}(\beta)F_{d,H}(T)\big).$$

Next, by choosing the following r^* in (1.11)

$$r^* := \left(\frac{2C_{1.12}\gamma_{d,H}(\beta)T^{2H}F_{d,H}(T)}{C_{1.11}}\right)^{1/2},$$

we have

$$\mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{R_T\geq r^*}\right] = q^{(>)}_{r^*,T} \leq \exp\left(-C_{1.11}\frac{(r^*)^2}{T^{2H}}\right) = \exp\big(-2C_{1.12}\gamma_{d,H}(\beta)F_{d,H}(T)\big).$$

Concerning (1.12) in Lemma 1.4. We have justified all inequalities (2.2)–(2.4), and therefore (2.1), with $X=R_T$ defined as in (1.4), $a=r_*$, $b=r_*$, and $c=C_{1.12}\gamma_{d,H}(\beta)F_{d,H}(T)$. This proves (1.6) with $\underline{R}_T$ and $\overline{R}_T$ defined as in (1.9). This completes the proof of Theorem 1.2. □

2.1. Upper bounds on $q^{(<)}_{r,T}, q^{(>)}_{r,T}$—Proof of Lemma 1.3.

(1) Fix an arbitrary $r\geq 1$. Suppose $R_T\geq r$. Then there must be a time $t_1\in[0,T]$ such that $\left|B^H_{t_1}-\overline{B}^H_T\right|\geq r$. This in turn implies that there exists $t_2\in[0,T]$ such that $\left|B^H_{t_1}-B^H_{t_2}\right|\geq r$. Then the triangle inequality shows that either $\left|B^H_{t_1}\right|\geq r/2$ or $\left|B^H_{t_2}\right|\geq r/2$. So we conclude that $\sup_{t\in[0,T]}|B_t|\geq r/2$. Hence, concerning $\mathcal{E}_T\leq 1$, and using [18, Theorem 4.1.1], we have

$$q^{(>)}_{r,T} = \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{A^{(>)}_{r,T}}\mathcal{E}_T\right] \leq \mathbb{P}\left(A^{(>)}_{r,T}\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]}\left|B^H_t\right|\geq r/2\right) \leq \exp\left(-C_{1.11}\frac{r^2}{T^{2H}}\right),$$

for some constant $C_{1.11}>0$. This proves (1.11).

(2) Fix an arbitrary $r>0$. Recall that $L_T(x)$ and $\mathcal{E}_T$ are defined in (1.1) and (1.2), respectively. We claim that

$$R_T\leq r \quad \text{implies} \quad -\log\mathcal{E}_T = \int_{\mathbb{R}^d} L_T(x)^2\mathrm{d}x \geq 2C_{1.10}\frac{T^2}{(r+1)^d}. \tag{2.5}$$

As a consequence, for all $r\geq 1$,

$$\begin{aligned} q^{(<)}_{r,T} = \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{A^{(<)}_{r,T}}\mathcal{E}_T\right] &\leq \sup_{\omega\in A^{(<)}_{r,T}}\mathcal{E}_T(\omega) = \exp\left(-\beta\inf_{\omega\in A^{(<)}_{r,T}}\int_{\mathbb{R}^d}L_T(z,\omega)^2\mathrm{d}z\right) \\ &\leq \exp\left(-2C_{1.10}\frac{\beta T^2}{(r+1)^d}\right) \leq \exp\left(-C_{1.10}\frac{\beta T^2}{r^d}\right). \end{aligned}$$

This confirms (1.10). Thus, it remains to prove (2.5). Assume that $R_T \le r$. Notice that

$$r^2 T \ge \int_0^T \left| B_t^H - \overline{B}_T^H \right|^2 \mathrm{d}t \ge \int_0^T 4r^2 \mathbf{1}_{\{|B_t^H - \overline{B}_T^H| > 2r\}} \mathrm{d}t$$
$$\ge 4r^2 \left| \left\{ t \in [0,T] : \left| B_t^H - \overline{B}_T^H \right| > 2r \right\} \right|,$$

where $|\cdot|$ denotes Lebesgue measure. Therefore,

$$\left| \left\{ t \in [0,T] : \left| B_t^H - \overline{B}_T^H \right| > 2r \right\} \right| \le \frac{T}{4}$$

and

$$\left| \left\{ t \in [0,T] : \left| B_t^H - \overline{B}_T^H \right| \le 2r \right\} \right| \ge \frac{3T}{4}.$$

It follows that

$$\int_{\mathbf{B}_{(2r+1)}(\overline{B}_T^H)} L_T(y)\mathrm{d}y \ge \frac{3T}{4}. \tag{2.6}$$

Denote by $K_d \coloneqq \pi^{d/2}/\Gamma(1+d/2)$ the volume of the unit ball in $\mathbb{R}^d$. Then

$$C_{1.10} = \frac{9}{2^{5+d} K_d}, \quad \text{and} \quad K_d(2r+1)^d = \left| \mathbf{B}_{2r+1}\left(\overline{B}_T^H\right) \right| = \int_{\mathbf{B}_{2r+1}\left(\overline{B}_T^H\right)} \mathrm{d}y.$$

Now, by the Cauchy-Schwarz inequality and (2.6),

$$K_d(2r+1)^d \int_{\mathbb{R}^d} L_T(y)^2 \mathrm{d}y \ge \int_{\mathbf{B}_{2r+1}(\overline{B}_T^H)} \mathrm{d}y \int_{\mathbf{B}_{2r+1}(\overline{B}_T^H)} L_T(y)^2 \mathrm{d}y$$
$$\ge \left(\int_{\mathbf{B}_{2r+1}(\overline{B}_T^H)} L_T(y)\mathrm{d}y \right)^2 \ge \frac{9T^2}{16}.$$

It follows that

$$\int_{\mathbb{R}^d} L_T(y)^2 \mathrm{d}y \ge \frac{9T^2}{16 K_d (2r+1)^d} \ge \frac{9}{2^{4+d}} \times \frac{T^2}{K_d(r+1)^d} = 2C_{1.10} \frac{T^2}{(r+1)^d}.$$

This proves claim (2.5). The proof of Lemma 1.3 is complete. □

2.2. **Lower bounds on Z_T—Proof of Lemma 1.4.** Now we study the term Z_T defined as in (1.3). Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^d$, and let $\mathbb{P}_T^\lambda$, $\mathcal{Q}_T$ be given as in (A.2). That is, the new measure $\mathbb{P}_T^\lambda$ adds a drift proportional to λ, and $\mathcal{Q}_T$ is the Radon-Nikodym change of measure term, see Theorem A.2. The drift enforces the behavior we think the self-repellent process should have. Then, we can write

$$Z_T = \mathbb{E}^{\mathbb{P}_T}[\mathcal{E}_T] = \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\mathcal{E}_T \cdot (\mathcal{Q}_T(\lambda M))^{-1} \right].$$

Applying Jensen's inequality to $\log Z_T$, we find

$$\log Z_T \ge \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\log\left(\mathcal{E}_T \cdot (\mathcal{Q}_T(\lambda M))^{-1} \right) \right] = -\left(I_1 + I_2 \right),$$

where

$$I_1 \coloneqq \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\beta \int_{\mathbb{R}^d} L_T(y)^2 \mathrm{d}y \right] \quad \text{and} \quad I_2 \coloneqq \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\log\left(\mathcal{Q}_T(\lambda M) \right) \right].$$

As a consequence, we need to establish upper bounds for both I_1 and I_2. In the proof below, the generic constant $C > 0$ may vary from line to line.

Upper bound for I_1. Due to the Girsanov formula for fBm's as stated in Theorem A.2, under $\mathbb{P}_T^\lambda$, $\{B_t^H : 0 \le t \le T\}$ has the same distribution as $\{B_t^H + \lambda t\mathbf{u} : 0 \le t \le T\}$ under $\mathbb{P}_T$. Fixing $T > e$. Let $g(t, \cdot)$ be the probability density of B_t^H for all $0 \le t \le T$. Since B^H has stationary increments, if $0 \le s < t$ then the probability density of $B_t^H - B_s^H$ is $g(t-s, \cdot)$. Note that for $x, y, z \in \mathbb{R}^d$, if $|x-z| < 1$ and $|y-z| < 1$, then $|x-y| < 2$. So we have

$$
\begin{aligned}
I_1 &\le C\beta \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, \mathbb{E}^{\mathbb{P}_T}\left[\mathbf{1}_{\mathbf{B}_2(0)}(B_t^H - B_s^H + (t-s)\lambda\mathbf{u})\right] \\
&\le C\beta \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \int_{\mathbf{B}_2(0)} \mathrm{d}z\, g(t-s, z-(t-s)\lambda\mathbf{u}) \\
&\le C\beta T \int_0^T \mathrm{d}r \int_{\mathbf{B}_2(0)} \mathrm{d}z\, g(r, z - r\lambda\mathbf{u}).
\end{aligned}
$$

Hence, choosing $\mathbf{u} = (1, 0, \ldots, 0)$, we have that

$$
I_1 \le C\beta T \int_0^T \mathrm{d}r \left[\int_{-2}^2 \mathrm{d}z_1\, r^{-H} \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \times \prod_{i=2}^d \int_{-2}^2 \mathrm{d}z_i\, r^{-H} \exp\left(-\frac{z_i^2}{r^{2H}}\right)\right].
$$

Notice that for all $r \ge 4/\lambda$, $z_1 \in [-2, 2]$, we have $|z_1| \le r\lambda/2$ and thus

$$
\exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \le \exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right).
$$

Therefore,

$$
\begin{aligned}
\int_{-2}^2 \mathrm{d}z_1\, \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \le\, & \mathbf{1}_{\{0 \le r < 4/\lambda\}} \int_{-2}^2 \mathrm{d}z_1\, \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \\
&+ 4 \times \mathbf{1}_{\{r \ge 4/\lambda\}} \exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right).
\end{aligned}
$$

If $\lambda \in (0, 1]$, it follows that

$$
\begin{aligned}
\mathbf{1}_{\{0 \le r < 4/\lambda\}} \int_{-2}^2 \mathrm{d}z_1\, \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \le\, & 4 \times \mathbf{1}_{\{0 \le r < 4/\lambda\}} \\
\le\, & C_* \mathbf{1}_{\{0 \le r < 4/\lambda\}} \exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right),
\end{aligned}
$$

where

$$
C_* := \sup_{\lambda \in (0,1]} \sup_{r \in (0, 4/\lambda)} 4\exp\left(\frac{1}{4}\lambda^2 r^{2(1-H)}\right) = 4\exp\left(2^{2(1-2H)}\right).
$$

On the other hand, if $\lambda > 1$, then we can write

$$
\begin{aligned}
\mathbf{1}_{\{0 \le r < 4/\lambda\}} \int_{-2}^2 \mathrm{d}z_1\, \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \le\, & \mathbf{1}_{\{0 \le r < 4/\lambda\}} \int_{-\infty}^{\infty} \mathrm{d}z_1\, \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \\
=\, & C \mathbf{1}_{\{0 \le r < 4/\lambda\}} r^H.
\end{aligned}
$$

Combining the above four cases shows that

$$
\int_{-2}^2 \mathrm{d}z_1\, \exp\left(-\frac{(z_1 - r\lambda)^2}{r^{2H}}\right) \le C\left(\exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right) + \mathbf{1}_{\{0 \le r < 4/\lambda < 4\}} r^H\right).
$$

Notice that

$$\int_{-2}^{2} \mathrm{d}z\, r^{-H} \exp\left(-\frac{z^2}{r^{2H}}\right) \le \min\left(4r^{-H}, \sqrt{\pi}\right) \le C\left(r^{-H} \wedge 1\right).$$

Therefore, we can write

$$\begin{aligned} I_1 \le & C\beta T \int_0^T \mathrm{d}r\, r^{-H} \left(1 \wedge r^{-H}\right)^{d-1} \left(\exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right) + \mathbf{1}_{\{0\le r<4/\lambda<4\}} r^H\right) \\ \le & C\beta T \left(I_{1,1} + I_{1,2} + I_{1,3}\right), \end{aligned}$$

where

$$\begin{aligned} I_{1,1} &:= \int_0^1 \mathrm{d}r\, r^{-H} \exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right), \\ I_{1,2} &:= \int_1^T \mathrm{d}r\, r^{-dH} \exp\left(-\frac{1}{4}\lambda^2 r^{2(1-H)}\right), \\ I_{1,3} &:= \mathbf{1}_{\{\lambda>1\}} \int_0^{4/\lambda} \mathrm{d}r \le C\left(1 \wedge \lambda^{-1}\right). \end{aligned}$$

Performing a change of variable $\frac{1}{4}\lambda^2 r^{2(1-H)} = s$, we can write

$$\begin{aligned} I_{1,1} =& C\lambda^{-1} \int_0^{\frac{\lambda^2}{4}} s^{-\frac{1}{2}} e^{-s} \mathrm{d}s \\ \le & C\left(\mathbf{1}_{\{0<\lambda\le 1\}} \lambda^{-1} \int_0^{\frac{\lambda^2}{4}} \mathrm{d}s\, s^{-\frac{1}{2}} + \mathbf{1}_{\{\lambda>1\}} \lambda^{-1} \int_0^\infty \mathrm{d}s\, s^{-\frac{1}{2}} e^{-s}\right) \le C\left(1 \wedge \lambda^{-1}\right), \end{aligned}$$

and

$$I_{1,2} = C\lambda^{-\frac{1-dH}{1-H}} \int_{\frac{\lambda^2}{4}}^{\frac{1}{4}\lambda^2 T^{2(1-H)}} \mathrm{d}s\, s^{\frac{1-dH}{2(1-H)}-1} e^{-s}.$$

Therefore, we need only estimate $I_{1,2}$. Assume $T \ge 1$, otherwise $I_{1,2} = 0$.

Case I: If $dH < 1$, then $\frac{1-dH}{2(1-H)} > 0$ and hence,

$$I_{1,2} \le C\lambda^{-\frac{1-dH}{1-H}} \int_0^\infty \mathrm{d}s\, s^{\frac{1-dH}{2(1-H)}-1} e^{-s} = C\lambda^{-\frac{1-dH}{1-H}}.$$

Case II: If $dH = 1$, then we have that $I_{1,2} \le C\left(\log T \wedge \lambda^{-2}\right)$, which is due to

$$I_{1,2} = C\int_{\frac{\lambda^2}{4}}^{\frac{1}{4}\lambda^2 T^{2(1-H)}} \mathrm{d}s\, s^{-1} e^{-s} \le C\lambda^{-2} \int_{\frac{\lambda^2}{4}}^{\frac{1}{4}\lambda^2 T^{2-2H}} \mathrm{d}s\, e^{-s} \le C\lambda^{-2} \quad \text{and}$$

$$I_{1,2} \le C\int_{\frac{\lambda^2}{4}}^{\frac{1}{4}\lambda^2 T^{2(1-H)}} \mathrm{d}s\, s^{-1} \le C\left(\log\left(\frac{\lambda^2}{4} T^{2-2H}\right) - \log\left(\frac{\lambda^2}{4}\right)\right) = C\log T.$$

Case III: If $dH > 1$, then we have

$$I_{1,2} \le C\min\left(\int_1^\infty \mathrm{d}r\, r^{-dH},\ \lambda^{-2} \int_0^\infty \mathrm{d}s\, e^{-s}\right) = C\left(1 \wedge \lambda^{-2}\right).$$

As a consequence, with $T > e$, we can write

$$I_1 \le I_1^*(\lambda) := \begin{cases} C\beta T \lambda^{-\frac{1-dH}{1-H}}, & dH < 1, \\ C\beta T \left(\log(T) \wedge \lambda^{-2} + 1 \wedge \lambda^{-1}\right), & dH = 1, \\ C\beta T \left(1 \wedge \lambda^{-1}\right), & dH > 1. \end{cases} \tag{2.7}$$

Upper bound for I_2. As for I_2, we need the Girsanov formula for martingales. Recall Theorem A.1 that for any unit vector $\mathbf{u} \in \mathbb{R}^d$, $M = M^{\mathbf{u}}$ defined as in (A.1) is a square-integrable martingale. The classical Girsanov formula for martingales implies that $\widetilde{M} := \left\{\widetilde{M}_t = M_t + \langle M \rangle_t\right\}$ is a martingale under the probability measure $\mathbb{P}_T^\lambda$ given by (A.2). As a consequence,

$$\begin{aligned} I_2 = \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\log\left(\mathcal{Q}_T(\lambda M)\right)\right] =& \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\lambda M_t - \frac{1}{2}\lambda^2 \langle M \rangle_t\right] = \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\lambda \widetilde{M}_t + \frac{1}{2}\lambda^2 \langle M \rangle_t\right] \\ =& \frac{1}{2}\lambda^2 \mathbb{E}^{\mathbb{P}_T^\lambda}\left[\langle M \rangle_t\right] = \frac{1}{2} C_H \lambda^2 t^{2(1-H)} =: I_2^*(\lambda). \end{aligned} \tag{2.8}$$

Matching bounds for I_1 and I_2. Recall that $\log(Z_t) \ge -(I_1 + I_2)$, and I_1 and I_2 are bounded by $I_1^*(\lambda)$ in (2.7) and $I_2^*(\lambda)$ in (2.8), respectively. Notably, I_1^* is a decreasing function of $\lambda \in (0, \infty)$, whereas I_2^* is an increasing function in the same range. Therefore, in order to optimize the lower bound for $\log(Z_T)$ based on (2.7) and (2.8), we need to find a suitable λ such that $I_1^*(\lambda)$ and $I_2^*(\lambda)$ coincide up to a constant. In the following, we omit the tedious computations required to identify the appropriate λ, and choose[1]

$$\lambda = \begin{cases} \beta^{\frac{1-H}{3-(d+2)H}} T^{-\frac{(1-2H)(1-H)}{3-(d+2)H}}, & dH < 1, \\ \beta^{1/3}\mathbf{1}_{\{\beta \ge 1\}} + \beta^{1/4}\mathbf{1}_{\{0<\beta\le 1\}}, & dH = 1, H = 1/2, \\ \beta^{1/2} T^{H-1/2}\sqrt{\log T}, & dH = 1, H < 1/2, \\ \beta^{1/2} T^{H-1/2}, & dH > 1, H < 1/2, \\ \beta^{1/3} T^{\frac{2H-1}{3}}, & dH > 1, H \ge 1/2. \end{cases}$$

Then, (1.12) follows immediately. The proof of Lemma 1.4 is complete. □

3. Discussion

In this section, we present some concrete examples and make some remarks on our results. When $d = 1$, our results are sharp. When $d \ge 2$, Theorem 1.2 provides some nontrivial lower and upper bounds, as illustrated in Table 3 and Figure 1. One may compare our results with the Brownian motion case given in [19], as detailed in the next example:

Example 3.1 (Brownian motion case)**.** In the Brownian motion case, as seen in Table 3 (a), if $H = 1/2$, we have

$$T^{1/d} \lesssim R_T \lesssim T, \quad \text{for large } T.$$

[1] Here, we leverage the fact that for any $\beta > 0$, the expression $\beta^{\gamma_1} T^{\gamma_2} \log(T)^{\gamma_3}$ behaves like $T^{\gamma_2} \log(T)^{\gamma_3}$ for large T, where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary numbers, with the constraint that γ_2 and γ_3 cannot both be zero.

This partially confirms the guess in [4, Equation (1.30)] in dimension 2, but is still far away from the ultimate conjecture that $R_T \asymp T^{3/4}$; see Equation (1.28) (*ibid.*). To the best of our understanding, the only known related finding is presented in [20], where it is shown that for d-dimensional self-avoiding random walk with the radius $R_T \gtrsim T^{2/3d}$. It is noteworthy that a direct comparison between our outcome and that of [20] is not feasible, as the latter focused on self-avoiding random walk, not the self-repellent Brownian motion examined in this paper. Additionally, the distance in [20] pertains *end-to-end distance*, which contrasts with that employed in our context.

	$d=1$	$d=2$	$d=3$	$d=4$	$d\geq 5$
Lower bound	1	$1/2$	$1/3$	$1/4$	$1/d$
Conjectured (shaded)/confirmed	1	$3/4$	0.58759700(40)	$1/2$ with $\log^{1/8}$	$1/2$
Upper bound	1				

(A) The Brownian motion case, i.e., $H = 1/2$. The lower and upper bounds correspond to the exponent of T in $\underline{R}_T$ and $\overline{R}_T$, as defined in (1.9). The conjectured and confirmed values are taken from [5, Table 1].

H	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$1/4$	$5/6$	$[7/16, 13/16]$	$[13/42, 11/14]$	$(\frac{1}{4}-, \frac{3}{4}+)$	$[1/5, 3/4]$	$[1/6, 3/4]$
$1/3$	$8/9$	$[7/15, 13/15]$	$(\frac{1}{3}-, \frac{5}{6}+)$	$[1/4, 5/6]$	$[1/5, 5/6]$	$[1/6, 5/6]$
$1/2$	1	$[1/2, 1]$	$[1/3, 1]$	$[1/4, 1]$	$[1/5, 1]$	$[1/6, 1]$
$2/3$	$10/9$	$[5/9, 10/9]$	$[10/27, 10/9]$	$[5/18, 10/9]$	$[2/9, 10/9]$	$[5/27, 10/9]$
$3/4$	$7/6$	$[7/12, 7/6]$	$[7/18, 7/6]$	$[7/24, 7/6]$	$[7/30, 7/6]$	$[7/36, 7/6]$

(B) The ranges (when $d \geq 2$) and the exact values (when $d = 1$) for various values of H and d. A gray background indicates cases with $dH < 1$. Red highlights the scenario where $dH = 1$ with darker one for the case $d = 2$ and lighter one for the cases $d \geq 3$. Cyan represents cases where $dH > 1$ with darker one for the cases $H < 1/2$ and lighter one for the cases $H \geq 1/2$.

TABLE 3. Exponents of T in R_T, as define in (1.4), for large T.

Remark 3.2. The Hurst parameter H does not need to be the same for each coordinate. The same strategy presented in this paper can be readily applied to other cases. Let $H_1, \ldots, H_d$ denote the Hurst parameter of $B^{H,1}, \ldots, B^{H,d}$, respectively. Then, the results in Theorem 1.2 still hold, with parameters depending on $H_1 + \cdots + H_d$, $\max\{H_1, \ldots, H_d\}$, and $\min\{H_1, \ldots, H_d\}$. For the sake of conciseness, we refrain from delving into the specifics in this paper.

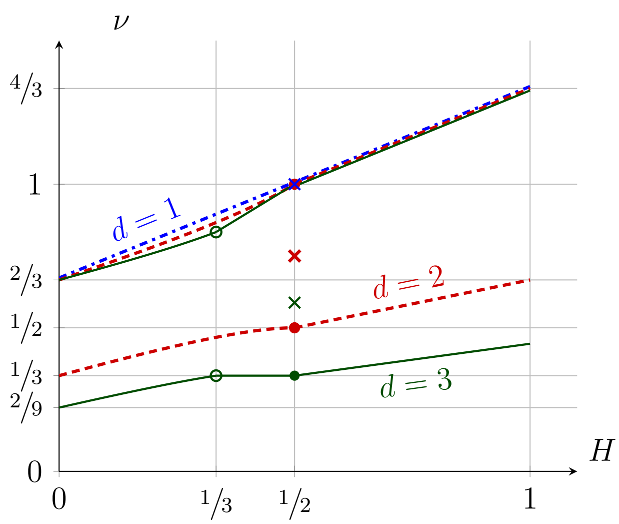

FIGURE 1. Plots of the exponent ν for $d = 1$ (the blue dash-dotted line), 2 (two red dashed lines, one for the upper bound, the other for the lower bound) and 3 (two green solid lines). The conjectured and confirmed values of μ when $H = 1/2$ are labeled by the cross mark ("x") for the cases $d = 1$ in blue (confirmed), $d = 2$ in red, and $d = 3$ in green. The two green circles for $d = 3$ at $H = 1/3$ refer to the case that exponent is subject to logarithm corrections.

Remark 3.3. Lemma 1.4 provides a sharp bound for the Brownian case $H = 1/2$. With $B = B^{1/2}$ denoting the d-dimensional Brownian motion, we can write

$$\begin{aligned}\int_{\mathbb{R}^d} L_T(y)^2 \mathrm{d}y &= \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2 \int_{\mathbb{R}^d} \mathrm{d}y\, \mathbf{1}_{\mathbf{B}_1(y)}(B_{t_1}) \mathbf{1}_{\mathbf{B}_1(y)}(B_{t_2}) \\ &= \int_0^T \mathrm{d}t_1 \int_0^T \mathrm{d}t_2\, \left|\mathbf{B}_1(B_{t_1}) \cap \mathbf{B}_1(B_{t_2})\right| \\ &\geq \sum_{k=0}^{\lfloor T \rfloor} \int_k^{k+1} \mathrm{d}t_1 \int_k^{k+1} \mathrm{d}t_2\, \left|\mathbf{B}_1(B_{t_1}) \cap \mathbf{B}_1(B_{t_2})\right|.\end{aligned}$$

Since $|\mathbf{B}_1(B_{t_1}) \cap \mathbf{B}_1(B_{t_2})|$ is a non-negative function of $B_{t_1} - B_{t_2}$, the summands in above expression are i.i.d. random variables. As a result, with $f(B_{t_1} - B_{t_2}) \coloneqq |\mathbf{B}_1(B_{t_1}) \cap \mathbf{B}_1(B_{t_2})|$,

$$\begin{aligned}Z_T =& \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\beta \int_{\mathbb{R}^d} \mathrm{d}y\, L_T(y)^2\right)\right] \\ \leq& \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\beta \sum_{k=0}^{\lfloor T \rfloor} \int_k^{k+1} \mathrm{d}t_1 \int_k^{k+1} \mathrm{d}t_2\, f(B_{t_1} - B_{t_2})\right)\right] \\ =& \prod_{k=0}^{\lfloor T \rfloor} \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\beta \int_k^{k+1} \mathrm{d}t_1 \int_k^{k+1} \mathrm{d}t_2\, f(B_{t_1} - B_{t_2})\right)\right] = e^{F_\beta \lfloor T \rfloor},\end{aligned}$$

where

$$F_\beta := \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\beta\int_0^1 \mathrm{d}t_1\int_0^1 \mathrm{d}t_2\, f(B_{t_1}-B_{t_2})\right)\right] < 0.$$

In other words, $Z_T \le \exp(-CT)$ with some $C>0$ for all $T>1$, and combining Lemma 1.4, we see that $\log Z_T \asymp -T$ as $T\to\infty$. Therefore, the lack of sharpness in Theorem 1.2 is likely attributed to the estimates in Lemma 1.3. We expect that this aspect can be resolved in future research.

Appendix A. Fractional Brownian motions and Girsanov theorem

In this section, we present some preliminaries about stochastic calculus for fBm's. For a more detailed account of this topic, we refer the interested readers to [21].

A d-dimensional stochastic process $B^H = \left\{\left(B_t^{H,1},\dots,B_t^{H,d}\right) : t\in\mathbb{R}_+\right\}$ is a called a *fraction Brownian motion* (*fBm*) with the Hurst parameters $H\in(0,1)$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, if

(i) $B^{H,i}$, $i=1,\dots,d$, are independent;
(ii) for each $1\le i\le d$, $\left\{B_t^{H,i} \colon t\in\mathbb{R}_+\right\}$ is a centered Gaussian family with covariance

$$\mathbb{E}\left[B_t^{H,i}B_s^{H,i}\right] = \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

Without loss of generality, we can assume that the filtration $\{\mathcal{F}_t \colon t\in\mathbb{R}_+\}$ is the canonical filtration generated by B^H.

Next, we define the integration of deterministic functions against fBm's. If ϕ is a smooth function on $\mathbb{R}_+$ with compact support, i.e., $\phi\in C_c^\infty(\mathbb{R}_+)$, then the integrals

$$B^{H,i}(\phi) := \int_0^\infty \phi(t)\mathrm{d}B_t^{H,i}, \qquad i=1,\dots,d,$$

are centered Gaussian random variables with the following covariance structure:

$$\mathbb{E}\left[B^{H,i}(\phi)B^{H,j}(\psi)\right] = \begin{cases} 0, & i\ne j,\\ H(2H-1)\displaystyle\iint_{\mathbb{R}_+^2}\mathrm{d}t\,\mathrm{d}s\,\phi(s)\psi(t)|t-s|^{2H-2}, & i=j,\end{cases}$$

for all $1\le i,j\le d$, and $\phi,\psi\in C_c^\infty(\mathbb{R}_+)$.

By typical approximation arguments, one can extend the integration to the Hilbert space $\mathcal{H}$ of functions on $\mathbb{R}_+$, with inner product,

$$\langle\phi,\psi\rangle_{\mathcal{H}} := \iint_{\mathbb{R}_+^2}\mathrm{d}t\,\mathrm{d}s\,\phi(s)\psi(t)|t-s|^{2H-2}.$$

In particular, for any $t\in\mathbb{R}_+$, the function $w(t,\cdot)$, given by

$$w(t,s) := c_1\, s^{1/2-H}(t-s)^{1/2-H}\mathbf{1}_{(0,t)}(s), \quad \text{for all } s\in\mathbb{R},$$

is an element of $\mathcal{H}$ (see [22, Proposition 2.1]), where, with $B(\cdot,\cdot)$ denoting the *Beta function*,

$$c_1 := \left[2H\times B\left(3/2-H, 1/2+H\right)\right]^{-1}.$$

This observation allows us to define the following Gaussian process $M = M^{\mathbf{u}} = \{M_t : t \in \mathbb{R}_+\}$ with parameter $\mathbf{u} = (u_1, \dots, u_d)$ being a unit vector in $\mathbb{R}^d$ as follows:

$$M_t \coloneqq \sum_{i=1}^d u_i \int_0^t w(t,s) \mathrm{d}B_s^{H,i}, \quad \text{for all } t \in \mathbb{R}_+. \tag{A.1}$$

The following theorem is a straightforward extension of [22, Theorem 3.1] from $d = 1$ to higher dimensional cases, thanks to the independence of the components of B^H:

Theorem A.1. *Let B^H be a d-dimensional fBm with $H \in (0,1)$, let $\mathbf{u}$ be a unit vector in $\mathbb{R}^d$, and let $M = M^{\mathbf{u}}$ be the Gaussian process given as in (A.1). Then M is a square-integrable martingale with quadratic variation*

$$\langle M, M \rangle_t = C_H t^{2(1-H)}, \ \forall t \in \mathbb{R}_+, \quad \textit{where} \quad C_H \coloneqq \frac{\Gamma(3/2 - H)}{4H(1-H)\Gamma(1/2 + H)\Gamma(2-2H)}.$$

For any $\lambda > 0$ and $T > 0$, denote

$$\mathcal{Q}_T(M) \coloneqq \exp\left(M_T - \frac{1}{2}\langle M, M \rangle_T \right),$$

and let $\mathbb{P}_T^\lambda = \mathbb{P}_T^{\lambda,\mathbf{u}}$ be a probability measure on $(\Omega, \mathcal{F}_T)$ that is equivalent to $\mathbb{P}_T$ with the Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_T^\lambda}{\mathrm{d}\mathbb{P}_T} \coloneqq \mathcal{Q}_T(\lambda M). \tag{A.2}$$

The next theorem, a *Girsanov formula for fBm's*, is a straightforward extension of [22, Theorem 4.1].

Theorem A.2. *Under probability $\mathbb{P}_T^\lambda$, the process $\left\{B_t^H : 0 \le t \le T\right\}$ is a d-dimensional fBm with a drift $\lambda\mathbf{u} \in \mathbb{R}^d$, i.e., the distribution of the process B^H up to time T under $\mathbb{P}_T^\lambda = \mathbb{P}_T^{\lambda,\mathbf{u}}$ is the same as $B^{H,\lambda,\mathbf{u},T} = \left\{B_t^H + \lambda t\,\mathbf{u} : 0 \le t \le T\right\}$ under $\mathbb{P}_T$.*

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References

[1] Domb, C., Joyce, G.S.: Cluster expansion for a polymer chain. J. Phys. C: Solid State Phys. **5**(9), 956 (1972) https://doi.org/10.1088/0022-3719/5/9/009

[2] Edwards, S.F.: The statistical mechanics of polymers with excluded volume. Proc. Phys. Soc. **85**, 613–624 (1965) https://doi.org/10.1007/s001090000086

[3] Madras, N., Slade, G.: The Self-avoiding Walk. Probability and its Applications, p. 425. Birkhäuser Boston, Inc., Boston (1993)

[4] Duminil-Copin, H., Goodman, J., Slade, G.: Lectures on self-avoiding walks. In: Probability and Statistical Physics in Two and More Dimensions. Clay Math. Proc., vol. 15, pp. 395–467. Amer. Math. Soc., Providence (2012)

[5] Slade, G.: Self-avoiding walk, spin systems and renormalization. Proc. A. **475**(2221), 20180549–21 (2019) https://doi.org/10.1098/rspa.2018.0549

[6] Biswas, P., Cherayil, B.J.: Dynamics of fractional brownian walks. J. Phys. Chem. **99**(2), 816–821 (1995) https://doi.org/10.1021/j100002a052

[7] Grothaus, M., Oliveira, M.J., Silva, J.L., Streit, L.: Self-avoiding fractional Brownian motion—the Edwards model. J. Stat. Phys. **145**(6), 1513–1523 (2011) https://doi.org/10.1007/s10955-011-0344-2

[8] Bornales, J., Oliveira, M.J., Streit, L.: Self-repelling fractional Brownian motion—a generalized Edwards model for chain polymers. In: Quantum Bio-informatics V. QP–PQ: Quantum Probab. White Noise Anal., vol. 30, pp. 389–401. World Sci. Publ., Hackensack (2013)

[9] Bock, W., Bornales, J.B., Cabahug, C.O., Eleutério, S., Streit, L.: Scaling properties of weakly self-avoiding fractional Brownian motion in one dimension. J. Stat. Phys. **161**(5), 1155–1162 (2015) https://doi.org/10.1007/s10955-015-1368-9

[10] Rosen, J.: The intersection local time of fractional Brownian motion in the plane. J. Multivariate Anal. **23**(1), 37–46 (1987) https://doi.org/10.1016/0047-259X(87)90176-X

[11] Hu, Y., Nualart, D.: Renormalized self-intersection local time for fractional Brownian motion. Ann. Probab. **33**(3), 948–983 (2005) https://doi.org/10.1214/009117905000000017

[12] Mueller, C., E., N.: The radius of a self-repelling star polymer. preprint arXiv:2306.01537 (2023)

[13] Fixman, M.: Radius of gyration of polymer chains. J. Chem. Phys. **36**(2), 306–310 (1962) https://doi.org/10.1063/1.1732501

[14] Bolthausen, E.: On self-repellent one-dimensional random walks. Probab. Theory Related Fields **86**(4), 423–441 (1990) https://doi.org/10.1007/BF01198167

[15] Greven, A., Hollander, F.: A variational characterization of the speed of a one-dimensional self-repellent random walk. Ann. Appl. Probab. **3**(4), 1067–1099 (1993) https://doi.org/10.1214/aoap/1177005273

[16] Hara, T., Slade, G.: Self-avoiding walk in five or more dimensions. I. The critical behaviour. Comm. Math. Phys. **147**(1), 101–136 (1992) https://doi.org/10.1007/BF02099530

[17] Brydges, D., Spencer, T.: Self-avoiding walk in 5 or more dimensions. Comm. Math. Phys. **97**(1-2), 125–148 (1985) https://doi.org/10.1007/BF01206182

[18] Adler, R.J., Taylor, J.E.: Random Fields and Geometry. Springer Monographs in Mathematics, p. 448. Springer, New York (2007)

[19] Hara, T., Slade, G.: Critical behaviour of self-avoiding walk in five or more dimensions. Bull. Amer. Math. Soc. (N.S.) **25**(2), 417–423 (1991) https://doi.org/10.1090/S0273-0979-1991-16085-4

[20] Madras, N.: A lower bound for the end-to-end distance of the self-avoiding walk. Canad. Math. Bull. **57**(1), 113–118 (2014) https://doi.org/10.4153/CMB-2012-022-6

[21] Biagini, F., Hu, Y., Øksendal, B., Zhang, T.: Stochastic Calculus for Fractional Brownian Motion and Applications. Probability and its Applications (New York), p. 329. Springer, London (2008)

[22] Norros, I., Valkeila, E., Virtamo, J.: An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli **5**(4), 571–587 (1999) https://doi.org/10.2307/3318691

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