Phase Analysis for a family of Stochastic Reaction-Diffusion Equations*

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Abstract

We consider a reaction-diffusion equation of the type

$$\partial_t \psi = \partial_x^2 \psi + V(\psi) + \lambda\sigma(\psi)\dot{W} \qquad \text{on } (0\,,\infty)\times\mathbb{T},$$

subject to a "nice" initial value and periodic boundary, where $\mathbb{T} = [-1\,,1]$ and $\dot{W}$ denotes space-time white noise. The reaction term $V:\mathbb{R}\to\mathbb{R}$ belongs to a large family of functions that includes Fisher–KPP nonlinearities $[V(x) = x(1-x)]$ as well as Allen-Cahn potentials $[V(x) = x(1-x)(1+x)]$, the multiplicative nonlinearity $\sigma:\mathbb{R}\to\mathbb{R}$ is non random and Lipschitz continuous, and $\lambda > 0$ is a non-random number that measures the strength of the effect of the noise $\dot{W}$.

The principal finding of this paper is that: (i) When λ is sufficiently large, the above equation has a unique invariant measure; and (ii) When λ is sufficiently small, the collection of all invariant measures is a non-trivial line segment, in particular infinite. This proves an earlier prediction of Zimmerman et al. (2000). Our methods also say a great deal about the structure of these invariant measures.

Keywords: Stochastic partial differential equations; invariant measures; phase transition.

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Contents

1 Introduction

Let $\dot W = \{\dot W(t\,,x)\}_{t\geqslant 0, x\in[-1,1]}$ denote a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}$ where

$$\mathbb{T} := [-1\,,1].$$

That is, $\dot W$ is a centered, generalized Gaussian random field with covariance measure

$$\mathrm{Cov}[\dot W(t\,,x)\,,\dot W(s\,,y)] = \delta_0(t-s)\delta_0(x-y) \text{ for all } s,t \geqslant 0 \text{ and } x,y \in \mathbb{T}.$$

The principal goal of this paper is to study the large-time behavior of the stochastic reaction-diffusion equation,

$$\partial_t \psi(t\,,x) = \partial_x^2 \psi(t\,,x) + V(\psi(t\,,x)) + \lambda\sigma(\psi(t\,,x))\dot{W}(t\,,x), \tag{1.1}$$

for $(t\,,x) \in (0\,,\infty)\times\mathbb{T}$, with periodic boundary conditions $\psi(t\,,-1) = \psi(t\,,1)$ for all $t>0$, and a nice initial profile $\psi_0 : \mathbb{T}\to[0\,,\infty)$. The functions σ and V are non-random, and fairly regular, and $\lambda > 0$ is a non-random quantity that represents the strength of the noise $\dot{W}$.

We will discuss the technical details about σ, ψ_0, V, ... in the next section. For now, we mention only that permissible choices of V include Fisher–KPP type non linearities $[V(z) = z(1-z)]$ as well as Allen-Cahn type potentials $[V(z) = z(1-z)(1+z)]$.

The main findings of this paper can be summarized as follows.

Informal Theorem. *Under nice regularity conditions on σ, V, and ψ_0:*

1. (1.1) *is well posed;*
2. (1.1) *has an invariant probability measure μ_0;*
3. [High-noise regime] *There exists a non-random number $\lambda_1 > 0$ such that the only invariant probability measure of* (1.1) *is μ_0 when $\lambda > \lambda_1$; and*
4. [Low-noise regime] *There exists a non-random number $\lambda_0 > 0$ such that if $\lambda < \lambda_0$, then* (1.1) *has infinitely-many invariant measures. Moreover, there exists a probability measure μ_1 – singular with respect to μ_0 – such that the line segment*

$$\mathscr{I\!M} := \{(1-a)\mu_0 + a\mu_1 : 0 \leqslant a \leqslant 1\} \tag{1.2}$$

coincides with all invariant probability measures of (1.1).

In their rigorous forms – see Theorem 2.4 below – assertions 3 and 4 together verify a corrected version of earlier predictions of Zimmerman et al. (2000), deduced originally via experiments and computer simulations. Equally significantly, our proof of the rigorous form of Item 4 clearly "explains" why there is phase transition in the low-noise regime.

We conclude the Introduction by setting forth some notation that will be used throughout the paper.

We will often write $\psi(t)$ in place of the mapping $x \mapsto \psi(t\,,x)$; we do likewise for other functions that may depend on extra parameters.

We always write $\mathbf{1}_G$ for the indicator function of any and every set G. That is, $\mathbf{1}_G(z)$ is equal to one if $z \in G$ and $\mathbf{1}_G(z) = 0$ otherwise. When G is an event in our underlying probability space $(\Omega\,,\mathcal{F},\mathrm{P})$, we follow common practice and omit the variable $z \in \Omega$ in the expression $\mathbf{1}_G(z)$.

For any Banach space $\mathbb{X}$, we let $C(\mathbb{X})$ denote the space of all continuous functions $f : \mathbb{X} \to \mathbb{R}$. The space $C(\mathbb{X})$ is always a Banach space endowed with the supremum norm,

$$\|f\|_{C(\mathbb{X})} := \sup_{x\in\mathbb{X}} |f(x)|.$$

We always write $C_+(\mathbb{X})$ for the cone of non-negative elements of $C(\mathbb{X})$ and $C_{>0}(\mathbb{X})$ for the cone of strictly positive elements of $C(\mathbb{X})$, i.e., $f(x) > 0$ for all $x \in \mathbb{T}$; $C_b(\mathbb{X})$ denotes the bounded elements of $C(\mathbb{X})$. The space $C_b(\mathbb{X})$ is always endowed with the topology of pointwise convergence. That is, we endow $C_b(\mathbb{X}) \subset \mathbb{R}^{\mathbb{X}}$ with relative topology, where $\mathbb{R}^{\mathbb{X}}$ is given the discrete topology. By $C(\mathbb{X}\,;\mathbb{Y})$ we mean the space of all continuous function on the Banach space $\mathbb{X}$ that take value in the Banach space $\mathbb{Y}$. We always topologize $C(\mathbb{X}\,;\mathbb{Y})$ using the norm defined by

$$\|f\|_{C(\mathbb{X};\mathbb{Y})} := \sup_{x\in\mathbb{X}} \|f(x)\|_{\mathbb{Y}}.$$

In particular, $C(\mathbb{X}\,;\mathbb{R}) = C(\mathbb{X})$.

We topologize $\mathbb{T} = [-1\,,1]$ so that it is the one-dimensional torus. That is, $\mathbb{T}$ is always endowed with addition mod 2, and ± 1 are identified with one another using the usual quotient topology on $\mathbb{T}$. Let us emphasize in particular that

$$\lim_{x\to\pm 1} f(x) = f(1) = f(-1) \qquad \text{for every } f \in C(\mathbb{T}).$$

Haar measure on $\mathbb{T}$ is always normalized to have total mass 2, and its infinitesimal is denoted by symbols such as $\mathrm{d}x, \mathrm{d}y, \ldots$ in the context of Lebesgue integration.

For every $\alpha \in (0\,,1)$, let $C^\alpha(\mathbb{T})$ denote the collection of all $f \in C(\mathbb{T})$ that satisfy $\|f\|_{C^\alpha(\mathbb{T})} < \infty$, where[1]

$$\|f\|_{C^\alpha(\mathbb{T})} := \|f\|_{C(\mathbb{T})} + \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \frac{|f(y)-f(x)|}{|y-x|^\alpha}.$$

Thus, $f \in C^\alpha(\mathbb{T})$ iff f is Hölder continuous with index α. In keeping with previous notation, $C^\alpha_+(\mathbb{T})$ denotes the cone of all non-negative elements of $C^\alpha(\mathbb{T})$.

The $L^k(\Omega)$-norm of a random variable $Z \in L^k(\Omega)$ is denoted by $\|Z\|_k := \{\mathrm{E}(|Z|^k)\}^{1/k}$ for all $1 \leqslant k < \infty$.

We let $\mathscr{F} := \{\mathscr{F}_t\}_{t\geqslant 0}$ designate the "Brownian filtration." That is, for all $t \geqslant 0$, $\mathscr{F}_t$ denotes the sub σ-algebra of $\mathcal{F}$ that is generated by all Wiener integrals of the form

$$\int_{(0,t)\times\mathbb{T}} f(x)\, W(\mathrm{d}s\,\mathrm{d}x),$$

as f ranges over $L^2(\mathbb{T})$. By augmenting $\mathscr{F}$ if need be, we may – and always will – assume that the filtration $\mathscr{F}$ satisfies the "usual conditions" of stochastic integration theory. That is,

$$\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s,$$

and $\mathscr{F}_t$ is complete for all $t \geqslant 0$. We will require the "usual conditions" because, as is well known (and not hard to prove), they ensure that the first hitting time of a closed set by a continuous, $\mathscr{F}$-adapted stochastic process is measurable.[2]

Finally, let us introduce two special elements $\mathbb{0}, \mathbb{1} \in C_+(\mathbb{T})$ as follows:

$$\mathbb{0}(x) := 0 \quad \text{and} \quad \mathbb{1}(x) := 1 \qquad \text{for all } x \in \mathbb{T}. \tag{1.3}$$

Thus, in particular, $\delta_{\mathbb{0}}$ and $\delta_{\mathbb{1}}$ denote the probability measures on $C_+(\mathbb{T})$ that put respective point masses on the constant functions $\mathbb{0}$ and $\mathbb{1}$. The measures δ_0 and $\delta_{\mathbb{0}}$ should not be mistaken for one another; the former is a probability measure on $\mathbb{R}$ and the latter is a probability measure on $C_+(\mathbb{T})$. Similar remarks apply to δ_1 and $\delta_{\mathbb{1}}$.

Throughout we assume that the underlying probability space is complete.

[1] Some authors use instead the norm defined by

$$|f|_{C^\alpha(\mathbb{T})} := |f(0)| + \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \frac{|f(y)-f(x)|}{|y-x|^\alpha}.$$

The two norms are equivalent, since $|f|_{C^\alpha(\mathbb{T})} \leqslant \|f\|_{C^\alpha(\mathbb{T})} \leqslant 2^\alpha |f|_{C^\alpha(\mathbb{T})}$.

[2] A much deeper theorem of Hunt (1957) extends this to cover the first hitting time of any Borel, and even analytic, set. We will not need that extension in the sequel.

2 The Main Results

In this section we introduce two theorems that make rigorous the Informal Theorem of §1. First, let us observe that the SPDE (1.1) is stated in terms of three functions σ (the "diffusion coefficient"), V (the "potential"), and ψ_0 (the "initial profile") which have not yet been described. Thus, we begin with a precise description of those functions.

2.1 Hypotheses on the Diffusion Coefficient

Throughout this paper, we choose and fix a globally Lipschitz continuous function $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(0) = 0.$$

Let us define

$$\mathrm{L}_\sigma := \inf_{a\in\mathbb{R}\setminus\{0\}} \left|\frac{\sigma(a)}{a}\right|, \qquad \mathrm{Lip}_\sigma := \sup_{\substack{a,b\in\mathbb{R}:\\ a\neq b}} \left|\frac{\sigma(b)-\sigma(a)}{b-a}\right|. \tag{2.1}$$

Evidently, $0 \leqslant \mathrm{L}_\sigma \leqslant \mathrm{Lip}_\sigma$, and

$$\mathrm{L}_\sigma|a| \leqslant |\sigma(a)| \leqslant \mathrm{Lip}_\sigma|a| \qquad \text{for all } a \in \mathbb{R}.$$

Because σ is Lipschitz continuous, it follows that $\mathrm{Lip}_\sigma < \infty$ and hence the second inequality above has content. We frequently assume that the first inequality does too. That is, we often suppose in addition that $\mathrm{L}_\sigma > 0$. We will make explicit mention whenever this assumption is in place.

2.2 Hypotheses on the Potential

Throughout the paper we are concerned with potentials V of the form,

$$V(x) = x - F(x) \qquad \text{for all } x \in \mathbb{R},$$

where $F : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy the following conditions:

(F1) $F \in C^2(\mathbb{R}_+)$, $F(0) = 0$, $F' \geqslant 0$;
(F2) $\limsup_{x\downarrow 0} F'(x) < 1$ and $\lim_{x\to\infty} F'(x) = \infty$; and
(F3) There exists a real number $m_0 > 1$ such that $F(x) = O(x^{m_0})$ as $x \to \infty$.

Examples are Allen-Cahn type potentials [$F(x) = x^3$], as well as Fisher-KPP type nonlinearities [$F(x) = x^2$].

We make references to $V(x)$ and $x - F(x)$ interchangeably throughout. We also make references to the following elementary properties of the function F without explicit mention.

Lemma 2.1. *V and F are locally Lipschitz and satisfy the following technical conditions:*

1. $\limsup_{x\downarrow 0}(F(x)/x) < 1$;
2. $\lim_{x\uparrow\infty}(F(x)/x) = \infty$;
3. $\sup_{x\geqslant 0} V(x) < \infty$; *and*
4. $\lim_{N\to\infty} \inf_{N\leqslant x<y\leqslant N+1}\{F(y) - F(x)\}/(y - x) = \infty$.

Proof. V and F are locally Lipschitz because F' is continuous. Thanks to **(F2)**, we can find $r_0 > 0$ such that $\sup_{(0,r_0)} F' < 1$. Apply the fundamental theorem of calculus to deduce part 1 from **(F1)**, and that

$$\frac{F(x)}{x} = \frac{1}{x}\int_0^x F'(a)\,\mathrm{d}a \geqslant \frac{1}{x}\int_{x/2}^x F'(a)\,\mathrm{d}a \geqslant \frac{1}{2}\inf_{a\geqslant x/2} F'(a) \qquad \text{for all } x > 0.$$

Let $x \to \infty$ and appeal to **(F2)** to arrive at part 2. Part 3 follows immediately from part 2 and the continuity, hence local boundedness, of the function V. Finally, we may observe that whenever $N \leqslant x < y \leqslant N+1$,

$$\frac{F(y)-F(x)}{y-x} \geqslant \inf_{x\leqslant z\leqslant y} F'(z) \geqslant \inf_{N\leqslant z\leqslant N+1} F'(z) \to \infty \qquad \text{as } N \to \infty,$$

thanks to mean value theorem and **(F2)**. This concludes our demonstration. □

Remark 2.2. The astute reader might have noticed that Lemma 2.1 requires only that F satisfies **(F1)** and **(F2)**. Condition **(F3)** will be used later on in Lemma 4.5 in order to establish quantitative, global-in-time, spatial continuity bounds for the solution to our SPDE (1.1).

2.3 Hypothesis on the initial profile

Throughout, we assume that

$$\psi_0 \in C_+(\mathbb{T}),$$

and ψ_0 is non random. This assumption is used everywhere in the paper and so we assume it here and throughout without explicit mention. We frequently will assume additionally that $\psi_0 \neq \mathbb{0}$ ($\mathbb{0}$ was define in (1.3)); equivalently, that $\psi_0 > 0$ on an open ball in $\mathbb{T}$. This assumption will be made explicitly every time it is needed.

2.4 The Main Results

We do not expect the solution ψ to the SPDE (1.1) to be differentiable in either variable. Therefore, (1.1) must be interpreted in the generalized sense; see Walsh (1986). From now on, we regard the SPDE (1.1) as shorthand for its mild – or integral – formulation which can be written as follows:

$$\begin{aligned}\psi(t\,,x) = (\mathcal{P}_t\psi_0)(x) + \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)V(\psi(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \\ + \lambda\int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y);\end{aligned} \tag{2.2}$$

where the function p denotes the heat kernel for the operator $\partial_t - \partial_x^2$ on $(0\,,\infty)\times\mathbb{T}$ with periodic boundary conditions, and $\{\mathcal{P}_t\}_{t\geqslant 0}$ denotes the associated heat semigroup. That is,

$$p_t(x\,,y) = \frac{1}{\sqrt{4\pi t}}\sum_{k=-\infty}^{\infty}\exp\left\{-\frac{(x-y+2k)^2}{4t}\right\} \quad \text{for all } t>0 \text{ and } x,y\in\mathbb{T}, \tag{2.3}$$

and

$$\mathcal{P}_0 f = f \quad \text{and} \quad (\mathcal{P}_t f)(x) = \int_{\mathbb{T}} p_t(x\,,y)f(y)\,\mathrm{d}y,$$

for all $(t\,,x)\in(0\,,\infty)\times\mathbb{T}$ and for every $f\in C(\mathbb{T})$ (say).

The first of our two main theorems is a standard existence and uniqueness theorem.

Theorem 2.3. *The random integral equation* (2.2) *has a predictable random-field solution* $\psi = \{\psi(t\,,x)\}_{t\in\mathbb{R}_+,x\in\mathbb{T}}$ *that is unique among all solutions that are a.s. in* $C_+(\mathbb{R}_+\times\mathbb{T})$*. Moreover,* $\psi(t)\in C^\alpha_+(\mathbb{T})$ *a.s. for every* $t>0$ *and* $\alpha\in(0\,,1/2)$*. If in addition* $\psi_0\neq\mathbb{0}$*, then* $\psi(t\,,x)>0$ *for all* $(t\,,x)\in(0\,,\infty)\times\mathbb{T}$ *a.s.*

The second result of this paper describes the invariant measure(s) of the solution to (1.1). This is a meaningful undertaking since, as we shall see in Proposition 4.13 below, the infinite-dimensional stochastic process $\{\psi(t)\}_{t\geqslant0}$ is a Feller process.

Recall the function $\mathbb{0}$ from (1.3), and recall also that $\delta_{\mathbb{0}}$ denotes point mass on $\mathbb{0}$. Because $\sigma(0)=0$, it is easy to see that if we replaced the initial profile ψ_0 with the initial profile $\mathbb{0}$, then the solution ψ would be identically zero. This is basically another way to say that $\delta_{\mathbb{0}}$ is always an invariant measure for (1.1). The next theorem explores the question of uniqueness for this invariant measure ["phase transition" refers to the lack of uniqueness of an invariant measure in this context].

Theorem 2.4. *If* $\mathrm{L}_\sigma>0$*, then there exist* $\lambda_1>\lambda_0>0$ *such that the following are valid independently of the choice of the initial profile* $\psi_0\in C_+(\mathbb{T})\setminus\{\mathbb{0}\}$*:*

1. *If* $\lambda\in(0\,,\lambda_0)$*, then:*
   (a) *There exists a unique probability measure* μ_+ *on* $C_+(\mathbb{T})$ *that is invariant for* (1.1) *and* $\mu_+\{\mathbb{0}\}=0$*. Moreover,* μ_+ *charges* $C_{>0}(\mathbb{T})$*;*
   (b) (Ergodic decomposition). *The set of all probability measures on* $C_+(\mathbb{T})$ *that are invariant for* (1.1) *is the collection* $\mathscr{IM}$ *[see* (1.2)*] of all convex combinations of* $\mu_1:=\mu_+$ *and* $\mu_0:=\delta_{\mathbb{0}}$*;*
   (c) *For every* $\alpha\in(0\,,1/2)$*,* μ_+ *is a probability measure on* $C^\alpha_+(\mathbb{T})$ *and*
$$\int\|\omega\|^k_{C^\alpha(\mathbb{T})}\,\mu_+(\mathrm{d}\omega)<\infty\qquad\textit{for every real number } k\geqslant2. \tag{2.4}$$
   (d) (Ergodic theorem). $\mu_+(\bullet)=\lim_{T\to\infty}T^{-1}\int_0^T\mathrm{P}\{\psi(t)\in\bullet\}\,\mathrm{d}t$*, where convergence holds in total variation; and*
2. *If* $\lambda>\lambda_1$*, then* $\delta_{\mathbb{0}}$ *is the only invariant measure for* (1.1)*, and almost surely,* $\lim_{t\to\infty}\psi(t)=\mathbb{0}$ *in* $C(\mathbb{T})$*. In fact,* $\limsup_{t\to\infty}t^{-1}\log\|\psi(t)\|_{C(\mathbb{T})}<0$ *a.s.*

Remark 2.5. As a by-product of our *a priori* estimates, we shall prove that (2.4) can sometimes be improved upon. For instance, if there exists $\nu>0$ such that $F(x)=x^{1+\nu}$ for all $x\geqslant0$, then it follows from Example 4.7 below [see also Example 4.4] that for every $\alpha\in(0\,,1/2)$ there exists a real number $q>0$ such that
$$\int\exp\left(q\|\omega\|^{\nu/2(1+\nu)}_{C^\alpha(\mathbb{T})}\right)\mu_+(\mathrm{d}\omega)<\infty.$$
Among others, this remark applies to the Fisher-KPP ($\nu=1$) and the Allen-Cahn ($\nu=2$) SPDEs.

Some of the content of Theorem 2.4 was predicted earlier by Zimmerman et al. (2000). One might ask the following open question.

Open Problem. Does there exist a unique number $\lambda_c\in[\lambda_0\,,\lambda_1]$ such that (1.1) has a unique invariant measure when $\lambda>\lambda_c$ and infinitely-many invariant measures when $0<\lambda<\lambda_c$?

A standard method for proving that critical exponents exist in interacting particle systems is to establish a suitable monotonicity property and use comparison arguments. Since ψ is not monotone in λ, comparison arguments are likely to not help with this problem. In this context, we hasten to add that we have also found no mention of this problem in the work of Zimmerman et al. (2000).

Part (c) of Theorem 2.4 and Remark 2.5 contain information about the non-trivial extremum μ_+ of the collection $\mathscr{I\!M}$ of all invariant measures of (1.1) when λ is small. More detailed information is presented in §10. For example, we prove that $\mu_+(C^{1/2}(\mathbb{T})) = 0$ under a mild additional constraint (10.1) on the nonlinearity F. This complements part (c) of Theorem 2.4, as the latter implies that $\mu_+(C^\alpha(\mathbb{T})) = 1$ for every $\alpha \in (0\,,1/2)$. It is also proved in §10 that the typical function in the support of μ_+ doubles the Hausdorff dimension of every non-random set in the same way as Brownian motion (McKean, 1955).

Let us conclude this section by mentioning that we will prove Theorem 2.3 in Section 3. Theorem 2.4 and Remark 2.5 are proved in Section 9, and the intervening five sections are devoted to developing the requisite technical results that support the argument of Section 9. A few additional technical results are included in the appendices.

3 Proof of Theorem 2.3

One of the first goals of this section is to establish the well-posedness of the SPDE (1.1) subject to periodic boundary, and non-random initial value $\psi(0) = \psi_0 \in C_+(\mathbb{T})$.

For the duration of the proof we will choose and fix a number $\alpha \in (0\,,1/2)$, and assume that

$$\lambda = 1.$$

This assumption can be made without incurring any loss in generality. For otherwise we can replace σ by $\bar\sigma := \lambda\sigma$ and observe that $\bar\sigma$ too is Lipschitz continuous and vanishes at the origin.

Standard well-posedness theory for SPDE requires Lipschitz continuous coefficients, but the diffusion term V is not Lipschitz. Thus, we will truncate V and take the limit as we remove the truncation. To that end, let us define for all integers $N \geqslant 1$ a function $V_N : \mathbb{R} \to \mathbb{R}$ as follows:

$$V_N(w) := \begin{cases} 0 & \text{if } w \leqslant 0,\\ V(w) & \text{if } 0 < w < N,\\ V(N) & \text{if } w \geqslant N.\end{cases} \tag{3.1}$$

Every V_N is a bounded and Lipschitz continuous function. Standard results (see Walsh, 1986, Chapter 3) ensure that for each integer $N \geqslant 1$ there exists a unique continuous random field ψ_N that solves the SPDE,

$$\partial_t \psi_N(t\,,x) = \partial_x^2 \psi_N(t\,,x) + V_N\left(\psi_N(t\,,x)\right) + \sigma\left(\psi_N(t\,,x)\right)\dot W(t\,,x), \tag{3.2}$$

for $(t\,,x) \in (0\,,\infty) \times \mathbb{T}$, and subject to periodic boundary conditions with $\psi_{N,0}(x) = \psi_N(0\,,x) = \psi_0(x)$ for all $x \in \mathbb{T}$. To be precise, ψ_N is the unique solution to the following mild formulation of (3.2) [compare with (2.2)]:

$$\begin{aligned}\psi_N(t\,,x) = (\mathcal{P}_t\psi_0)(x) + \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y) V_N(\psi_N(s\,,y))\,\mathrm{d}s\,\mathrm{d}y&\\ + \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y).&\end{aligned} \tag{3.3}$$

where, as in (2.2) and (2.3), p denotes the heat kernel and $\{\mathcal{P}_t\}_{t\geqslant 0}$ the corresponding heat semigroup. We pause to mention a technical aside: The standard literature on SPDE does not treat $\mathbb{T}$ as the torus, rather as an interval in $\mathbb{R}$. Therefore, "continuity" in that setting does not quite mean what it does in the present setting. For this reason, we complete the above picture by claiming, and subsequently proving, that $(t\,,x) \mapsto \psi_N(t\,,x)$ is indeed a.s. continuous in the present sense for every $N \geqslant 1$. Because of the Kolmogorov continuity theorem, this claim follows as soon as we prove that $\psi_N(t\,,1) = \psi_N(t\,,-1)$ a.s. for every $t > 0$ and $N \in \mathbb{N}$. But this follows immediately from (3.3) and the elementary fact that $p_t(x\,,y) = p_t(x+2\,,y)$ for every $x, y \in \mathbb{T}$, where we recall addition in $\mathbb{T}$ is always addition mod 2. We return to the bulk of the argument.

Because $V_N(0) = 0$, $\sigma(0) = 0$, and $\psi_0 \geqslant 0$, the comparison theorem for SPDEs (Lemma 3.3) tells us that with probability one,

$$\psi_N(t\,,x) \geqslant 0 \qquad \text{for all } t \geqslant 0,\ x \in \mathbb{T}, \text{ and } N \geqslant 1. \tag{3.4}$$

Next, let us observe that the V_N's have the following consistency property: $V_N(x) = V_{N+1}(x)$ for all $x \in [0\,,N]$. Furthermore, Part 4 of Lemma 2.1 ensures that there exists a non-random integer $N_0 > 1$ such that

$$V_N(x) \geqslant V_{N+1}(x) \quad \text{for all } x \in \mathbb{R} \text{ and all } N \geqslant N_0.$$

Thus, a second appeal to the comparison theorem for SPDEs (Lemma 3.3) yields the following a.s bound:

$$\psi_N(t\,,x) \geqslant \psi_{N+1}(t\,,x) \quad \text{for all } t \geqslant 0,\ x \in \mathbb{T}, \text{ and all } N \geqslant N_0.$$

In particular,

$$\psi(t\,,x) := \lim_{N\to\infty} \psi_N(t\,,x)$$

exists for all $t \geqslant 0$ and $x \in \mathbb{T}$ off a single P-null set. Since the infinite-dimensional process $t \mapsto \psi_N(t)$ is progressively measurable with respect to the underlying Brownian filtration $\mathscr{F} := \{\mathscr{F}_t\}_{t\geqslant 0}$, so is the random process $t \mapsto \psi(t)$.

Define

$$T_N := \inf\left\{t \geqslant 0 : \|\psi_N(t)\|_{C(\mathbb{T})} \geqslant N\right\} \qquad \text{for all } N \in \mathbb{N},$$

where $\inf\varnothing := \infty$. Since $\{f \in C(\mathbb{T}) : \|f\|_{C(\mathbb{T})} \geqslant N\}$ is closed for every $N \in \mathbb{N}$, and because $\mathscr{F}$ satisfies the "usuall conditions" of stochastic integration theory, every T_N is an $\mathscr{F}$-stopping time. The uniqueness assertion for each ψ_N, and elementary properties of the Walsh stochastic integral (Walsh, 1986), together imply that

$$\psi_N(t) = \psi_{N+1}(t) \qquad \text{for all } t \in [0\,,T_N),$$

almost surely for all large N. We can iterate this to see that $\psi_N(t) = \psi_{N+M}(t)$ for all $t \in [0\,,T_N)$ a.s. for all large N and $M \geqslant 1$. Let $M \to \infty$ to find that

$$\psi(t) = \psi_N(t) \qquad \text{for all } t \in [0\,,T_N),$$

almost surely for every $N \geqslant 1$. Because ψ_N is a.s. continuous, it follows that T_N can also be realized as the first time $t \geqslant 0$ that $\|\psi(t)\|_{C(\mathbb{T})} \geqslant N$. In particular, $T_N \leqslant T_{N+1}$ a.s. for all $N \geqslant 1$ and hence,

$$T_\infty := \lim_{N\to\infty} T_N \quad \text{exists a.s.}$$

Moreover, it is immediate that ψ is almost surely continuous on $(0\,,T_\infty) \times \mathbb{T}$ a.s.

Proposition 3.1. *$T_\infty = \lim_{N\to\infty} T_N = \infty$ a.s. Therefore, $\psi \in C_+(\mathbb{R}_+ \times \mathbb{T})$ a.s.*

Proof. Since $\psi_N \geqslant 0$ a.s. [see (3.4)] and $V_N(w) \leqslant w$ for all $w \geqslant 0$, the comparison theorem for SPDEs (Lemma 3.3) ensures that $0 \leqslant \psi_N(t\,,x) \leqslant u(t\,,x)$ for all $t \geqslant 0$ and $x \in \mathbb{T}$, where u denotes the unique, continuous mild solution to the SPDE ,

$$\partial_t u(t\,,x) = \partial_x^2 u(t\,,x) + u(t\,,x) + \sigma(u(t\,,x))\dot{W}(t\,,x),$$

for $t > 0$ and $x \in \mathbb{T}$, subject to periodic boundary conditions and initial profile $u(0) = \sup_{x\in\mathbb{T}} \psi_0(x)$. See Walsh (1986).

Define

$$g(t\,,w) := \mathrm{e}^{-t}\sigma\left(\mathrm{e}^t w\right) \qquad \text{for all } t \geqslant 0 \text{ and } w \in \mathbb{R}.$$

It is easy to see that we may write $u(t\,,x) = \mathrm{e}^t U(t\,,x)$, where U denotes the unique, continuous mild solution to the SPDE,

$$\partial_t U(t\,,x) = \partial_x^2 U(t\,,x) + g(t\,,U(t\,,x))\dot{W}(t\,,x),$$

for $t > 0$ and $x \in \mathbb{T}$, subject to same boundary and initial values as u was. This elementary fact holds because:

1. The fundamental solution to the linear operator $\varphi \mapsto \partial_t\varphi - \partial_x^2\varphi - \varphi$ is $\exp(t)$ times the fundamental solution to the heat operator $\varphi \mapsto \partial_t\varphi - \partial_x^2\varphi$; and
2. $|g(t\,,w) - g(t\,,z)| \leqslant \mathrm{Lip}_\sigma|w - z|$ uniformly for all $t \geqslant 0$ and $w, z \in \mathbb{R}$. Therefore, we can apply standard methods from SPDEs to establish the said results about the random field U; see Walsh (1986, Chapter 3).

Because u and U are also nonnegative a.s., we can combine the preceding comparison arguments in order to see that

$$\sup_{N\in\mathbb{N}} \mathrm{E}\left(\sup_{t\in[0,T]} \|\psi_N(t)\|_{C(\mathbb{T})}^k\right) \leqslant \mathrm{e}^{kT}\mathrm{E}\left(\sup_{t\in[0,T]} \|U(t)\|_{C(\mathbb{T})}^k\right), \tag{3.5}$$

for all real numbers $k \geqslant 2$ and $T > 0$.

Now the proof of Proposition 4.1 in Khoshnevisan et al. (2020) can easily be repurposed in order to show that there exist real numbers $D_1, D_2 > 0$ (depending only on Lip_σ and $\|\psi_0\|_{C(\mathbb{T})}$) such that

$$\mathrm{E}\left(|U(t\,,x)|^k\right) \leqslant D_1^k \exp\{D_2 k^3 t\},$$

uniformly for all $x \in \mathbb{T}$ and $t \geqslant 0$. Also, standard methods (see Khoshnevisan et al., 2020, Remark 4.3) can be employed to show that

$$\sup_{\substack{x,y\in\mathbb{T}\\ 0\leqslant s,t\leqslant T\\ (t,x)\neq(s,y)}} \left\|\frac{|U(t\,,x) - U(s\,,y)|}{|x-y|^{1/2} + |s-t|^{1/4}}\right\|_k < \infty \qquad \text{for every } T > 0 \text{ and } k \geqslant 2. \tag{3.6}$$

We may take the supremum over all $s, t \in [0\,,T]$ (and not $s, t \in [t_0\,,T]$) because we are not maximizing over an auxiliary parameter λ in the present setting. In any case, we may combine the preceding in order to deduce that

$$\sup_{t\in[0,T]} \|U(t)\|_{C(\mathbb{T})} \in L^k(\Omega) \qquad \text{for every } k \geqslant 2 \text{ and } T > 0. \tag{3.7}$$

This and (3.5) in turn imply that

$$\mathrm{P}\{T_N \leqslant T\} = \mathrm{P}\left\{\sup_{t\in[0,T]} \|\psi_N(t)\|_{C(\mathbb{T})} \geqslant N\right\} = O(N^{-k}) \qquad \text{as } N \to \infty, \tag{3.8}$$

which is more than good enough to complete the proof, since we have demonstrated already that $\psi_N \in C_+(\mathbb{T})$ a.s. and $\psi = \psi_N$ a.s. on $[0\,,T_N) \times \mathbb{T}$ for every $N \in \mathbb{N}$. □

We can now return to the proof of Theorem 2.3 and establish that the continuous, progressively-measurable process ψ is indeed the solution to (1.1).

As part of the proof of Proposition 3.1 we showed that

$$\sup_{t\in[0,T]} \|\psi(t)\|_{C(\mathbb{T})} \in \bigcap_{k\geqslant 2} L^k(\Omega),$$

for all real numbers $T > 0$; see for example (3.8). Since σ is Lipschitz, this fact and a standard appeal to the Burkholder-Davis-Gundy inequality together show that

$$\mathcal{I}(t\,,x) := \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y) \tag{3.9}$$

is a well-defined Walsh stochastic integral for every $(t\,,x) \in \mathbb{R}_+ \times \mathbb{T}$, and $\mathcal{I}$ is continuous a.s. Furthermore, elementary properties of stochastic integrals show that, because $\psi = \psi_N$ on $[0\,,T_N)\times\mathbb{T}$,

$$\mathcal{I}(t\,,x) = \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_N(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y),$$

for all $(t\,,x) \in [0\,,T_N) \times \mathbb{T}$, almost surely. This and (3.3) together imply that, with probability one, ψ solves (2.2) simultaneously for all $(t\,,x) \in [0\,,T_N) \times \mathbb{T}$. Let $N \to \infty$ and appeal to Proposition 3.1 in order to see that, indeed, (1.1) has a continuous mild solution ψ. The uniqueness of this ψ follows from the uniqueness of the ψ_N's.

Next we prove that $\psi(t\,,x) \geqslant 0$ a.s. for every $t > 0$ and $x \in \mathbb{T}$. Indeed, for every $N \in \mathbb{N}$,

$$\mathrm{P}\left\{\psi(t\,,x) < 0\ ,\ T_N > t\right\} = \mathrm{P}\left\{\psi_N(t\,,x) < 0\ ,\ T_N > t\right\} = 0,$$

since ψ_N solves an SPDE with Lipschitz-continuous coefficients; see Mueller (1991) and Shiga (1994). Since $\lim_{N\to\infty} \mathrm{P}\{T_N \leqslant t\} = 0$, this proves that $\psi(t\,,x) \geqslant 0$ a.s.

Suppose in addition that $\psi_0 \neq \mathbb{0}$. Thanks to the comparison theorem (Mueller, 1991; Shiga, 1994), ψ is everywhere bounded below by the solution to (1.1). [Apply comparison first to ψ_N and then use the facts that $\psi = \psi_N$ on $[0\,,T_N) \times \mathbb{T}$ and $\lim_{N\to\infty} T_N = \infty$ a.s.] Therefore, the strict positivity theorem of Mueller (1991) implies that

$$\mathrm{P}\left\{\inf_{s\in[0,t]}\inf_{x\in\mathbb{T}} \psi_N(s\,,x) \leqslant 0\right\} = 0,$$

for all $t > 0$ and $N \in \mathbb{N}$; see also Mueller and Nualart (2008) and Conus et al. (2012, eq. (5.15)). In particular, it follows that

$$\mathrm{P}\left\{\inf_{x\in\mathbb{T}} \psi(t\,,x) \leqslant 0\right\} = \lim_{N\to\infty} \mathrm{P}\left\{\inf_{x\in\mathbb{T}} \psi_N(t\,,x) \leqslant 0\ ,\ T_N > t\right\} = 0,$$

for all $t \geqslant 0$. This completes all but one part of the proof of Theorem 2.3: It remains only to prove that $\mathrm{P}\{\psi(t) \in C^\alpha(\mathbb{T})\} = 1$ for every $\alpha \in (0\,,1/2)$ and $t > 0$.[3] This fact follows immediately from the continuity of $\psi(t)$ – shown earlier here – and the fact that

$$\mathrm{E}\left(\|\psi(t)\|_{C^\alpha(\mathbb{T})}\right) < \infty \qquad \text{for every } \alpha \in (0\,,1/2) \text{ and } t > 0;$$

[3] This is a somewhat subtle statement. For example, the condition "$t > 0$" cannot in general be replaced by "$t \geqslant 0$," as $\psi(0)$ need not be in $C^\alpha(\mathbb{T})$ for any $\alpha \in (0\,,1/2)$.

see Proposition 4.6 below. An inspection of the proof of Proposition 4.6 shows that our reasoning is not circular. Thus, we may conclude the proof of Theorem 2.3. □

We pause to observe that the proof of Proposition 3.1 can be slightly generalized in order to yield information about the rate at which T_N goes to infinity as $N \to \infty$. Although we will not need the following in the sequel, we state and prove it for potential future uses.

Corollary 3.2. *There exists a real number $L \geqslant 0$ such that*

$$\mathrm{P}\{T_N \leqslant T\} \leqslant L \exp\left\{-\frac{(\log N)^{3/2}}{L\sqrt{T}}\right\} \qquad \text{for all } N \in \mathbb{N} \text{ and } T > 0. \tag{3.10}$$

In particular,[4]

$$T_N = \Omega\left(\frac{(\log N)^3}{(\log\log N)^2}\right) \qquad \text{as } N \to \infty \text{ a.s.}$$

Proof. One can combine (3.6) and (3.7), and apply a chaining argument or see Khoshnevisan et al. (2020, Proposition 5.8), in order to see that there exists a number $L_1 > 0$ such that

$$\mathrm{E}\left(\sup_{t\in[0,T]} \|U(t)\|_{C(\mathbb{T})}^k\right) \leqslant L_1^k \mathrm{e}^{L_1 k^3 T} \qquad \text{for all } T > 0 \text{ and } k \geqslant 2.$$

Therefore, we may deduce from (3.5) that

$$\sup_{N\in\mathbb{N}} \mathrm{E}\left(\sup_{t\in[0,T]} \|\psi_N(t)\|_{C(\mathbb{T})}^k\right) \leqslant L_1^k \mathrm{e}^{L_2 k^3 T} \qquad \text{for all } T > 0 \text{ and } k \geqslant 2,$$

for a suitably large number $L_2 > 1$ that does not depend on $(k\,, T)$. In particular, the Chebyshev inequality yields a number $L_3 > 0$ such that

$$\mathrm{P}\{T_N \leqslant T\} \leqslant \inf_{k\geqslant 2} \frac{L_1^k \mathrm{e}^{L_2 k^3 T}}{N^k} \leqslant \exp\left\{-\frac{(\log N)^{3/2}}{L_3\sqrt{T}}\right\} \qquad \text{for all } T > 0,\ N > L_1^2.$$

This implies (3.10) for all $N \in \mathbb{N}$, provided that we choose a sufficiently large L. The lower bound for the growth rate of T_N follows from that probability bound, a suitable appeal to the Borel-Cantelli lemma along the subsequence $N = 2^n$ $[n \in \mathbb{N}]$, and monotonicity. We omit the details. □

We now return to the main discussion and conclude the section by observing that the proof of the positivity portion of Theorem 2.3 used a localization argument – via stopping times $\{T_N\}_{N=1}^\infty$ – that reduced the positivity of the solution of (1.1) to the positivity of the solution of an SPDE with Lipschitz-continuous coefficients. The very same localization argument can be used, in conjunction with the comparison theorem of SPDEs with Lipschitz-continuous coefficients (Mueller, 1991; Shiga, 1994), in order to yield the following.

Lemma 3.3. *For every $i = 1, 2$, let $\psi^{(i)}$ denote the continuous solution to the following SPDE with periodic boundary conditions, as in (1.1):*

$$\begin{bmatrix} \partial_t u(t\,,x) = \partial_x^2 u(t\,,x) + a_i(u(t\,,x)) + \sigma(u(t\,,x))\dot{W}(t\,,x) & \text{for } (t\,,x) \in (0\,,\infty) \times \mathbb{T}; \\ \text{subject to} \quad u(0\,,x) = \psi_0^{(i)}(x) & \text{for all } x \in \mathbb{T}; \end{bmatrix}$$

[4] We recall that $a_N = \Omega(b_N)$ as $N \to \infty$ for positive a_N and b_N's iff $\liminf_{N\to\infty}(a_N/b_N) > 0$.

where $\psi_0^{(i)} \in C(\mathbb{T})$ *is non random, and* $a_i : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous. Suppose, in addition, that* $\psi_0^{(1)} \leqslant \psi_0^{(2)}$ *everywhere on* $\mathbb{T}$*, and* $a_1 \leqslant a_2$ *everywhere on* $\mathbb{R}$*. Then,*

$$\mathrm{P}\left\{\psi^{(1)}(t\,,x) \leqslant \psi^{(2)}(t\,,x) \quad \textit{for all } (t\,,x) \in \mathbb{R}_+ \times \mathbb{T}\right\} = 1.$$

We will repeatedly appeal to this comparison result.

4 Existence of Invariant Measures

We now begin the proof of Theorem 2.4. Throughout, let ψ denote the unique mild solution to (1.1) with values in $C_+(\mathbb{R}_+ \times \mathbb{T})$, whose existence was established in Theorem 2.3.

4.1 Tightness

The first stage of our demonstration of Theorem 2.4 is proof of tightness in a suitable space. We begin the proof of tightness with a few technical lemmas.

Lemma 4.1. *There exists a real number* $K > 0$ *such that*

$$\int_0^\infty \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ |p_s(x\,,y) - p_s(z\,,y)| \leqslant K|x - z| \log_+(1/|x - z|) \qquad \textit{for all } x, z \in \mathbb{T},$$

where $\log_+ a := \log(\mathrm{e} \vee a)$ *for all* $a \geqslant 0$.

Proof. Consider the free-space heat kernel for the operator $\partial_t - \partial_x^2$, given by

$$G_t(x) := \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R}.$$

In this way we can see from (2.3) that $p_t(x\,,y) = \sum_{k=-\infty}^{\infty} G_t(2k + x - y)$ for all $t > 0$ and $x, y \in \mathbb{T}$. Thus, we find that

$$p_t(x\,,y) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} G_t(2w + x - y)\mathrm{e}^{2\pi i w k}\, \mathrm{d}w = \frac{1}{2} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi^2 k^2 t - i\pi(x-y)k}, \tag{4.1}$$

by the Poisson summation formula (see Feller, 1971, p. 630). It follows that

$$|p_t(x\,,y) - p_t(z\,,y)| \leqslant \frac{1}{2} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi^2 k^2 t} \left|1 - \mathrm{e}^{i\pi(x-z)k}\right| = \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi^2 k^2 t} \sqrt{1 - \cos(\pi k|x - z|)},$$

uniformly for all $t > 0$ and $x, y, z \in \mathbb{T}$. Since $1 - \cos a \leqslant a^2 \wedge 1$ for every $a \in \mathbb{R}$,

$$|p_t(x\,,y) - p_t(z\,,y)| \leqslant \sqrt{2} \sum_{k=1}^{\infty} \mathrm{e}^{-\pi^2 k^2 t} \left((|x - z|\pi k) \wedge 1\right), \tag{4.2}$$

where the implied constant is universal and finite. Integrate both sides over $y \in \mathbb{T}$ and $t \geqslant 0$, and apply Tonelli's theorem to find that

$$\int_0^\infty \mathrm{d}t \int_{\mathbb{T}} \mathrm{d}y\ |p_t(x\,,y) - p_t(z\,,y)| \leqslant \frac{2\sqrt{2}}{\pi^2} \sum_{k=1}^{\infty} \frac{(|x - z|\pi k) \wedge 1}{k^2}.$$

Consider separately the cases that $|\pi k| \leqslant |x-z|^{-1}$ and $|\pi k| > |x-z|^{-1}$ in order to finish in the case that $|x-z| \leqslant 1$. If $|x-z| > 1$, then we use the trivial bound $|x-z| \leqslant 2$ to see that

$$C := \sup_{x,z\in\mathbb{T}} \int_0^\infty \mathrm{d}t \int_{\mathbb{T}} \mathrm{d}y \; |p_t(x\,,y) - p_t(z\,,y)| < \infty.$$

The case $|x-z| > 1$ follows from this, since $C \leqslant C|x-z|\log_+(1/|x-z|)$. □

Lemma 4.2. *There exists a real number $K > 0$ such that*

$$\int_0^\infty \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y \, |p_s(x\,,y) - p_s(z\,,y)|^2 \leqslant K|x-z| \qquad \textit{for all } x, z \in \mathbb{T}.$$

Proof. Use (4.1) to obtain that

$$p_t(x\,,y) - p_t(z\,,y) = \frac{1}{2}\sum_{k=-\infty}^{\infty} \mathrm{e}^{-\pi^2k^2t}\mathrm{e}^{-i\pi yk}\left(\mathrm{e}^{-i\pi xk} - \mathrm{e}^{-i\pi zk}\right).$$

The orthogonality of $\left\{\mathrm{e}^{-i\pi yk} : k \in \mathbb{Z}\right\}$ in $L^2([-1,1])$ and the elementary bound used for (4.2) give

$$\begin{aligned} \int_{\mathbb{T}} \mathrm{d}y \, |p_t(x\,,y) - p_t(z\,,y)|^2 &= \sum_{k=-\infty}^{\infty} \mathrm{e}^{-2\pi^2k^2t}\left[1 - \cos\left(\pi k(x-z)\right)\right] \qquad (4.3)\\ &\leqslant \sum_{k=-\infty}^{\infty} \mathrm{e}^{-2\pi^2k^2t}[\left(\pi^2k^2(x-z)^2\right) \wedge 1]. \end{aligned}$$

Integrate both sides over $t \geqslant 0$ to have

$$\int_0^\infty \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}y \, |p_s(x\,,y) - p_s(z\,,y)|^2 \leqslant \sum_{k=-\infty}^{\infty} \frac{(\pi^2(x-z)^2k^2) \wedge 1}{2\pi^2k^2t}.$$

Like in lemma 4.1, we consider the cases that $|\pi k| \leqslant |x-z|^{-1}$ and $|\pi k| > |x-z|^{-1}$ to finish the proof.

□

Lemma 4.3. *Let $\gamma := (64\mathrm{Lip}_\sigma^2)^2 \vee \frac{1}{4}$. For every real number $k \geqslant 2$,*

$$\sup_{t\geqslant 0}\sup_{x\in\mathbb{T}} \mathrm{E}\left(|\psi(t\,,x)|^k\right) \leqslant \left[2\|\psi_0\|_{C(\mathbb{T})} + 4\mathcal{R}(k)\right]^k \quad \textit{where} \quad \mathcal{R}(k) := \sup_{y\geqslant 0}\frac{V(y) + \gamma k^2 y}{1+\gamma k^2},$$

and $\mathcal{R}(k) < \infty$. Furthermore, $\mathcal{R}(k) \uparrow \infty$ as $k \uparrow \infty$.

Example 4.4. Suppose there exist real numbers $A, \nu > 0$ such that $F(x) \geqslant Ax^{1+\nu}$ for all $x \geqslant 0$. This happens with an identity and with $\nu = 1$ in the Fisher-KPP case and $\nu = 2$ in the Allen-Cahn case. Then, $\mathcal{R}(k) \leqslant \text{const} \cdot k^{2/\nu}$. Thus, Lemma 4.3 asserts that there exists a positive real number $c = c(A\,,\nu\,,\psi_0)$ such that $\mathrm{E}(|\psi(t\,,x)|^k) \leqslant c^k k^{2k/\nu}$ uniformly for all $k \geqslant 2$, $t \geqslant 0$, and $x \in \mathbb{T}$.

Proof. Throughout, we choose and fix an integer $k \geqslant 2$. Recall the properties of the function F, hence also V, from §2.2. On one hand, Lemma 2.1 ensures that there exists $\rho = \rho(k) > 0$ such that $(1+\gamma k^2)y \leqslant F(y)$ for all $y \geqslant \rho$. On the other hand, $(1+\gamma k^2)y \leqslant (1+\gamma k^2)\rho$ when $y \in [0\,,\rho)$. Therefore, $(1+\gamma k^2)y \leqslant (1+\gamma k^2)\rho + F(y)$ for all $y \geqslant 0$. This is another way to say that

$$V(y) \leqslant (1+\gamma k^2)\rho - \gamma k^2 y \qquad \text{for all } y \geqslant 0.$$

In particular, $\mathcal{R}(k) < \infty$ because $\mathcal{R}(k)$ is the smallest such ρ. A comparison lemma (see Lemma 3.3) now ensures that

$$\mathrm{P}\left\{\psi(t\,,x) \leqslant \varphi(t\,,x) \text{ for all } t \geqslant 0 \text{ and } x \in \mathbb{T}\right\} = 1, \tag{4.4}$$

where φ solves the following semi-linear SPDE,

$$\partial_t \varphi(t\,,x) = \partial_x^2 \varphi(t\,,x) + (1+\gamma k^2)\mathcal{R}(k) - \gamma k^2 \varphi(t\,,x) + \lambda\sigma(\varphi(t\,,x))\dot{W}(t\,,x), \tag{4.5}$$

subject to a periodic boundary condition and the same initial condition $\varphi(0) = \psi_0$ as for ψ. One can understand (4.5) in mild form in two different ways: One way is to write (4.5) in integral form using the Green's function for the heat operator $\partial_t - \partial_x^2$, as is done in Walsh (1986). An equivalent but slightly different way is to write (4.5) in mild form in terms of the fundamental solution $\tilde{p}$ of the perturbed operator $f \mapsto \partial_t f - \partial_x^2 f - \gamma k^2 f$. Note that

$$\tilde{p}_t(x\,,y) = \mathrm{e}^{-\gamma k^2 t} p_t(x\,,y) \qquad \text{for every } t > 0 \text{ and } x, y \in [-1\,,1],$$

where p continues to denote the heat kernel for $\partial_t - \partial_x^2$; see (2.3). It follows from this discussion that φ is the unique solution to the random integral equation,

$$\begin{aligned}\varphi(t\,,x) &= \mathrm{e}^{-\gamma k^2 t}(\mathcal{P}_t\psi_0)(x) + (1+\gamma k^2)\mathcal{R}(k)\int_{(0,t)\times\mathbb{T}} \tilde{p}_{t-s}(x\,,y)\,\mathrm{d}s\,\mathrm{d}y + \lambda\mathcal{J}(t\,,x)\\ &= \mathrm{e}^{-\gamma k^2 t}(\mathcal{P}_t\psi_0)(x) + \frac{(1+\gamma k^2)\mathcal{R}(k)}{\gamma k^2}\left(1 - \mathrm{e}^{-\gamma k^2 t}\right) + \lambda\mathcal{J}(t\,,x),\end{aligned} \tag{4.6}$$

where

$$\mathcal{J}(t\,,x) := \int_{(0,t)\times\mathbb{T}} \mathrm{e}^{-\gamma k^2(t-s)} p_{t-s}(x\,,y)\sigma(\varphi(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y).$$

General theory (Walsh, 1986; Dalang, 1999) tells us that φ exists and solves the integral equation (4.6) uniquely among all continuous predictable random fields. Moreover,

$$\sup_{t\in(0,T)}\sup_{x\in\mathbb{T}} \mathrm{E}\left(|\varphi(t\,,x)|^k\right) < \infty \qquad \text{for all } T > 0.$$

We now estimate the above moments slightly more carefully.

Because

$$0 \leqslant \mathrm{e}^{-\gamma k^2 t}(\mathcal{P}_t\psi_0)(x) + \frac{(1+\gamma k^2)\mathcal{R}(k)}{\gamma k^2}\left(1 - \mathrm{e}^{-\gamma k^2 t}\right) \leqslant \|\psi_0\|_{C(\mathbb{T})} + 2\mathcal{R}(k) =: L,$$

uniformly over all $t \geqslant 0$, an application of the Burkholder-Davis-Gundy (BDG) inequality to the a.s. identity (4.6) yields

$$\mathrm{E}\left(|\varphi(t\,,x)|^k\right) \leqslant 2^{k-1}L^k + 2^{k-1}A_k\mathrm{E}\left(|\langle\mathcal{J}\rangle_{t,x}|^{k/2}\right),$$

where A_k denotes the optimal constant in the BDG inequality (Burkholder et al., 1972), and

$$\langle\mathcal{J}\rangle_{t,x} := \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \mathrm{e}^{-2\gamma k^2(t-s)} p_{t-s}^2(x\,,y)\sigma^2(\varphi(s\,,y)).$$

Thanks to (2.1), $\sigma^2(\varphi(s\,,y)) \leqslant \mathrm{Lip}_\sigma^2|\varphi(s\,,y)|^2$ a.s. Therefore, Minkowski's inequality yields

$$\begin{aligned}\|\langle\mathcal{J}\rangle_{t,x}\|_{k/2} &\leqslant \mathrm{Lip}_\sigma^2 \sup_{s\in(0,t)}\sup_{y\in\mathbb{T}}\|\varphi(s\,,y)\|_k^2\int_0^t \mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \mathrm{e}^{-2\gamma k^2(t-s)}p_{t-s}^2(x\,,y)\\ &\leqslant 2\mathrm{Lip}_\sigma^2 \sup_{s\in(0,t)}\sup_{y\in\mathbb{T}}\|\varphi(s\,,y)\|_k^2\int_0^t\left(1\vee(2s)^{-1/2}\right)\mathrm{e}^{-2\gamma k^2 s}\,\mathrm{d}s\\ &\leqslant \frac{4\mathrm{Lip}_\sigma^2}{\sqrt{\gamma k^2}}\sup_{s\in(0,t)}\sup_{y\in\mathbb{T}}\|\varphi(s\,,y)\|_k^2,\end{aligned}$$

using the facts that: (a) $p_s(x\,,y) \leqslant 2(1\vee s^{-1/2})$ (Khoshnevisan et al., 2020, Lemma B.1); (b) $p_{2(t-s)}(x\,,x) = \int_{\mathbb{T}} p_{t-s}^2(x\,,y)\mathrm{d}y$; and (c)

$$\int_0^t\left(1\vee(2s)^{-1/2}\right)\mathrm{e}^{-2\gamma k^2 s}\,\mathrm{d}s \leqslant \int_0^{1/2}(2s)^{-1/2}\mathrm{e}^{-2\gamma k^2 s}\,\mathrm{d}s + \int_{1/2}^\infty \mathrm{e}^{-2\gamma k^2 s}\,\mathrm{d}s \leqslant 2(\gamma k^2)^{-1/2}.$$

Here, we used the fact that $\gamma k^2 \geqslant \sqrt{\gamma k^2}$ for all $k \geqslant 2$, thanks to the assumption that $\gamma = (64\mathrm{Lip}_\sigma^2)^2 \vee \frac14$. According to Carlen and Kree (1991), $A_k \leqslant (4k)^{k/2}$ for all $k \geqslant 2$. Thus, we combine to find that

$$\begin{aligned}\mathrm{E}\left(|\varphi(t\,,x)|^k\right) &\leqslant 2^{k-1}L^k + 2^{2k-1}k^{k/2}\left[\frac{4\mathrm{Lip}_\sigma^2}{\sqrt{\gamma k^2}}\sup_{s\in(0,t)}\sup_{y\in\mathbb{T}}\|\varphi(s\,,y)\|_k^2\right]^{k/2}\\ &\leqslant 2^{k-1}L^k + \tfrac12\sup_{s\in(0,t)}\sup_{y\in\mathbb{T}}\mathrm{E}\left(|\varphi(s\,,y)|^k\right),\end{aligned}$$

using the fact that $\gamma \geqslant (64\mathrm{Lip}_\sigma^2)^2$. This immediately yields

$$\sup_{s\in(0,t)}\sup_{y\in\mathbb{T}}\mathrm{E}\left(|\varphi(s\,,y)|^k\right) \leqslant (2L)^k.$$

Let $t\uparrow\infty$ and recall (4.4) in order to deduce the announced bound for the moments of $\psi(t\,,x)$.

In order to complete the proof, we observe that $\mathcal{R}$ is nondecreasing, and for every $m \geqslant 0$,

$$\lim_{k\to\infty}\mathcal{R}(k) \geqslant \liminf_{k\to\infty}\frac{V(m)+\gamma k^2 m}{1+\gamma k^2} = m.$$

Let $m\to\infty$ to see that $\lim_{k\to\infty}\mathcal{R}(k) = \infty$. □

Lemma 4.5. *Recall the constant $m_0 > 1$ from* **(F3)** *and the function $\mathcal{R}$ from Lemma 4.3. For every $\tau > 0$, there exists $L_0 = L_0(\tau\,,\mathrm{Lip}_\sigma) > 0$ – independent of ψ_0 – such that*

$$\sup_{t\geqslant\tau}\mathrm{E}\left(|\psi(t\,,x)-\psi(t\,,z)|^k\right) \leqslant L_0^k\left(k^{k/2}\left[\|\psi_0\|_{C(\mathbb{T})}+\mathcal{R}(k)\right]^k + \left[\|\psi_0\|_{C(\mathbb{T})}+\mathcal{R}(m_0k)\right]^{m_0k}\right)|x-z|^{k/2},$$

uniformly for all $k \geqslant 2$ and $x, z \in \mathbb{T}$. If, in addition, $\psi_0 \in C_+^\alpha(\mathbb{T})$ for some $\alpha \in (0\,,1/2)$, then for every $k \geqslant 2$ there exists $L_k > 0$ – independent of ψ_0 – such that

$$\sup_{t\geqslant0}\mathrm{E}\left(|\psi(t\,,x)-\psi(t\,,z)|^k\right) \leqslant L_k\left\{\|\psi_0\|_{C^\alpha(\mathbb{T})}^k + 1\right\}|x-z|^{\alpha k},$$

uniformly for all $x, z \in \mathbb{T}$.

Proof. Choose and fix $t \geqslant \tau > 0$ and $x, z \in \mathbb{T}$. Thanks to (2.2), we can write

$$\|\psi(t\,,x) - \psi(t\,,z)\|_k \leqslant I_1 + I_2 + \lambda I_3,$$

where

$$\begin{aligned} I_1 &:= |(\mathcal{P}_t\psi_0)(x) - (\mathcal{P}_t\psi_0)(z)|, \\ I_2 &:= \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y \ \ |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|\, \|V(\psi(s\,,y))\|_k, \\ I_3 &:= \|\mathcal{I}(t\,,x) - \mathcal{I}(t\,,y)\|_k, \end{aligned}$$

and where $\mathcal{I}$ is the random field that was defined by a stochastic convolution in (3.9). We estimate I_1, I_2, and I_3 separately and in this order.

We apply the triangle inequality in conjunction with (4.2) to find that

$$\begin{aligned} I_1 &\leqslant \|\psi_0\|_{C(\mathbb{T})} \int_{\mathbb{T}} |p_t(x\,,y) - p_t(z\,,y)|\, \mathrm{d}y \leqslant \text{const} \cdot \|\psi_0\|_{C(\mathbb{T})} |x-z| \sum_{k=1}^{\infty} k\mathrm{e}^{-\pi^2 k^2 \tau} \\ &\leqslant \text{const} \cdot \|\psi_0\|_{C(\mathbb{T})} |x - z|, \end{aligned}$$

uniformly for all $x, z \in \mathbb{T}$, where the implied constant also does not depend on ψ_0.

(F3) ensures that there exists a number $A > 0$ such that

$$|V(z)| \leqslant A(|z| + |z|^{m_0}) \qquad \text{for all } z \geqslant 0.$$

Consequently,

$$\begin{aligned} \mathrm{E}\left(|V(\psi(s\,,y))|^k\right) &\leqslant 2^{k-1}A^k \left\{\mathrm{E}\left(|\psi(s\,,y)|^k\right) + \mathrm{E}\left(|\psi(s\,,y)|^{m_0 k}\right)\right\} \\ &\leqslant 2^{k-1}A^k \left\{\left[2\|\psi_0\|_{C(\mathbb{T})} + 4\mathcal{R}(k)\right]^k + \left[2\|\psi_0\|_{C(\mathbb{T})} + 4\mathcal{R}(m_0 k)\right]^{m_0 k}\right\} \\ &\leqslant C_3^k \left\{\left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(k)\right]^k + \left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(m_0 k)\right]^{m_0 k}\right\}, \end{aligned} \tag{4.7}$$

uniformly for all $s > 0$, $y \in \mathbb{T}$, and $k \geqslant 2$, where $C_3 = C_3(F) > 0$. In this way, we find that

$$\begin{aligned} I_2 &\leqslant C_4 \left\{\left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(k)\right] + \left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(m_0 k)\right]^{m_0}\right\} \int_0^\infty \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y \ \ |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)| \\ &\leqslant C_5 \left\{\left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(k)\right] + \left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(m_0 k)\right]^{m_0}\right\} |x - z| \log_+(1/|x-z|), \end{aligned}$$

where $C_4, C_5 > 0$ do not depend on $(k\,,x\,,z\,,\psi_0)$; see Lemma 4.1.

Finally, we apply the BDG inequality (see Burkholder et al., 1972), using the Carlen and Kree (1991) bound for the optimal BDG constant, in order to see that

$$\begin{aligned} I_3^k &\leqslant (4k)^{k/2} \mathrm{Lip}_\sigma^k \left\| \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y \ \ |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|^2\, |\psi(s\,,y)|^2 \right\|_{k/2}^{k/2} \\ &\leqslant (4k)^{k/2} \mathrm{Lip}_\sigma^k \left(\int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y \ \ |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|^2\, \|\psi(s\,,y)\|_k^2 \right)^{k/2}. \end{aligned}$$

We have used the Minkowski inequality in the final bound. Apply Lemma 4.3 above together with Lemma 4.2 in order to find that there exists a number $C_5 = C_5(\mathrm{Lip}_\sigma) > 0$ such that

$$\begin{aligned} I_3 &\leqslant 2\sqrt{k}\mathrm{Lip}_\sigma \left[2\|\psi_0\|_{C(\mathbb{T})} + 4\mathcal{R}(k)\right] \left(\int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y \ \left|p_{t-s}(x\,,y) - p_{t-s}(z\,,y)\right|^2\right)^{1/2} \\ &\leqslant C_5\sqrt{k}\left[\|\psi_0\|_{C(\mathbb{T})} + \mathcal{R}(k)\right] |x-z|^{1/2}. \end{aligned}$$

uniformly for all $t > 0$ and $x, z \in \mathbb{T}$. The first part of the lemma follows from combining the preceding estimates for I_1, I_2, and I_3.

Next, suppose additionally that $\psi_0 \in C^\alpha_+(\mathbb{T})$ for some $\alpha \in (0\,,1/2)$. Then, the estimate for I_1 can be improved upon as follows: We may write

$$I_1 = \left|\mathrm{E}\left[\psi_0(\beta(t)+x) - \psi_0(\beta(t)+z)\right]\right|,$$

for a standard Brownian motion β on $\mathbb{T}$. In this way we may write

$$I_1 \leqslant \mathrm{E}\left|\psi_0(\beta(t)+x) - \psi_0(\beta(t)+z)\right| \leqslant \|\psi_0\|_{C^\alpha(\mathbb{T})}|x-z|^\alpha,$$

uniformly for all $t \geqslant 0$ and $x, z \in \mathbb{T}$. The estimates for I_2 and I_3 remain the same. Combine things to finish the proof. □

We are ready for the tightness result.

Proposition 4.6. *For every $\tau > 0$ and $\alpha \in (0\,,1/2)$, there exists a number $L_1 = L_1(\|\psi_0\|_{C(\mathbb{T})}\,,\tau\,,\alpha) > 0$ such that $a \mapsto L_1(a\,,\tau\,,\alpha)$ is non decreasing and*

$$\sup_{t\geqslant\tau} \mathrm{E}\left(\|\psi(t)\|^k_{C^\alpha(\mathbb{T})}\right) \leqslant L_1^k\left(\sqrt{k}\,\mathcal{R}(k) + [\mathcal{R}(m_0k)]^{m_0}\right)^k, \tag{4.8}$$

uniformly for all real numbers $k \geqslant 2$. Consequently, the laws of $\{\psi(t)\}_{t\geqslant\tau}$ are tight on $C^\alpha_+(\mathbb{T})$. If, in addition, $\psi_0 \in C^\alpha_+(\mathbb{T})$ for some $\alpha \in (0\,,1/2)$, then

$$\sup_{t\geqslant 0} \mathrm{E}\left(\|\psi(t)\|^k_{C^\alpha(\mathbb{T})}\right) < \infty \quad \textit{for all } k \geqslant 2. \tag{4.9}$$

Proof. Related results appear in Cerrai (2003, 2005); we prefer to give a more detailed proof.

We can combine Lemmas 4.3 and 4.5 with a quantitative form of Kolmogorov's continuity theorem (see Khoshnevisan et al. (2020, Proposition 5.8) or Dalang et al. (2009, Theorem 4.3)) to deduce (4.8) and (4.9). On one hand, the Arzelà-Ascoli theorem implies that the set

$$\Gamma_n := \{f \in C^\alpha_+(\mathbb{T}) : \|f\|_{C^\alpha(\mathbb{T})} \leqslant n\}$$

is compact for every $n \in \mathbb{N}$. On the other hand, (4.8) and Markov's inequality together imply that

$$\lim_{n\to\infty} \limsup_{t\to\infty} \mathrm{P}\left\{\psi(t) \not\in \Gamma_n\right\} = 0.$$

This readily implies tightness, and concludes the proof. □

Example 4.7. Let us continue with Example 4.4 and suppose there exist real numbers $a, A, \nu > 0$ such that

$$Ax^{1+\nu} \geqslant F(x) \geqslant ax^{1+\nu} \qquad \text{for all } x \geqslant 0,$$

so that $\mathcal{R}(k) \leqslant \text{const} \cdot k^{2/\nu}$ and $m_0 = 1 + \nu$. According to Proposition 4.6, for every $\alpha \in (0\,,1/2)$ there exists a positive real number $L \geqslant 0$ such that

$$\sup_{t \geqslant 1} \mathrm{E}\left(\|\psi(t)\|^k_{C^\alpha(\mathbb{T})} \right) \leqslant L^k k^{2(1+\nu)k/\nu} \quad \text{for all } k \geqslant 2.$$

It follows readily from this and Stirling's formula that there exists $q = q(\alpha\,,\nu) > 0$ such that

$$\sup_{t \geqslant 1} \mathrm{E}\left[\exp\left(q \|\psi(t)\|^{\nu/2(1+\nu)}_{C^\alpha(\mathbb{T})} \right) \right] < \infty.$$

4.2 Temporal continuity

The following is very well known for SPDEs with Lipschitz-continuous coefficients. We will prove that the following formulation holds in the present case that V is not globally Lipschitz continuous exactly as it holds in the case that V were replaced by a globally Lipschitz function.

Proposition 4.8. *For every $\theta \in (0\,,1/4)$ and $Q > 1$ there exists a number $L = L(\theta\,,Q) > 0$ such that for all $T > 0$, and $k \geqslant 2$,*

$$\mathrm{E}\left(\sup_{t \in [T, T+Q]} \sup_{s \in (0,1]} \left\| \frac{\psi(t+s) - \psi(t)}{s^\theta} \right\|^k_{C(\mathbb{T})} \right) \leqslant L^k \left\{ T^{-3/4} \|\psi_0\|_{C(\mathbb{T})} + \mathcal{A}_k + \mathcal{B}_k \sqrt{k} \right\}^k,$$

where $\mathcal{A}_k := \sup_{r \geqslant 0} \sup_{y \in \mathbb{T}} \|V(\psi(r\,,y)\|_k$ and $\mathcal{B}_k := \sup_{r \geqslant 0} \sup_{z \in \mathbb{T}} \|\psi(r\,,z)\|_k$ are finite; see (4.7) and Lemma 4.3. Also, for every $\theta \in (0\,,1/4)$, $Q > 1$, and $k \geqslant 2$ there exists a number $K = K(\theta\,,Q\,,k) > 0$ – independently of ψ_0 – such that

$$\sup_{T > 0} \mathrm{E}\left(\sup_{t \in [T, T+Q]} \sup_{s \in [0,\varepsilon]} \|\psi(t+s) - \psi(t)\|^k_{C(\mathbb{T})} \right) \\ \leqslant K \left\{ \|\psi_0\|_{C(\mathbb{T})} + 1 \right\}^k \varepsilon^{\theta k} + K \sup_{s \in [0,\varepsilon]} \|\mathcal{P}_s \psi_0 - \psi_0\|^k_{C(\mathbb{T})}.$$

uniformly for every $\varepsilon \in (0\,,1)$.

As we shall see from the proof, one can also keep track of the dependence of the constant K on $(\theta\,,Q\,,k)$. We omit the details. The proof of Proposition 4.8 itself will proceed in a standard manner, but uses the Poisson summation formula at a key juncture, as did the proof of Proposition 4.6.

Recall (2.2) and (3.9), and write

$$|\psi(t+\varepsilon\,,x) - \psi(t\,,x)| \leqslant I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &:= |(\mathcal{P}_{t+\varepsilon}\psi_0)(x) - (\mathcal{P}_t\psi_0)(x)|\,,\\
I_2 &:= \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ |V(\psi(s\,,y))|\, |p_{t+\varepsilon-s}(x\,,y) - p_{t-s}(x\,,y)|\,,\\
I_3 &:= \int_t^{t+\varepsilon} \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ |V(\psi(s\,,y))| p_{t+\varepsilon-s}(x\,,y),\\
I_4 &:= |\mathcal{I}(t+\varepsilon\,,x) - \mathcal{I}(t\,,x)|\,.
\end{aligned}$$

We estimate the $L^k(\Omega)$-norm of $I_1, \ldots, I_4$ in this order. Proposition 4.8 follows from Lemmas 4.9, 4.10, 4.11, and 4.12, Proposition 4.6, and a chaining argument.

Lemma 4.9. *There exists a number $K > 0$ such that*

$$I_1 \leqslant K\|\psi_0\|_{C(\mathbb{T})}\left[\frac{1}{t^{1/4}}\min\left(1\,,\frac{\varepsilon}{t}\right)^{1/2}\wedge 1\right],$$

uniformly for all $x \in \mathbb{T}$ and $\varepsilon, t > 0$.

Proof. By the Cauchy-Schwarz inequality,

$$I_1 \leqslant \|\psi_0\|_{C(\mathbb{T})}\int_{\mathbb{T}}|p_{t+\varepsilon}(x\,,y) - p_t(x\,,y)|\,\mathrm{d}y \leqslant \|\psi_0\|_{C(\mathbb{T})}\sqrt{2\int_{\mathbb{T}}|p_{t+\varepsilon}(x\,,y) - p_t(x\,,y)|^2\,\mathrm{d}y}.$$

Therefore the result follows from Khoshnevisan et al. (2020, Lemma B.6). ☐

Lemma 4.10. *There exists a finite number $K > 0$ such that $\|I_2\|_k \leqslant K\mathcal{A}_k\sqrt{\varepsilon}$, uniformly for all $k \geqslant 2$, $x \in \mathbb{T}$, and $\varepsilon \in (0\,,1)$, and $t > 0$.*

Proof. Minkowski's inequality yields

$$\|I_2\|_k \leqslant \mathcal{A}_k\int_{\mathbb{T}}\mathrm{d}y\int_0^\infty \mathrm{d}s\ |p_{s+\varepsilon}(x\,,y) - p_s(x\,,y)|. \tag{4.10}$$

We may use the Poisson summation formula, as we did for Lemma 4.1, in order to see that for all $s, \varepsilon > 0$ and $x, y \in \mathbb{T}$,

$$|p_{s+\varepsilon}(x\,,y) - p_s(x\,,y)| \leqslant \frac{1}{2}\sum_{k\in\mathbb{Z}\setminus\{0\}}\mathrm{e}^{-\pi^2k^2s}\left|1 - \mathrm{e}^{-2\pi^2k^2\varepsilon}\right|.$$

Because $1 - \exp(-q) \leqslant (1 \wedge q)$ for all $q > 0$, it follows that

$$\sup_{x,y\in\mathbb{T}}\int_0^\infty |p_{s+\varepsilon}(x\,,y) - p_s(x\,,y)|\,\mathrm{d}s \leqslant \sum_{k=1}^\infty\left(k^{-2}\wedge\varepsilon\right),$$

uniformly for all $\varepsilon > 0$. This and (4.10) readily imply the lemma. ☐

Lemma 4.11. $\|I_3\|_k \leqslant \mathcal{A}_k\varepsilon$, *uniformly for all $k \geqslant 2$, $x \in \mathbb{T}$, and $\varepsilon, t > 0$.*

Proof. Another appeal to Minkowski's inequality yields $\|I_3\|_k \leqslant \mathcal{A}_k\int_t^{t+\varepsilon}\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \ p_{t+\varepsilon-s}(x\,,y)$, which is equal to $\mathcal{A}_k\varepsilon$. ☐

Lemma 4.12. *There exists a finite number $K > 0$ such that $\|I_4\|_k \leqslant K\mathcal{B}_k\sqrt{k}\,\varepsilon^{1/4}$ uniformly for all $k \geqslant 2$, $x \in \mathbb{T}$, and $\varepsilon, t > 0$.*

Proof. We can write $I_4 = I_{4,1} + I_{4,2}$, where

$$I_{4,1} := \int_{(0,t)\times\mathbb{T}}\left(p_{t+\varepsilon-s}(x\,,y) - p_{t-s}(x\,,y)\right)\sigma(\psi(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y),$$
$$I_{4,2} := \int_{(t,t+\varepsilon)\times\mathbb{T}}p_{t+\varepsilon-s}(x\,,y)\sigma(\psi(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y).$$

We estimate $I_{4,1}$ and $I_{4,2}$ separately, and in this order, using the BDG inequalities in the same manner as in the proof of Lemma 4.3. Indeed,

$$\begin{aligned}\|I_{4,1}\|_k^2 &\leqslant A_k^{2/k}\left\|\int_0^t \mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \left(p_{t+\varepsilon-s}(x\,,y)-p_{t-s}(x\,,y)\right)^2|\sigma(\psi(s\,,y))|^2\right\|_{k/2}\\ &\leqslant A_k^{2/k}\int_0^t \mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \left(p_{t+\varepsilon-s}(x\,,y)-p_{t-s}(x\,,y)\right)^2\|\sigma(\psi(s\,,y))\|_k^2,\end{aligned}$$

where A_k denotes, as before, the optimal constant of the $L^k(\Omega)$-form of the BDG inequality. Thus, Lemma B.6 of Khoshnevisan et al. (2020) implies that

$$\|I_{4,1}\|_k \leqslant L_1 A_k^{1/k}\sup_{r\geqslant 0}\sup_{z\in\mathbb{T}}\|\sigma(\psi(r\,,z))\|_k\sqrt{\int_0^\infty \frac{1}{s^{1/2}}\min\left(1\,,\frac{\varepsilon}{s}\right)\mathrm{d}s} \leqslant L_2 A_k^{1/k}\mathcal{B}_k\,\varepsilon^{1/4}.$$

We estimate $I_{4,2}$ using similar techniques. Namely,

$$\begin{aligned}\|I_{4,2}\|_k^2 &\leqslant A_k^{2/k}\left\|\int_t^{t+\varepsilon}\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \left(p_{t+\varepsilon-s}(x\,,y)\right)^2|\sigma(\psi(s\,,y))|^2\right\|_{k/2}\\ &\leqslant A_k^{2/k}\mathrm{Lip}_\sigma^2\mathcal{B}_k^2\int_0^\varepsilon \mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \left(p_s(x\,,y)\right)^2.\end{aligned}$$

To finish our estimate for $I_{4,2}$, we may apply the semigroup property and the symmetry of the heat kernel to see that

$$\int_0^\varepsilon \mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\ \left(p_s(x\,,y)\right)^2 = \int_0^\varepsilon p_{2s}(0\,,0)\,\mathrm{d}s \leqslant \sqrt{2}\int_0^\varepsilon \frac{\mathrm{d}s}{\sqrt{s}} = 2\sqrt{2\varepsilon};$$

see Khoshnevisan et al. (2020, Lemma B.1) for the last bound. Finally, we combine the two bounds for $I_{4,1}$ and $I_{4,2}$ and appeal to an inequality of Carlen and Kree (1991) that asserts that $A_k^{1/k}\leqslant 2\sqrt{k}$ for every $k\geqslant 2$. □

Proof of Proposition 4.8. The first assertion of the proposition follows immediately from Lemmas 4.9–4.12 and a chaining argument. We prove the second assertion of the proposition.

Define

$$\varphi(t\,,x) := \psi(t\,,x) - (\mathcal{P}_t\psi_0)(x) \qquad \text{for all } t\geqslant 0 \text{ and } x\in\mathbb{T}.$$

We repeat the proof of Lemma 4.5 in order to see that for every $k\geqslant 2$ there exists $L_k>0$ – independently of ψ_0 – such that

$$\sup_{t\geqslant 0}\mathrm{E}\left(|\varphi(t\,,x)-\varphi(t\,,z)|^k\right) \leqslant L_k\left(\|\psi_0\|_{C(\mathbb{T})}+1\right)^k|x-z|^{k/2},$$

uniformly for all $x,z\in\mathbb{T}$. The difference between the above inequality and that of Lemma 4.5 is that we are allowed to take a supremum over all $t\geqslant 0$ [and not just $t\geqslant\tau>0$]; this is because $\mathcal{P}_t\psi_0$ has been subtracted from ψ [to yield φ]. Lemma 4.3 tells us that for every $k\geqslant 2$ there exists $L_k'>0$ – independently of ψ_0 – such that $\mathcal{A}_k^k\vee\mathcal{B}_k^k\leqslant L_k'(\|\psi_0\|_{C(\mathbb{T})}+1)^k$. Therefore, Lemmas 4.10, 4.11, and 4.12 tell us that for every $k\geqslant 2$ there exists $L_k''>0$ – independently of ψ_0 – such that

$$\sup_{t\geqslant 0}\sup_{x\in\mathbb{T}}\mathrm{E}\left(|\varphi(t+\varepsilon\,,x)-\varphi(t\,,x)|^k\right) \leqslant L_k''\left(\|\psi_0\|_{C(\mathbb{T})}+1\right)^k\varepsilon^{k/4},$$

uniformly for every $\varepsilon \in (0\,,1)$. Apply these inequalities, together with a chaining argument, in order to see that for every $\theta \in (0\,,1/4)$ and $k \geqslant 2$ there exists $L_{k,\theta} > 0$ – independently of ψ_0 – such that

$$\sup_{T>0} \mathrm{E}\left(\sup_{t\in[T,T+Q]} \sup_{s\in[0,\varepsilon]} \|\varphi(t+s) - \varphi(t)\|_{C(\mathbb{T})}^k \right) \leqslant L_{k,\theta} \left\{ \|\psi_0\|_{C^\alpha(\mathbb{T})} + 1 \right\}^k \varepsilon^{\theta k},$$

uniformly for every $\varepsilon \in (0\,,1)$. This completes the proof of the proposition because

$$\|\psi(t+s) - \psi(t)\|_{C(\mathbb{T})}^k \leqslant 2^k \|\varphi(t+s) - \varphi(t)\|_{C(\mathbb{T})}^k + 2^k \|\mathcal{P}_{t+s}\psi_0 - \mathcal{P}_t\psi_0\|_{C(\mathbb{T})}^k,$$

and

$$\|\mathcal{P}_{t+s}\psi_0 - \mathcal{P}_t\psi_0\|_{C(\mathbb{T})} \leqslant \|\mathcal{P}_t\left(\mathcal{P}_s\psi_0 - \psi_0\right)\|_{C(\mathbb{T})} \leqslant \|\mathcal{P}_s\psi_0 - \psi_0\|_{C(\mathbb{T})},$$

thanks to the semigroup property of $\{\mathcal{P}_r\}_{r\geqslant 0}$. □

4.3 The Feller property

Let ψ denote the solution to (1.1). The existence, as well as regularity, of ψ has been established already in Theorem 2.3. We now study the Markov properties of the infinite-dimensional process $\psi = \{\psi(t)\}_{t\geqslant 0}$.

As is customary in Markov process theory, let P_μ denote the law of the random field ψ starting according to initial measure μ on $C_+(\mathbb{T})$, and let E_μ denote the corresponding expectation operator. Until now, P and E referred to $\mathrm{P}_{\psi_0} := \mathrm{P}_{\delta_{\psi_0}}$ and $\mathrm{E}_{\psi_0} := \mathrm{E}_{\delta_{\psi_0}}$ for a given (fixed) function $\psi_0 \in C_+(\mathbb{T})$. This is customary in Markov process theory, and we will use both notations without further explanation.

For every $\Phi \in C_b(C_+(\mathbb{T}))$, define

$$(P_t\Phi)(\psi_0) := \mathrm{E}_{\psi_0}[\Phi(\psi(t))] \qquad \text{for every } t \geqslant 0 \text{ and } \psi_0 \in C_+(\mathbb{T}).$$

Proposition 4.13. *The $C_+(\mathbb{T})$-valued stochastic process $\{\psi(t)\}_{t\geqslant 0}$ is Feller; that is, $\{P_t\}_{t\geqslant 0}$ is a continuous semigroup of positive linear operators from $C_b(C_+(\mathbb{T}))$ to $C_b(C_+(\mathbb{T}))$.*

Proof. The proof is similar to the derivation of Proposition 5.6 by Cerrai (2003). Throughout, we choose and fix some $\Phi \in C_b(C_+(\mathbb{T}))$.

Recall the approximations $\{\psi_N\}_{N=1}^\infty$ of ψ from the proof of Theorem 2.3. A key feature of every ψ_N is that it solves a Walsh-type SPDE of the form (3.2) with Lipschitz-continuous coefficients (see Dalang, 1999; Walsh, 1986). In particular, $\{\psi_N(t)\}_{t\geqslant 0}$ is a Feller process for every $N \geqslant 1$ (see, for example the method of Nualart and Pardoux (1999)).

Let $\mathrm{P}_\mu^N, \mathrm{E}_\mu^N, P_t^N, \cdots$ denote the same quantities as $\mathrm{P}_\mu, \mathrm{E}_\mu, P_t, \cdots$, except with the random field ψ replaced everywhere by the random field ψ_N. It follows from the dominated convergence theorem that

$$P_t\Phi = \lim_{N\to\infty} P_t^N\Phi \qquad \text{for every } t \geqslant 0.$$

The Feller property of ψ_N implies, among other things, that

$$P_{t+s}^N\Phi = P_t^N(P_s^N\Phi) \qquad \text{for all } t, s \geqslant 0.$$

Let $N \to \infty$ in order to deduce the semigroup property of $\{P_t\}_{t\geqslant 0}$ from the above and the positivity of the operator P_t for every $t \geqslant 0$.

We next turn to the more interesting "Feller property." Note that $\psi = \{\psi(t)\}_{t\geqslant 0}$ is a Feller process if: (1) $P_t\Phi \in C_b(C_+(\mathbb{T}))$ for every $t > 0$ [P_0 is manifestly the identity operator]; and (2) the Markovian semigroup $\{P_t\}_{t\geqslant 0}$ is stochastically continuous.

We prove (1) and (2) separately and in this order.

For every $\varphi_0 \in C_+(\mathbb{T})$, let φ denote the unique solution to (1.1), using the same underlying white noise $\dot{W}$, subject to the fixed initial data φ_0. [The notation is consistent with our choice of $(\psi_0\,,\psi)$]. As first step of the proof, we are going to prove that

$$\lim_{\varphi_0\to\psi_0} \mathrm{E}\left(\|\psi(t)-\varphi(t)\|_{C(\mathbb{T})}\right) = 0. \tag{4.11}$$

This and the bounded convergence theorem together imply that $P_t\Phi \in C_b(C_+(\mathbb{T}))$ for every $t \geqslant 0$, which proves (1).

Next, let us recall that φ_N denotes the solution to (3.2) for every $N \geqslant 1$, started at a given φ_0, and recall that

$$T_N(\varphi_0) := \inf\left\{t \geqslant 0 :\ \|\varphi(t)\|_{C(\mathbb{T})} \geqslant N\right\},$$

with $\inf\varnothing := \infty$. [Thus, the stopping time T_N of the proof of Theorem 2.3 can now also be written as $T_N(\psi_0)$.] Then, $\lim_{N\to\infty} T_N(\varphi_0) = \infty$ a.s., thanks to Proposition 3.1. Because $\varphi(t) = \varphi_N(t)$ for all $t \in (0\,,T_N(\varphi_0)]$ a.s.,

$$\begin{aligned}\mathrm{E}\left(\|\psi(t)-\varphi(t)\|_{C(\mathbb{T})}\right) &\leqslant \mathrm{E}\left(\|\psi(t)-\psi_N(t)\|_{C(\mathbb{T})}\right) + \mathrm{E}\left(\|\varphi(t)-\varphi_N(t)\|_{C(\mathbb{T})}\right) + \mathrm{E}\left(\|\psi_N(t)-\varphi_N(t)\|_{C(\mathbb{T})}\right)\\ &\leqslant K_t\mathrm{P}\{T_N(\varphi_0)\wedge T_N(\psi_0)\leqslant t\} + \mathrm{E}\left(\|\psi_N(t)-\varphi_N(t)\|_{C(\mathbb{T})}\right),\end{aligned}$$

where

$$K_t := \mathrm{E}\left(\|\psi(t)\|_{C(\mathbb{T})}\right) + \mathrm{E}\left(\|\varphi(t)\|_{C(\mathbb{T})}\right) + \sup_{N\geqslant 1}\mathrm{E}\left(\|\psi_N(t)\|_{C(\mathbb{T})}\right) + \sup_{N\geqslant 1}\mathrm{E}\left(\|\varphi_N(t)\|_{C(\mathbb{T})}\right).$$

On one hand, Proposition 4.6 and its proof together show that $K_t < \infty$; in fact, the proof shows that $\sup_{t\geqslant\tau} K_t < \infty$ for every $\tau > 0$. On the other hand, since the drift [V_N] and diffusion [σ] terms in (3.2) are both Lipschitz continuous, a standard regularity estimate such as the one used in the proof of Proposition 4.6 implies that for every integer $N \geqslant 1$ and for all $t > 0$ there exists a number $K_{N,t} > 0$ – independent of $(\varphi_0\,,\psi_0)$ – such that

$$\mathrm{E}\left(\|\psi_N(t)-\varphi_N(t)\|_{C(\mathbb{T})}\right) \leqslant K_{N,t}\|\psi_0-\varphi_0\|_{C(\mathbb{T})}.$$

Thus, we find that

$$\mathrm{E}\left(\|\psi(t)-\varphi(t)\|_{C(\mathbb{T})}\right) \leqslant K_t\mathrm{P}\{T_N(\varphi_0)\wedge T_N(\psi_0)\leqslant t\} + K_{N,t}\|\psi_0-\varphi_0\|_{C(\mathbb{T})}.$$

We summarize by emphasizing that the preceding holds for every $t \geqslant 0$, $\varphi_0, \psi_0 \in C_+(\mathbb{T})$, and $N \in \mathbb{N}$, and that K_N and $K_{N,t}$ do not depend on the choice of $(\varphi_0\,,\psi_0)$.

Now let us choose and fix an arbitrary $\varepsilon > 0$ and $\varphi_0 \in C_+(\mathbb{T})$ such that $\|\varphi_0-\psi_0\|_{C(\mathbb{T})} \leqslant \varepsilon$. Since $T_N(\varphi_0)\wedge T_N(\psi_0) \to \infty$ a.s. as $N \to \infty$, we now choose and fix $N \in \mathbb{N}$ large enough to ensure that

$$\mathrm{P}\{T_N(\varphi_0)\wedge T_N(\psi_0)\leqslant t\} \leqslant \varepsilon,$$

whence

$$\mathrm{E}\left(\|\psi(t)-\varphi(t)\|_{C(\mathbb{T})}\right) \leqslant (K_t + K_{N,t})\varepsilon.$$

Since ε is arbitrary, this proves (4.11), whence also that $P_t\Phi \in C_b(C_+(\mathbb{T}))$ for every $t \geqslant 0$.

Finally, we verify that the Markovian semigroup $\{P_t\}_{t\geqslant 0}$ is stochastically continuous. Choose and fix some $\Phi \in C_b(C(\mathbb{T}))$. Then, for all $\psi_0 \in C_+(\mathbb{T})$ and $s, t \geqslant 0$,

$$|(P_{t+s}\Phi)(\psi_0) - (P_s\Phi)(\psi_0)| \leqslant \mathrm{E}_{\psi_0}\left(|\Phi(\psi_N(t+s)) - \Phi(\psi_N(s))|\right) + 2\|\Phi\|_{C(C(\mathbb{T}))}\mathrm{P}\left\{T_N(\psi_0) \leqslant t+s\right\}.$$

Because the coefficients V_N and σ in the SPDE (3.2) are Lipschitz continuous, the random field ψ_N is uniformly continuous in $L^1(\Omega)$; see Dalang (1999). Thus, the bounded convergence theorem implies that

$$\lim_{t\downarrow 0}|(P_{t+s}\Phi)(\psi_0) - (P_s\Phi)(\psi_0)| \leqslant 2\|\Phi\|_{C(C(\mathbb{T}))}\lim_{N\to\infty}\mathrm{P}\left\{T_N(\psi_0) \leqslant s\right\} = 0.$$

This verifies the desired stochastic continuity of $t \mapsto P_t$, and completes the proof. □

4.4 The Krylov-Bogoliubov argument

Define for every number $t \geqslant 0$ and all Borel sets $\Gamma \subset C_+(\mathbb{T})$ a probability measure νP_t as follows:

$$(\nu P_t)(\Gamma) := \mathrm{P}_\nu\{\psi(t) \in \Gamma\} = \int_{C_+(\mathbb{T})} \nu(\mathrm{d}\psi_0)\ (P_t\mathbf{1}_\Gamma)(\psi_0),$$

for every Borel regular probability measure ν on $C_+(\mathbb{T})$. Thus, $(t\,,\nu) \mapsto \nu P_t$ defines the transition functions of the $C_+(\mathbb{T})$-valued Markov process $\psi := \{\psi(t)\}_{t\geqslant 0}$ which solves uniquely the stochastic PDE (1.1). Recall that ν is an *invariant measure* for ψ [equivalently, for (1.1)] if

$$\nu P_t = \nu \qquad \text{for every } t \geqslant 0.$$

Recall also that $\delta_{\mathbb{0}}$ is always an invariant measure for (1.1), and of course is concentrated on the trivial solution, $\psi(t) = \mathbb{0}$ for all $t \geqslant 0$.

Our next effort is toward proving that if $\psi_0 \neq \mathbb{0}$, $\mathrm{L}_\sigma > 0$ [see (2.1)], and λ is sufficiently small, then there also exists a non-trivial invariant measure $\mu_+ \perp \delta_{\mathbb{0}}$. The standard way to do this sort of thing is to appeal to the Krylov-Bogoliubov argument (see Da Prato and Zabczyk, 1996, Corollary 3.1.2), which we shall recall. But first, let us state and prove a simple consequence of Theorem 2.3.

Lemma 4.14. *Let ν be any probability measure on $C_+(\mathbb{T})$ that is invariant for (1.1). Then,*

$$\nu\left\{\omega \in C_+(\mathbb{T}) : \inf_{x\in\mathbb{T}}\omega(x) = 0 < \sup_{x\in\mathbb{T}}\omega(x)\right\} = 0.$$

Among other things, Lemma 4.14 tells us that if ν is an invariant measure for (1.1) that satisfies $\nu\{\mathbb{0}\} = 0$, then

$$\nu\left\{\omega \in C_+(\mathbb{T}) : \inf_{x\in\mathbb{T}}\omega(x) > 0\right\} = 1.$$

Proof of Lemma 4.14. We can always decompose ν as

$$\nu = \eta\delta_{\mathbb{0}} + (1-\eta)\tilde{\nu},$$

where $\tilde{\nu}$ is a probability measure on $C_+(\mathbb{T})$ such that $\tilde{\nu}\{\mathbb{0}\} = 0$ and $\eta \in [0\,,1]$. Therefore, it suffices to prove the lemma with ν replaced by $\tilde{\nu}$. Alternatively, we can relabel $[\tilde{\nu} \to \nu]$ to see that we may – and will – assume without loss of generality that $\nu\{\mathbb{0}\} = 0$.

If $\psi(0) = \omega \in C_+(\mathbb{T}) \setminus \{\mathbb{0}\}$, then Theorem 2.3 ensures that

$$\mathrm{P}_\omega\left\{\psi(t\,,x) > 0 \text{ for all } t > 0 \text{ and } x \in \mathbb{T}\right\} = 1,$$

whence

$$\mathrm{P}_\omega\left\{\inf_{x\in\mathbb{T}}\psi(t\,,x)>0\right\}=1\quad\text{for all } t>0 \text{ and } \omega\in C_+(\mathbb{T})\setminus\{0\}.$$

[Note the exchange of P_ω with the "for all $t>0$" quantifier.] Integrate over all such ω [dν] and consider $t=1$ in order to see that

$$\mathrm{P}_\nu\left\{\inf_{x\in\mathbb{T}}\psi(1\,,x)>0\right\}=1.$$

This proves the lemma, since the P_ν-law of $\psi(1)$ is ν. □

When $\psi_0\in C_+(\mathbb{T})\setminus\{\mathbb{0}\}$ and $\mathrm{L}_\sigma>0$, Theorem 2.3 and Proposition 4.13 together imply that ψ a Feller process, taking values in the cone $C_{>0}(\mathbb{T})$ of all $f\in C(\mathbb{T})$ such that $f(x)>0$ for all $x\in\mathbb{T}$, endowed with relative topology.

We can now recall the following specialization of the Krylov-Bogoliubov theorem (Da Prato and Zabczyk, 1996, Corollary 3.1.2).

Lemma 4.15 (A Krylov–Bogoliubov theorem)**.** *Suppose there exists a probability measure ν on $C_{>0}(\mathbb{T})$ such that the probability measures*

$$\left\{\frac{1}{T}\int_0^T(\nu P_s)\,\mathrm{d}s\right\}_{T>0}$$

has a tight infinite subsequence in $C_{>0}(\mathbb{T})$. Then, ψ has an invariant measure which is a probability measure on $C_{>0}(\mathbb{T})$.

Because $\mathbb{0}\notin C_{>0}(\mathbb{T})$, the preceding invariant measure cannot be $\delta_{\mathbb{0}}$. This is the desired non-triviality result. Thus, our next challenge is to verify the conditions of Lemma 4.15. We apply that lemma with $\nu:=\delta_{\mathbb{1}}$. In order to discuss tightness, we need a family of compact subsets of $C_{>0}(\mathbb{T})$, which we build next.

Choose and fix some $\alpha\in(0\,,1/2)$, and define for all $\varepsilon,\delta\in(0\,,1)$,

$$A(\varepsilon):=\left\{f\in C^\alpha(\mathbb{T}):\ \inf_{x\in\mathbb{T}}f(x)\geqslant\varepsilon\right\}\quad\text{and}\quad B(\delta):=\left\{f\in C^\alpha(\mathbb{T}):\ \|f\|_{C^\alpha(\mathbb{T})}\leqslant 1/\delta\right\}.$$

According to the Arzèla-Ascoli theorem, every $B(\delta)$ is compact [in the topology of $C(\mathbb{T})$]. Since every $A(\varepsilon)$ is closed, $A(\varepsilon)\cap B(\delta)\subset C(\mathbb{T})$ is compact, and of course also in $C_{>0}(\mathbb{T})$. We propose to prove that if $\psi_0=\mathbb{1}$, $\mathrm{L}_\sigma>0$, and λ is sufficiently small, then

$$\lim_{\varepsilon,\delta\downarrow 0}\limsup_{T\to\infty}\frac{1}{T}\,\mathrm{E}_{\mathbb{1}}\left[\int_0^T\mathbf{1}_{\{\psi(t)\notin A(\varepsilon)\cap B(\delta)\}}\,\mathrm{d}t\right]=0. \tag{4.12}$$

Given that (4.12) is true, Lemma 4.15 [with $\nu=\delta_{\mathbb{1}}$] readily implies the following.

Proposition 4.16. *If $\psi_0=\mathbb{1}$, $\mathrm{L}_\sigma>0$, and λ is sufficiently small, then ψ has an invariant measure μ_+ on $C_{>0}(\mathbb{T})$.*

Proposition 4.6 and Chebyshev's inequality together imply that

$$\limsup_{T\to\infty}\frac{1}{T}\,\mathrm{E}_{\mathbb{1}}\left[\int_0^T\mathbf{1}_{\{\psi(t)\notin B(\delta)\}}\,\mathrm{d}t\right]\leqslant\sup_{t\geqslant 1}\mathrm{P}_{\mathbb{1}}\left\{\|\psi(t)\|_{C^\alpha(\mathbb{T})}>1/\delta\right\}=o(1)\qquad\text{as }\delta\downarrow 0.$$

Therefore, (4.12) – whence also Proposition 4.16 – follows as soon as we establish the following:

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty}\frac{1}{T}\,\mathrm{E}_{\mathbb{1}}\left[\int_0^T \mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t\right]=0.$$

Recall that Fatou's lemma can be recast as follows: If $\{\xi_T\}_{T>1}$ is a process that take values in $[0\,,1]$, then

$$\limsup_{T\to\infty}\mathrm{E}(\xi_T)\leqslant \mathrm{E}\left(\limsup_{T\to\infty}\xi_T\right).$$

We apply this with $\xi_T := T^{-1}\int_0^T \mathbf{1}_{\{\inf_{\mathbb{T}}\psi(t)<\varepsilon\}}\,\mathrm{d}t$, and then apply the monotone convergence theorem in order to be able to deduce the above, whence also Proposition 4.16, from the following assertion:

$$\mathrm{P}_{\mathbb{1}}\left\{\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t=0\right\}=1. \tag{4.13}$$

We prove (4.13) in the next few sections. This will complete our proof of Proposition 4.16.

5 A Random Walk Argument

For every continuous space-time process $h=\{h(t\,,x)\}_{t\geqslant 0,x\in\mathbb{T}}$ define

$$L_t(h):=\inf_{x\in\mathbb{T}}h(t\,,x)\quad\text{and}\quad U_t(h):=\sup_{x\in\mathbb{T}}h(t\,,x)\qquad\text{for every } t\geqslant 0.$$

If, in particular, $h\geqslant 0$ then $U_t(h)=\|h(t)\|_{C(\mathbb{T})}$ for all $t\geqslant 0$. Our goal is to construct comparison processes which give a series of lower bounds for $L_t(\psi)$, all the time controlling $U_t(\psi)$, where ψ denote a solution to (1.1) with the initial profile $\psi_0=\mathbb{1}$. Our argument repeatedly uses the comparison principle for SPDEs in the form of Lemma 3.3.

5.1 An associated chain

By the definition of F, we can choose and fix a strictly negative integer $M\in-\mathbb{N}$, sufficiently negative to ensure that there exists $c\in(0\,,1)$ such that $(1-c)v\leqslant V(v)=v-F(v)\leqslant v$ for all $v\in(0\,,2^{M+1})$; see Lemma 2.1. Of course the second inequality holds for all $v>0$. For simplicity, we assume that $c=1/2$. It will not be difficult to study the general case $c\in(0\,,1)$ after we adjust the argument to follow. In other words, we will proceed with the assumption that

$$\tfrac{1}{2}v\leqslant V(v)=v-F(v)\leqslant v\qquad\text{for all } v\in(0\,,2^{M+1}). \tag{5.1}$$

From now on, the symbol "M" will be used only for this purpose.

As was mentioned in the preamble to this section, throughout we let ψ denote a solution to (1.1) with the initial profile $\psi_0=\mathbb{1}$. Now we set up our random walk comparison.

We will define $\mathscr{F}$-stopping times $0=\tau_0<\tau_1<\cdots$ and comparison processes $v_0,v_1,\ldots$ such that

$$\psi(t)\geqslant v_n(t)\quad\text{for } t\in[\tau_n\,,\tau_{n+1}),\text{ everywhere on }\mathbb{T}. \tag{5.2}$$

For this reason, we will define $v_n(t)$ only for $t\geqslant\tau_n$.

To start the process, let us define

$$\tau_0:=0\quad\text{and}\quad v_0(0\,,x):=2^{M-2}\text{ for all } x\in\mathbb{T},$$

Now we proceed inductively on n. Suppose that we have defined τ_n and $v_n(\tau_n) = \{v_n(\tau_n\,,x)\}_{x\in\mathbb{T}}$. Let $\{\theta_t\}_{t\geqslant 0}$ denote the standard shift operator on our probability space. Informally speaking, this means that $\theta_t \dot{W}(s\,,x) := \dot{W}(s+t\,,x)$ for all $s,t \geqslant 0$ and $x \in \mathbb{T}$. More precisely, θ_t is defined via

$$\int_{\mathbb{R}_+\times\mathbb{T}} h(s\,,x)\,\theta_t W(\mathrm{d}s\,\mathrm{d}x) := \int_{(t,\infty)\times\mathbb{T}} h(s-t\,,x)\,W(\mathrm{d}s\,\mathrm{d}x) \quad \text{for all } t \geqslant 0 \text{ and } h \in L^2(\mathbb{R}_+\times\mathbb{T}).$$

With the above shifts in mind, we define w_n for $n \geqslant 0$ as the unique adapted and continuous solution to the following SPDE:

$$\begin{bmatrix} \partial_t w_n(t\,,x) = \partial_x^2 w_n(t\,,x) + \frac12 L_{\tau_n}(v_n) + \lambda\sigma(w_n(t\,,x))\theta_{\tau_n}\dot{W}(t\,,x) & \text{for } (t\,,x) \in (0\,,\infty)\times\mathbb{T}, \\ \text{subject to } w_n(0\,,x) = L_{\tau_n}(v_n) & \text{for all } x \in \mathbb{T}. \end{bmatrix}$$

Define τ_{n+1} to be the smallest $t+\tau_n > \tau_n$ that at least one of the following occurs:

1. $L_t(w_n) = 2L_{\tau_n}(v_n)$
2. $L_t(w_n) = \frac12 L_{\tau_n}(v_n)$
3. $U_t(w_n) = 4L_{\tau_n}(v_n)$.

If such a t does not exist, then $\tau_{n+1} := \infty$. If such a t does exist, then we let

$$v_n(\tau_n + t\,,x) := w_n(t\,,x) \qquad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{T}.$$

In case 1, that is a.s. on $\{L_{\tau_{n+1}}(v_n) = 2L_{\tau_n}(v_n)\} \cup \{\tau_{n+1} < \infty\}$, we let

$$v_{n+1}(\tau_{n+1}, x) := \begin{cases} 2L_{\tau_n}(v_n) & \text{if } L_{\tau_n}(v_n) \leqslant 2^{M-2}, \\ 2^{M-2} & \text{if } L_{\tau_n}(v_n) \geqslant 2^{M-1}, \end{cases}$$

for every $x \in \mathbb{T}$. And in cases 2 and 3, that is a.s. on

$$\left\{L_{\tau_{n+1}}(v_n) = \tfrac12 L_{\tau_n}(v_n)\right\} \cup \{U_t(v_n) = 4L_{\tau_n}(v_n)\} \cup \{\tau_{n+1} < \infty\},$$

we define

$$v_{n+1}(\tau_{n+1}, x) := \tfrac12 L_{\tau_n}(v_n) \qquad \text{for every } x \in \mathbb{T}.$$

If $\tau_{n+1} < \infty$ a.s., then by (5.1), the Markov property, and the comparison theorem for SPDEs, (5.2) holds almost surely. This finishes our inductive construction, provided that we can prove the following.

Lemma 5.1. *If* $\mathrm{L}_\sigma > 0$, *then* $\mathrm{P}_{\mathbb{1}}\{\tau_{n+1} < \infty\} = 1$ *for all* $n \in \mathbb{Z}_+$.

Proof. Since $\tau_0 = 0$, we may [and will] assume that we have proved that $\mathrm{P}_{\mathbb{1}}\{\tau_n < \infty\} = 1$ for some $n \in \mathbb{Z}_+$, and proceed to prove that $\mathrm{P}_{\mathbb{1}}\{\tau_{n+1} < \infty\} = 1$. With this in mind, choose and fix some $n \in \mathbb{Z}_+$ and suppose that $\tau_n < \infty$ a.s. Then, for every $t > 0$ and $x \in \mathbb{T}$,

$$w_n(t\,,x) = \left(1 + \frac{t}{2}\right) L_{\tau_n}(v_n) + \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(w_n(s\,,y))\,\theta_{\tau_n} W(\mathrm{d}s\,\mathrm{d}y) \qquad \text{a.s.,}$$

and

$$\tau_{n+1} \leqslant \hat{\tau}_{n+1} := \inf\left\{t > 0 :\ L_t(w_n) \text{ or } U_t(w_n) \notin \left[\tfrac12 L_{\tau_n}(v_n)\,, 4L_{\tau_n}(v_n)\right]\right\},$$

where $\inf\varnothing = \infty$. Note that $\hat{\tau}_{n+1}$ is a stopping time with respect to the filtration $\mathscr{F}$. It remains to prove that $\hat{\tau}_{n+1} < \infty$ $\mathrm{P}_{\mathbb{1}}$-a.s.

Define

$$\hat{w}_n(t) := \int_{\mathbb{T}} w_n(t\,,x)\,\mathrm{d}x \qquad \text{for all } t \geqslant 0.$$

A stochastic Fubini argument yields

$$\hat{w}_n(t) = (2+t)L_{\tau_n}(v_n) + \lambda\mathcal{M}_t \qquad \text{for all } t \geqslant 0, \tag{5.3}$$

where

$$\mathcal{M}_t := \int_{(0,t)\times\mathbb{T}} \sigma(w_n(s\,,y))\,\theta_{\tau_n} W(\mathrm{d}s\,\mathrm{d}y) \qquad \text{for all } t \geqslant 0$$

defines a continuous $L^2(\mathrm{P})$-martingale. Since σ is Lipschitz and $\sigma(0) = 0$, it follows that $|\sigma(z)| \leqslant \mathrm{Lip}_\sigma|z|$ for all $z \in \mathbb{R}$. Therefore, the quadratic variation of $\mathcal{M}$ is given by

$$\langle\mathcal{M}\rangle_t = \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \sigma^2(w_n(s\,,y)) \leqslant \mathrm{Lip}_\sigma^2 \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \ |w_n(s\,,y)|^2 \quad \text{for all } t \geqslant 0.$$

It follows from this inequality that

$$\langle\mathcal{M}\rangle_t \leqslant 32[\mathrm{Lip}_\sigma L_{\tau_n}(v_n)]^2 t \qquad \text{for all } t \geqslant 0 \text{ a.s. on } \{\hat{\tau}_{n+1} = \infty\}.$$

The law of large numbers for continuous $L^2(\mathrm{P})$-martingales then implies that

$$\limsup_{t\to\infty} \left|\frac{\mathcal{M}_t}{t}\right| \leqslant 32\,[\mathrm{Lip}_\sigma L_{\tau_n}(v_n)]^2 \lim_{t\to\infty} \left|\frac{\mathcal{M}_t}{\langle\mathcal{M}\rangle_t}\right| = 0 \qquad \text{a.s. on } \{\hat{\tau}_{n+1} = \infty\} \cap \mathcal{E}, \tag{5.4}$$

for the event $\mathcal{E} := \{\lim_{t\to\infty}\langle\mathcal{M}\rangle_t = \infty\}$. Since $\mathrm{L}_\sigma > 0$, the inequality $|\sigma(z)| \geqslant \mathrm{L}_\sigma|z|$ – valid for all $z \in \mathbb{R}$ – has content, and implies that $\langle\mathcal{M}\rangle_t \geqslant \frac{1}{2}\mathrm{L}_\sigma^2 L_{\tau_n}^2(v_n)t$ for all $t \geqslant 0$ a.s. on $\{\hat{\tau}_{n+1} = \infty\}$. In particular,

$$\mathrm{P}_{\mathbb{1}}\left(\{\hat{\tau}_{n+1} = \infty\} \cap \mathcal{E}\right) = \mathrm{P}_{\mathbb{1}}\{\hat{\tau}_{n+1} = \infty\}.$$

This fact, (5.4), and (5.3) together imply that $\lim_{t\to\infty} t^{-1}\hat{w}_n(t) = L_{\tau_n}(v_n) > 0$ a.s. on $\{\hat{\tau}_{n+1} = \infty\}$, whence

$$\mathrm{P}_{\mathbb{1}}\left\{\sup_{t\geqslant 0}\hat{w}_n(t) = \infty\ ,\ \ \hat{\tau}_{n+1} = \infty\right\} = \mathrm{P}_{\mathbb{1}}\left\{\hat{\tau}_{n+1} = \infty\right\}.$$

Since

$$\sup_{t\geqslant 0}\hat{w}_n(t) \leqslant 4L_{\tau_n}(v_n) < \infty \quad \text{a.s. on} \quad \{\hat{\tau}_{n+1} = \infty\},$$

it follows that $\mathrm{P}_{\mathbb{1}}\{\hat{\tau}_{n+1} = \infty\} = 0$, as was claimed. □

Now we define an embedded "reflected chain" X, along with the length of time τ_n for step n:

$$\begin{aligned} X_n &= \log_2 L_{\tau_n}(v_n), && \text{for } n \geqslant 0,\\ \ell_n &= \tau_n - \tau_{n-1}, && \text{for } n \geqslant 1. \end{aligned}$$

Let us pause to explain why we refer to X as a "reflected chain." Firstly, the strong Markov property of every infinite-dimensional diffusion $\{w_n(t)\}_{t\geqslant 0}$ implies that for every $n \in \mathbb{Z}_+$, and given the value of X_n, the random variable X_{n+1} is conditionally independent of $\mathscr{F}_{\tau_n}$, and clearly $X_0 = M-2$; thus, X is a time-inhomogenous Markov chain that starts at $M-2$. Secondly, the definition of X implies immediately that X moves in three distinct ways: $X_{n+1} - X_n = \pm 1$ for all $n \geqslant 0$, and in all cases, except that $X_{n+1} - X_n = M - 2 - X_n \leqslant -1$ a.s. on $\{X_n \geqslant M-1\}$. We find that $X_0 = M-2$ and from time one onward, X moves in increments of ± 1 and is reflected at $M-1$ by an increment of -1 to stay in $\mathbb{Z}\cap(-\infty\,,M-1]$ henceforth. The latter exception ensures that $X_n \leqslant M-1$ for all $n\geqslant 1$, and explains the use of "reflection at $M-1$." Finally, we will soon demonstrate that, if the parameter λ in (1.1) is small enough, then

$$\mathrm{P}(X_{n+1}-X_n=+1 \mid X_n) \geqslant \tfrac{2}{3} \quad \text{for all } n\geqslant 1,\ \mathrm{P}_{\mathbb{1}}\text{-a.s. on the event } \{X_n<M-1\}.$$

This will imply that X moves upward at least as fast as a random walk with upward drift, when it is not being reflected. We now return to the main part of the discussion.

In order to study the walk X we make some definitions. For simplicity, define for all $n\in\mathbb{N}$, $t\geqslant 0$, and $x\in\mathbb{T}$,

$$\mathcal{I}_n(t\,,x) := \lambda\int_{(0,t)\times\mathbb{T}} p_{t-s}(x-y)\sigma(v_n(\tau_n+s\,,y))\,\theta_{\tau_n}W(\mathrm{d}y\,\mathrm{d}s).$$

Note that, in the definition of $\mathcal{I}_n(t\,,x)$, the parameter t refers to the time beyond τ_n. Thanks to the mild formulation of w_n, and hence also v_n, the following is valid a.s. for every $t\in[0\,,\ell_{n+1}]$:

$$\begin{aligned} w_n(t\,,x) = v_n(\tau_n+t\,,x) &= 2^{X_n} + t2^{X_n-1} + \mathcal{I}_n(t\,,x)\\ &= \left(1+\frac{t}{2}\right)L_{\tau_n}(v_n) + \mathcal{I}_n(t\,,x). \end{aligned} \tag{5.5}$$

Next we estimate $\mathcal{I}_n$. For possibly random numbers $T\geqslant 0$ and $\delta>0$, consider the event

$$A_n(T,\lambda\,,\delta) := \left\{\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{T}}|\mathcal{I}_n(t\,,x)|\leqslant\delta\right\}.$$

By (2.1) and by the definition of τ_n and τ_{n+1},

$$|\sigma(v_n(t\,,x))| \leqslant 4\mathrm{Lip}_\sigma L_{\tau_n}(v_n) \quad \text{for every } t\in[\tau_n\,,\tau_{n+1}], \tag{5.6}$$

almost surely. We define a truncated version of σ and corresponding versions of $\mathcal{I}$ and A as follows:

$$\sigma_n(t\,,x) = \sigma(v_n(\tau_n+t\,,x))\mathbf{1}_{(0,\ell_{n+1})}(t), \tag{5.7}$$

$$\mathcal{I}_n^{\mathrm{tr}}(t\,,x) = \lambda\int_{(0,t)\times\mathbb{T}} p_{t-s}(x-y)\sigma_n(s\,,y)\,\theta_{\tau_n}W(\mathrm{d}y\,\mathrm{d}s), \tag{5.8}$$

$$A_n^{\mathrm{tr}}(T,\lambda\,,\delta) = \left\{\sup_{t\in[0,T]}\sup_{x\in\mathbb{T}}|\mathcal{I}_n^{\mathrm{tr}}(t\,,x)|\leqslant\delta\right\}.$$

Since $\sigma_n=0$ beyond time ℓ_{n+1}, elementary properties of the Walsh stochastic integral ensure that a.s. on the event $\{\ell_{n+1}<T\}$,

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{T}}|\mathcal{I}_n^{\mathrm{tr}}(t\,,x)|\leqslant\delta \quad\Leftrightarrow\quad \sup_{0\leqslant t\leqslant \ell_{n+1}}\sup_{x\in\mathbb{T}}|\mathcal{I}_n(t\,,x)|\leqslant\delta.$$

Because the underlying probability space is assumed to be complete, it follows that

$$A_n^{\mathrm{tr}}(T,\lambda\,,\delta)\cap\{\ell_{n+1}<T\} = A_n(\ell_{n+1}\,,\lambda\,,\delta)\cap\{\ell_{n+1}<T\} \qquad \text{a.s.}$$

Next we give a probabilistic estimate for $\mathcal{I}_n^{\mathrm{tr}}(t\,,x)$. To keep the flow of the argument, we postpone the proof until the end of the paper.

Lemma 5.2. *There exist numbers $C_0, C_1 > 0$ such that*

$$\mathrm{P}\left(\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{T}} |\mathcal{I}_n^{tr}(t\,,x)| > \rho \;\middle|\; \mathscr{F}_{\tau_n} \right) \leqslant C_0 \exp\left(-\frac{C_1\rho^2}{16T^{1/2}\lambda^2 \mathrm{Lip}_\sigma^2 L_{\tau_n}^2(v_n)} \right),$$

$\mathrm{P}_{\mathbb{1}}$*-a.s. for every $\rho, T > 0$ and $n \geqslant 1$.*

For convenience let us define $\mathscr{P}_n$ to be the conditional measure,

$$\mathscr{P}_n(\cdot) = \mathrm{P}(\,\cdot \mid \mathscr{F}_{\tau_n}).$$

We now select

$$\delta := \tfrac{1}{4} L_{\tau_n}(v_n). \tag{5.9}$$

Note that δ is random, but $\mathscr{F}_{\tau_n}$-measurable. Moreover, (5.5) assures us that, as long as $T \geqslant 3$, the following holds a.s. on $A_n^{\mathrm{tr}}(T, \lambda\,, \delta) \cap \{\ell_{n+1} > T\}$:

$$v_n(\tau_n + T, x) \geqslant \tfrac{5}{2} L_{\tau_n}(v_n) - \delta > 2L_{\tau_n}(v_n).$$

This proves that $A_n^{\mathrm{tr}}(T, \lambda\,, \delta) = A_n^{\mathrm{tr}}(T, \lambda\,, \delta) \cap \{\ell_{n+1} \leqslant T\}$ a.s. for (say) $T = 3$. In addition, (5.5) ensures that a.s. on $A_n^{\mathrm{tr}}(T, \lambda\,, \delta)$,

$$\tfrac{1}{2} L_{\tau_n}(v_n) < L_{\tau_n}(v_n) - \delta \leqslant v_n(\tau_n + t\,, x) \leqslant \tfrac{5}{2} L_{\tau_n}(v_n) + \delta < 4L_{\tau_n}(v_n),$$

for all $t \in [0\,, \ell_{n+1}]$ and $x \in \mathbb{T}$ a.s. That is, $\ell_{n+1} \leqslant T$ a.s. on $A_n^{\mathrm{tr}}(T, \lambda\,, \delta)$, and we are in case 1 (i.e., $L_t(w_n) = 2L_{\tau_n}(v_n)$). This is another way to say that the random walk X moves up except possibly at the reflecting boundary.

We apply Lemma 5.2, conditionally on $\mathscr{F}_{\tau_n}$, in order to see that

$$\mathscr{P}_n\left(A_n^{\mathrm{tr}}(T, \lambda\,, \delta)^c\right) \leqslant C_0 \exp\left(-\frac{C_1\delta^2}{T^{1/2}\lambda^2 (4\mathrm{Lip}_\sigma L_{\tau_n}(v_n))^2} \right) = C_0 \exp\left(-\frac{C_1}{256\sqrt{T}\,\mathrm{Lip}_\sigma^2 \lambda^2} \right), \tag{5.10}$$

$\mathrm{P}_{\mathbb{1}}$-a.s. uniformly for all $n \geqslant 1$, $T > 0$, and every initial choice of $\lambda > 0$ in (1.1). We emphasize that the right-most quantity in (5.10) is non random. In any case, we apply the above with $T = 3$ in order to see that there exists a non-random number $\lambda_1 = \lambda_1(C_1\,, \sigma) \in (0\,, 1)$ such that

$$\mathscr{P}_n\big(X_{n+1} = (X_n + 1)\big)1_{\{X_n \leqslant M-2\}} \geqslant \mathscr{P}_n(A_n^{\mathrm{tr}}(3\,, \lambda\,, \delta))1_{\{X_n \leqslant M-2\}} \geqslant \frac{2}{3} 1_{\{X_n \leqslant M-2\}}, \tag{5.11}$$

$\mathrm{P}_{\mathbb{1}}$-a.s. for every $\lambda \in (0\,, \lambda_1)$ and $n \geqslant 1$. This proves that the random walk X has an upward drift, thereby concluding our random walk argument.

5.2 A reduction

In this subsection we reduce our proof of the existence of non-trivial invariant measures [Proposition 4.16] to condition (5.12) below. That condition will be verified in the following subsection provided additionally that $\mathrm{L}_\sigma > 0$ and λ is sufficiently small.

Proposition 5.3. *Suppose $\lambda \in (0\,, 1)$ is small enough to ensure that (5.11) holds. Then, Condition (4.13) – whence also Proposition 4.16 – follows provided that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{\{X_{j+1} < -k\}} = 0 \qquad \textit{a.s.} \tag{5.12}$$

Proposition 5.3 hinges on two coupling lemmas that respectively bound $\{\ell_n\}_{n=1}^\infty$ from above and from below by "better behaved" sequences of random variables.

Lemma 5.4. *Suppose $\lambda \in (0\,,1)$ is small enough to ensure that (5.11) holds. Then, there exists a sequence $\{\underline{\ell}_n\}_{n=1}^\infty$ of i.i.d. random variables such that $\ell_n \geqslant \underline{\ell}_n \geqslant 0$ a.s. for all $n \geqslant 1$, and $0 < \mathrm{E}_{\mathbb{1}}[\underline{\ell}_1] \leqslant 1$.*

Proof. Choose and fix a small enough λ, as designated, and recall the $\mathscr{F}_{\tau_n}$-measurable random variable δ from (5.9) for every $n \in \mathbb{Z}_+$. For every $T > 0$, let

$$R(T) := C_0 \exp\left(-\frac{C_1}{256\sqrt{T}\,\mathrm{Lip}_\sigma^2}\right)$$

denote the supremum of the right-hand side of (5.10) over all $\lambda \in (0\,,1)$. Throughout, we choose and fix $T \in (0\,,1)$ such that $R(T) < 1$.

Because $A_n^{\mathrm{tr}}(T,\lambda\,,\delta)^c$ is independent of $\mathscr{F}_{\tau_n}$, we can enlarge the event $A_n^{\mathrm{tr}}(T,\lambda\,,\delta)^c$ to an event $A_n^{(1)}(T)^c$ whose probability is exactly equal to $R(T) < 1$, and such that $A_n^{(1)}(T)$ is independent of $\mathscr{F}_{\tau_n}$. By virtue of construction,

$$A_n^{\mathrm{tr}}(T,\lambda\,,\delta) \supseteq A_n^{(1)}(T).$$

In addition, using (5.5), we see that a.s. on $A_n^{\mathrm{tr}}(T,\lambda\,,\delta) \cap \{\ell_{n+1} < T\}$, we have

$$\sup_{t\in[0,\ell_{n+1}]}\sup_{x\in\mathbb{T}} w_n(t\,,x) < 2L_{\tau_n}(v_n),$$
$$\inf_{t\in[0,\ell_{n+1}]}\inf_{x\in\mathbb{T}} w_n(t\,,x) > \tfrac{1}{2}L_{\tau_n}(v_n).$$

Thus, $A_n^{\mathrm{tr}}(T,\lambda\,,\delta) = A_n^{\mathrm{tr}}(T,\lambda\,,\delta) \cap \{\ell_{n+1} \geqslant T\}$, and $\ell_{n+1} \geqslant T$ a.s. on $A_n^{\mathrm{tr}}(T,\lambda\,,\delta)$. With this in mind, we can define

$$\underline{\ell}_{n+1} := T\mathbf{1}_{A_n^{(1)}(T)} \leqslant T\mathbf{1}_{A_n^{\mathrm{tr}}(T,\lambda,\delta)} \leqslant \ell_{n+1} \wedge T \leqslant \ell_{n+1} \wedge 1.$$

This is the desired sequence, with the announced properties. ☐

Lemma 5.4 provides a suitable lower sequence for $\{\ell_n\}_{n=1}^\infty$. The following matches that result with a corresponding upper sequence.

Lemma 5.5. *There exists a sequence $\{\bar{\ell}_n\}_{n=1}^\infty$ of almost surely nonnegative i.i.d. random variables such that with probability one for all $n \geqslant 1$ we have $\ell_n \leqslant \bar{\ell}_n$. Finally, $0 < \mathrm{E}_{\mathbb{1}}(|\bar{\ell}_1|^k) < \infty$ for every real number $k \geqslant 2$.*

Proof. By (5.5), if $\ell_{n+1} > T$, then

$$2^{X_n} + T2^{X_n-1} + \mathcal{I}_n^{\mathrm{tr}}(T,x) \leqslant 2^{X_n+2}.$$

Thus, if $\ell_{n+1} > T$ and T is large enough, we have

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{T}} |\mathcal{I}_n^{\mathrm{tr}}(t\,,x)| > 2^{X_n} + T2^{X_n-1} - 2^{X_n+2}.$$

That is, $\mathcal{I}_n^{\mathrm{tr}}(t\,,x)$ must counteract the drift which is pushing v_{n+1} out of the interval around $2^{X_n} = L_{\tau_n}(v_n)$.

Choose T_0 so large that $T > T_0$ implies that

$$2^{X_n} + T2^{X_n-1} - 2^{X_n+2} > T2^{X_n-2} = \frac{TL_{\tau_n}(v_n)}{4}.$$

Then using Lemma 5.2 we can choose T_0 so large that for $T > T_0$, the following holds $\mathrm{P}_{\mathbb{1}}$-a.s.:

$$\begin{aligned}\mathrm{P}(\ell_{n+1} > T \mid \mathscr{F}_{\tau_n}) &\leqslant \mathrm{P}\left(\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{T}} |\mathcal{I}_n^{\mathrm{tr}}(t\,,x)| > \frac{TL_{\tau_n}(v_n)}{4} \;\middle|\; \mathscr{F}_{\tau_n}\right)\\ &\leqslant \left[c_0 \exp\left(-\frac{c_1 T^{3/2}}{\lambda^2}\right)\right] \wedge 1;\end{aligned}$$

where c_0, c_1 are non-random, positive real numbers that do not depend on $(n\,,\lambda)$. By the strong Markov property, ℓ_{n+1} is independent of $\mathscr{F}_{\tau_n}$ though it is measurable with respect to $\mathscr{F}_{\tau_{n+1}}$ by virtue of construction. Therefore, Lemma A.2 ensures that we can find a random variable $\bar{\ell}_{n+1}$, independent of $\mathscr{F}_{\tau_n}$, such that $\ell_{n+1} \leqslant \bar{\ell}_{n+1}$ and

$$\mathrm{P}_{\mathbb{1}}\left\{\bar{\ell}_{n+1} > t\right\} = c_0 \exp\left(-\frac{c_1 t^{3/2}}{\lambda^2}\right) \wedge 1 \qquad \text{for all } t \geqslant T_0,$$

and $\{\bar{\ell}_m\}_{m=1}^{\infty}$ is i.i.d. $[\mathrm{P}_{\mathbb{1}}]$. Finally, we can study the moments of ℓ_1 as follows: On one hand, Lemma 5.4 ensures that

$$\|\bar{\ell}_1\|_2 \geqslant \mathrm{E}_{\mathbb{1}}(\bar{\ell}_1) \geqslant \mathrm{E}_{\mathbb{1}}(\ell_1) \geqslant \mathrm{E}_{\mathbb{1}}(\underline{\ell}_1) > 0.$$

On the other hand,

$$\mathrm{E}_{\mathbb{1}}(|\bar{\ell}_1|^k) = k\int_0^\infty t^{k-1}\mathrm{P}_{\mathbb{1}}\{\bar{\ell}_{n+1} > t\}\,\mathrm{d}t \leqslant T_0^k + kc_0\int_{T_0}^\infty t^{k-1}\exp\left(-\frac{c_1 t^{3/2}}{\lambda^2}\right)\mathrm{d}t < \infty,$$

for every $k \geqslant 2$. □

We are ready to prove Proposition 5.3.

Proof of Proposition 5.3. Thanks to (5.2) we may observe that for every $n \in \mathbb{N}$ and $\varepsilon \in (0\,,1/8)$,

$$\begin{aligned}\frac{1}{\tau_n}\int_0^{\tau_n} \mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t &= \frac{1}{\tau_n}\sum_{j=0}^{n-1}\int_{\tau_j}^{\tau_{j+1}} \mathbf{1}_{\{L_t(\psi)<\varepsilon\}}\,\mathrm{d}t\\ &\leqslant \frac{1}{\tau_n}\sum_{j=0}^{n-1}\int_{\tau_j}^{\tau_{j+1}} \mathbf{1}_{\{L_t(v_j)<\varepsilon\}}\,\mathrm{d}t\\ &\leqslant \frac{1}{\tau_n}\sum_{j=0}^{n-1}\int_{\tau_j}^{\tau_{j+1}} \mathbf{1}_{\{U_t(v_j)<8\varepsilon\}}\,\mathrm{d}t\end{aligned}$$

$\mathrm{P}_{\mathbb{1}}$-almost surely. The very construction of the stopping times $\tau_1, \tau_2, \ldots$ ensures that if there exists $t \in [\tau_j\,,\tau_{j+1}]$, and a realization of the process ψ, such that $L_t(v_j) < \varepsilon$, then certainly $U_t(v_j) \leqslant 8\varepsilon < 1$ for all $t \in [\tau_j\,,\tau_{j+1}]$ [for that realization of ψ], whence also $L_{\tau_{j+1}}(v_{j+1}) \leqslant 8\varepsilon < 1$, for the same realization of ψ. In this way, we find that

$$\frac{1}{\tau_n}\int_0^{\tau_n} \mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t \leqslant \frac{1}{\tau_n}\sum_{j=0}^{n-1}\ell_{j+1}\mathbf{1}_{\{X_{j+1}\leqslant -|\log_2(8\varepsilon)|\}} \qquad \mathrm{P}_{\mathbb{1}}\text{-a.s.}$$

If λ is sufficiently small, then Lemma 5.4 and the strong law of large numbers together imply that

$$\liminf_{n\to\infty}\frac{\tau_n}{n}=\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\ell_j\geqslant\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\underline{\ell}_j=\mathrm{E}(\underline{\ell}_1)>0\qquad\mathrm{P}_{\mathbb{1}}\text{-a.s.}\tag{5.13}$$

Similarly, Lemma 5.5 ensures that

$$\limsup_{n\to\infty}\frac{\tau_n}{n}\leqslant\mathrm{E}(\bar{\ell}_1)<\infty\qquad\mathrm{P}_{\mathbb{1}}\text{-a.s.}\tag{5.14}$$

The Cauchy-Schwarz inequality and (5.13) together yield the following: $\mathrm{P}_{\mathbb{1}}$-a.s.,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n}\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t&\leqslant\frac{1}{\mathrm{E}_{\mathbb{1}}(\underline{\ell}_1)}\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ell_{j+1}\mathbf{1}_{\{X_{j+1}\leqslant-|\log_2(8\varepsilon)|\}}\\&\leqslant\frac{1}{\mathrm{E}_{\mathbb{1}}(\underline{\ell}_1)}\sqrt{\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\ell_{j+1}^2}\sqrt{\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}_{\{X_{j+1}\leqslant-|\log_2(8\varepsilon)|\}}}.\end{aligned}$$

Thanks to Lemma 5.5, we may deduce from the above that $\mathrm{P}_{\mathbb{1}}$-a.s.,

$$\limsup_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n}\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t\leqslant\frac{\|\bar{\ell}_1\|_2}{\|\underline{\ell}_1\|_1}\sqrt{\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}_{\{X_{j+1}\leqslant-|\log_2(8\varepsilon)|\}}}.$$

This proves that (5.12) implies (4.13), except the non-random averaging variable $T\to\infty$ is replaced by the random averaging variable $\tau_n\to\infty$. In order to complete the proof, let us choose and fix 2 numbers a and b such that

$$\mathrm{E}(\bar{\ell}_1)>b\geqslant a>\mathrm{E}(\underline{\ell}_1).$$

For every $T\gg 1$ let $n=n(T)=\lceil T/a\rceil$, so that $a(n-1)<T\leqslant an$ and $n\geqslant 3$, whence $(n-1)^{-1}\leqslant 1/2$. By enlarging $n(T)$ further to a finite random number, if need be, (5.13) and (5.14) together ensure that

$$an\leqslant\tau_n\leqslant bn,$$

for all T sufficiently large. In this way we find that, for all sufficiently large $T\gg 1$,

$$\begin{aligned}\frac{1}{T}\int_0^T\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t&\leqslant\frac{1}{a(n-1)}\int_0^{an}\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t\\&\leqslant\frac{2}{an}\int_0^{an}\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t\\&\leqslant\frac{2b}{a\tau_n}\int_0^{\tau_n}\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t.\end{aligned}$$

Now let $T\to\infty$ first, and then $a\to\mathrm{E}(\underline{\ell}_1)$ and $b\to\mathrm{E}(\bar{\ell}_1)$ in order to see that $\mathrm{P}_{\mathbb{1}}$-a.s.,

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\mathbf{1}_{\{\inf_{x\in\mathbb{T}}\psi(t,x)<\varepsilon\}}\,\mathrm{d}t\leqslant\frac{2\|\bar{\ell}_1\|_2^2}{\|\underline{\ell}_1\|_1^2}\sqrt{\limsup_{m\to\infty}\frac{1}{m}\sum_{j=0}^{m-1}\mathbf{1}_{\{X_{j+1}\leqslant-|\log_2(8\varepsilon)|\}}}.\tag{5.15}$$

This proves Proposition 5.3. □

5.3 Proof of Proposition 4.16

We are ready to begin completing the proof of Proposition 4.16, which assures us of the existence of non-trivial invariant measures when $\mathrm{L}_\sigma > 0$ and λ is sufficiently small. Our method requires an analysis of the excursions of the chain X from the level $M-1$. The construction of the chain X ensures that

$$\mathrm{P}_{\mathbb{1}}\{X_0 = M-2\} = 1.$$

Moreover, $\mathrm{P}_{\mathbb{1}}\{|X_{n+1} - X_n| = 1\} = 1$ for every $n \geqslant 1$.

Set $\alpha_0 := 0$ and for all $n \in \mathbb{Z}_+$ define

$$\alpha_{n+1} := \inf\{j > \alpha_n :\, X_j = M-1\},$$

where $\inf \varnothing := \infty$. Then, the α_n's are stopping times in the filtration $\{\mathscr{F}_{\tau_n}\}_{n=0}^\infty$.

Lemma 5.6. $\mathrm{P}_{\mathbb{1}}\{\alpha_n < \infty\} = 1$ *for every* $n \geqslant 1$.

Proof. The proof works by coupling the inhomogeneous Markov chain X to an infinite family of independent, biased random walks. This coupling is the motivation behind the title of this section ("a random walk argument"), and will be useful in the sequel as well.

First we prove that $\mathrm{P}_{\mathbb{1}}\{\alpha_1 < \infty\} = 1$. Define a random walk $\{Y_n^{(1)}\}_{n=1}^\infty$ on $\mathbb{Z}$ as follows:

1. $Y_0^{(1)} := X_0 = M-2$;
2. Iteratively define $Y_n^{(1)}$ for every $n \geqslant 1$ as follows:
   (a) When $X_{n+1} - X_n = -1$, set $Y_{n+1}^{(1)} = Y_n^{(1)} - 1$;
   (b) Next, let us introduce new variables $\{\Delta_m\}_{m=1}^\infty$ that are independent of one another, such that a.s. on $\{X_m \leqslant M-2\}$,
   $$\mathrm{P}\{\Delta_m = +1\} = 1 - \mathrm{P}\{\Delta_m = -1\} = \frac{2}{3\mathrm{P}(X_{m+1} - X_m = +1 \mid \mathscr{F}_{\tau_m})},$$
   for every $m \geqslant 0$ such that $\mathrm{P}_{\mathbb{1}}\{X_m \leqslant M-2\} > 0$. For all other values of m, $\mathrm{P}\{\Delta_m = +1\} = 0$. The preceding is a well-defined construction thanks to (5.11). Now we set $Y_{n+1}^{(1)} := Y_n^{(1)} + \Delta_n$ whenever $X_{n+1} = X_n + 1$ and $\alpha_1 > n$; and
   (c) Let $\{Z_m\}_{m=1}^\infty$ be an independent, biased, simple random walk on $\mathbb{Z}$ whose left-right probabilities given by $\mathrm{P}\{Z_1 = +1\} = 1 - \mathrm{P}\{Z_1 = -1\} = 2/3$. Finally, define $Y_{n+1}^{(1)} := Y_n^{(1)} + Z_{n+1-\alpha_1}$ whenever $X_{n+1} = X_n + 1$ and $\alpha_1 \leqslant n$.

The above construction shows that $\{Y_n^{(1)}\}_{n=1}^\infty$ is a simple random walk on $\mathbb{Z}$ such that:

(i) $Y_0^{(1)} = M-2$;
(ii) $\mathrm{P}_{\mathbb{1}}\{Y_{n+1}^{(1)} - Y_n^{(1)} = +1\} = 1 - \mathrm{P}_{\mathbb{1}}\{Y_{n+1}^{(1)} - Y_n^{(1)} = -1\} = 2/3$ for all $n \geqslant 0$; and
(iii) $\mathrm{P}_{\mathbb{1}}\{Y_n^{(1)} \leqslant X_n \text{ for all } 1 \leqslant n \leqslant \alpha_1\} = 1$, where $Y_\infty^{(1)} := X_\infty := M-1$ to make the notation work out correctly in case $\mathrm{P}_{\mathbb{1}}\{\alpha_1 = \infty\} > 0$ [which we are about to rule out].

Since $Y^{(1)}$ has an upward drift and starts at $M-2$, it almost surely reaches $M-1$ in finite time. Because of item (iii) above, α_1 is not greater than the first hitting time of $M-2$ by $Y^{(1)}$. This proves that $\mathrm{P}_{\mathbb{1}}\{\alpha_1 < \infty\}$; in fact, that

$$\limsup_{m\to\infty} m^{-1} \log \mathrm{P}_{\mathbb{1}}\{\alpha_1 > m\} < 0.$$

To complete the proof, we work by induction. Suppose we have proved that $\mathrm{P}_{\mathbb{1}}\{\alpha_i < \infty\} = 1$ for some $i \geqslant 1$. We recycle the preceding random walk construction to produce a random walk $Y^{(i+1)}$ on $\mathbb{Z}$ that is *independent* of $Y^{(1)}, \ldots, Y^{(i)}$ and:[5]

[5] In this proof we do not use the additional fact that $Y^{(1)}, Y^{(2)}, \ldots$ are independent from one another, but we will use that fact later on.

(i) $Y^{(i+1)}_0 = M-2$;

(ii) $\mathrm{P}_{\mathbb{1}}\{Y^{(i+1)}_{n+1} - Y^{(i+1)}_n = +1\} = 1 - \mathrm{P}_{\mathbb{1}}\{Y^{(i+1)}_{n+1} - Y^{(i+1)}_n = -1\} = 2/3$ for all $n \geqslant 0$; and

(iii) $\mathrm{P}_{\mathbb{1}}\{Y^{(i+1)}_n \leqslant X_{n+\alpha_i}$ for all $1 \leqslant n \leqslant \alpha_{i+1} - \alpha_i\} = 1$ where $Y^{(i+1)}_\infty := X_\infty := M-1$.

Since $Y^{(i+1)}$ starts at $M-2$ and has an upward drift, it a.s. reaches $M-1$ in finite time. Therefore, the same argument that proved that $\mathrm{P}_{\mathbb{1}}\{\alpha_1 < \infty\} = 1$ now implies that $n \mapsto X_{n+\alpha_i}$ reaches $M-1$ in a.s.-finite time, whence $\mathrm{P}_{\mathbb{1}}\{\alpha_{i+1} < \infty\} = 1$. □

Next we prove that α_n is asymptotically of sharp order n as $n \to \infty$. We will state and prove the complete result, though we need only the following asymptotic lower bound on α_n/n.

Lemma 5.7. $\mathrm{P}_{\mathbb{1}}$-*almost surely,*

$$\frac{2}{3} \leqslant \liminf_{n\to\infty} \frac{\alpha_n}{n} \leqslant \limsup_{n\to\infty} \frac{\alpha_n}{n} \leqslant 3.$$

Proof. Recall the independent biased random walks $Y^{(1)}, Y^{(2)}, \ldots$ of the proof of Lemma 5.6, and define for every $k \in \mathbb{N}$,

$$\beta_k := \inf\left\{ j \geqslant 1 : \, Y^{(k)}_j = M-1 \right\},$$

where $\inf \varnothing = \infty$. Choose and fix an integer $k \geqslant 1$. Evidently, $\beta_1, \beta_2, \ldots$ are i.i.d. random variables. And since $Y^{(k)}_1 = M-2$, $Y^{(k)}$ has positive upward drift, and $Y^{(1)}_n - (n/3)$ defines a mean-zero martingale, a gambler's ruin computation shows that $\mathrm{E}_{\mathbb{1}}(\beta_1) = 3$.

Define $\alpha_0 := 0$. Because $Y^{(k)}_n \leqslant X_{n+\alpha_{k-1}}$ $\mathrm{P}_{\mathbb{1}}$-a.s. for all $n \in \mathbb{N}$, it follows from a little book keeping that

$$\beta_k \geqslant \inf\{n \geqslant 1 : \, X_{n+\alpha_{k-1}} = M-1\}.$$

Apply induction on k to see that $\beta_k \geqslant \alpha_k - \alpha_{k-1}$ for all $k \in \mathbb{N}$, $\mathrm{P}_{\mathbb{1}}$-a.s. Thus, the strong law of large numbers implies that

$$\limsup_{n\to\infty} \frac{\alpha_n}{n} = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^n (\alpha_k - \alpha_{k-1}) \leqslant \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \beta_k = \mathrm{E}_{\mathbb{1}}(\beta_1) = 3 \qquad \mathrm{P}_{\mathbb{1}}\text{-a.s.}$$

For the converse bound we might observe that, if $\beta_k = 1$, then $X_{1+\alpha_{k-1}} = M-1$ and hence $\alpha_k - \alpha_{k-1} = 1$. Therefore,

$$\liminf_{n\to\infty} \frac{\alpha_n}{n} = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^n (\alpha_k - \alpha_{k-1}) \geqslant \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \mathbf{1}_{\{\beta_k = 1\}} = \frac{2}{3} \qquad \mathrm{P}_{\mathbb{1}}\text{-a.s.},$$

thanks to the strong law of large numbers. The lemma follows. □

We are ready to conclude this subsection by verifying Proposition 4.16; namely, that if $\mathrm{L}_\sigma > 0$ and λ is sufficiently small (which we assume is the case), then there exists a non-trivial invariant measure.

Proof of Proposition 4.16. Recall the i.i.d. random walks $Y^{(1)}, Y^{(2)}, \ldots$ from the proof of Lemma 5.6. The very construction of the $Y^{(i)}$'s implies that

$$\sum_{j=1}^{\alpha_n} \mathbf{1}_{\{X_j \leqslant -k\}} = \sum_{\ell=1}^{n} \sum_{j=1}^{\alpha_\ell - \alpha_{\ell-1}} \mathbf{1}_{\{X_{j+\alpha_{\ell-1}} \leqslant -k\}} \leqslant \sum_{\ell=1}^{n} \sum_{j=1}^{\infty} \mathbf{1}_{\{Y^{(\ell)}_j \leqslant -k\}} =: \sum_{\ell=1}^{n} \chi_\ell,$$

notation being clear. Now, $\chi_1, \chi_2, \ldots$ are i.i.d., and a standard computation shows that

$$\mathrm{E}_{\mathbb{1}}[\chi_1] \leqslant 17 \cdot 2^{-(k-M+2)/2};$$

see Lemma A.3 for example. Therefore, Kolmogorov's strong law of large numbers implies that

$$\limsup_{n\to\infty} n^{-1} \sum_{j=1}^{\alpha_{2n}} \mathbf{1}_{\{X_j \leqslant -k\}} \leqslant 34 \cdot 2^{-(k-M+2)/2},$$

$\mathrm{P}_{\mathbb{1}}$-a.s. Lemma 5.7 ensures that $\alpha_{2n} \geqslant n$ for all n large. Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{\{X_j \leqslant -k\}} \leqslant 17 \cdot 2^{-(k-M)/2} \qquad \mathrm{P}_{\mathbb{1}}\text{-a.s.} \tag{5.16}$$

This implies (5.12). Therefore, Proposition 4.16 follows from Proposition 5.3. □

6 A Support Theorem

In general, a "support theorem" for a probability measure ν is a full, or sometimes a partial, description of the support of the measure ν. In this section we provide a partial support theorem that describes the support of the law of $\psi(t)$, at least for small values of t, where ψ solves the SPDE (1.1) starting from a given function $\psi_0 \in C_{>0}(\mathbb{T})$.

Proposition 6.1. *Choose and fix non-random number $A > A_0 > 0$, and $\alpha \in (0\,,1/2)$. Then, for every non-random $\psi_0 \in C_+(\mathbb{T})$ with $\frac12 A_0 \leqslant \inf_{x\in\mathbb{T}} \psi_0(x) \leqslant \|\psi_0\|_{C^\alpha(\mathbb{T})} \leqslant A$, and for all $\delta > 0$, there exists $t_0 = t_0(A\,,A_0\,,\alpha\,,\delta) > 0$ and a strictly positive number $\mathfrak{p}_{A,A_0}(t_0\,,\alpha\,,\delta)$ – dependent on $(A\,,A_0\,,t_0\,,\alpha\,,\delta)$ but otherwise independently of ψ_0 – such that the solution ψ to (1.1) with initial profile ψ_0 satisfies*

$$\mathrm{P}\left\{\sup_{x\in\mathbb{T}} |\psi(t_0\,,x) - A_0| \leqslant \delta\,, \|\psi(t_0)\|_{C^{\alpha/2}(\mathbb{T})} \leqslant A+1\right\} \geqslant \mathfrak{p}_{A,A_0}(t_0\,,\alpha\,,\delta).$$

Proof. It clearly suffices to prove the result when δ is small. Therefore, we may [and will] assume without any loss in generality that

$$\delta < \frac{1}{16} \wedge \frac{A_0}{2} \wedge \left(\frac{A_0}{2A\mathscr{C}_\alpha}\right)^{2/\alpha}, \tag{6.1}$$

is sufficiently small (but fixed), where

$$\mathscr{C}_\alpha := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^\alpha \mathrm{e}^{-x^2/2}\,\mathrm{d}x = \frac{2^{\alpha/2}}{\sqrt{\pi}} \Gamma\left(\frac{1+\alpha}{2}\right). \tag{6.2}$$

The essence of the idea is quite natural: By regularity estimates on the paths of the solution [Propositions 4.6 and 4.8] we may choose t sufficiently small to ensure that

$$\mathrm{P}\left\{\|\psi(t) - \psi_0\|_{C(\mathbb{T})} \leqslant \delta\,, \|\psi(t)\|_{C^{\alpha/2}(\mathbb{T})} \leqslant A_0 + 1\right\} > 0.$$

Then, we apply Girsanov's theorem (Allouba, 1998; Da Prato and Zabczyk, 1996) to shift the center of the above radius-δ ball in $C(\mathbb{T})$. Because of the multiplicative nature of the noise in (1.1), and since σ vanishes at zero, the said appeal to Girsanov's theorem is somewhat non trivial. Therefore, we write a detailed proof.

Fix a real number $k \geqslant 2$, and recall the random field $\mathcal{I}$ from (3.9). We may apply the BDG inequality, in a manner similar to our method of proof of Lemma 4.5, in order to see that there exists a real number $c_k > 0$ such that simultaneously for every $x \in \mathbb{T}$ and $t > 0$,

$$\begin{aligned}\|\mathcal{I}(t\,,x)\|_k^2 &\leqslant c_k \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ [p_{t-s}(x\,,y)]^2 \|\sigma(\psi(s\,,y))\|_k^2 \\ &\leqslant c_k \mathrm{Lip}_\sigma^2 \sup_{r\geqslant 0}\sup_{z\in\mathbb{T}} \|\psi(r\,,z)\|_k^2 \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ [p_s(x\,,y)]^2 \\ &= c_k \mathrm{Lip}_\sigma^2 \sup_{r\geqslant 0}\sup_{z\in\mathbb{T}} \|\psi(r\,,z)\|_k^2 \int_0^t p_{2s}(0\,,0)\,\mathrm{d}s;\end{aligned}$$

we have appealed to the semigroup property of the heat kernel in order to deduce the last inequality. Now, Lemma B.1 of Khoshnevisan et al. (2020) tells us that $p_{2s}(0\,,0) \leqslant 2\max\{s^{-1/2}\,,1\}$, and Proposition 4.6 implies that $\sup_{r\geqslant 0}\sup_{z\in\mathbb{T}} \|\psi(r\,,z)\|_k < \infty$. Therefore, there exists $c_k' > 0$ such that

$$\sup_{x\in\mathbb{T}} \|\mathcal{I}(t\,,x)\|_k \leqslant c_k' \max\left\{t^{1/4}\,,t^{1/2}\right\} \qquad \text{for all } t > 0. \tag{6.3}$$

Next, we might observe from Lemma 4.5 and the proof of Proposition together that there exists $c_k'' > 0$ such that

$$\|\mathcal{I}(t\,,x) - \mathcal{I}(s\,,x')\|_k \leqslant c_k'' \left\{|x-x'|^{1/2} + |t-s|^{1/4}\right\}, \tag{6.4}$$

uniformly for all $s,t > 0$ and $x,x' \in \mathbb{T}$. Therefore, we may apply chaining together with (6.3) and (6.4) in order to find that there exists $C_k > 0$ such that

$$\mathrm{E}\left(\sup_{s\in(0,t)} \|\mathcal{I}(s)\|_{C^{\alpha/2}(\mathbb{T})}^k\right) \leqslant C_k t^{k/5} \qquad \text{for all } t \in (0\,,1]. \tag{6.5}$$

The careful reader might find that we have made a few arbitrary choices here: The $C^{\alpha/2}(\mathbb{T})$-norm can be replaced by a $C^{\beta}(\mathbb{T})$-norm for any $\beta \in (0\,,\alpha)$, and $t^{k/5}$ can be replaced by $t^{\theta k}$ for any $\theta \in (0\,,\frac14)$. Of course, in that case, $C_k = C_k(\theta\,,\beta)$.

Next, we consider the events

$$\boldsymbol{\mathcal{G}}_t := \left\{\sup_{s\in(0,t)} \|\mathcal{I}(s)\|_{C^{\alpha/2}(\mathbb{T})} \leqslant \frac{\delta}{10\lambda}\right\},$$

as t roams over $(0\,,1]$. According to Chebyshev's inequality, and thanks to (6.5),

$$\mathrm{P}\left(\boldsymbol{\mathcal{G}}_t^c\right) \leqslant \frac{10^k\lambda^k}{\delta^k} C_k t^{\theta k} \qquad \text{for all } t \in (0\,,1].$$

Since $\sup_{w\geqslant 0} V(w) < \infty$ [see Lemma 2.1], it follows that, for all sufficiently small values of $t_0 \in (0\,,\delta)$,

$$\begin{aligned}&\mathrm{P}\left(\boldsymbol{\mathcal{G}}_{t_0}\right) \geqslant \frac12, && 2\sup_{w\geqslant 0} V(w) t_0 < \frac{\delta}{10},\\ &2\sup_{|w|<A+1} |V(w)| t_0 \leqslant \frac{\delta}{10}, && \sup_{|w|<A+1} |V(w)| \sum_{k=1}^\infty \frac{k^2 t_0 \wedge 1}{k^{(4-\alpha)/2}} \leqslant \frac{1}{20},\\ &(A\mathscr{C}_\alpha t_0)^{\alpha/2} \leqslant \frac{\delta}{10}.\end{aligned} \tag{6.6}$$

From now on, we select and fix such a $t_0 = t_0(\delta\,,A) \in (0\,,\delta)$.

According to (2.2), with probability one,

$$\psi(s\,,x) = (\mathcal{P}_s\psi_0)(x) + \int_0^s \mathrm{d}r \int_{\mathbb{T}} \mathrm{d}y\ p_{s-r}(x\,,y)V(\psi(r\,,y)) + \lambda\mathcal{I}(s\,,x) \qquad \text{for all } s > 0 \text{ and } x \in \mathbb{T}.$$

Let W denote the Brownian sheet that is naturally associated to the white noise $\dot{W}$; that is,

$$W(s\,,x) := \int_{(0,s)\times[-1,x]} \dot{W}(\mathrm{d}r\,\mathrm{d}y) \qquad \text{for all } s > 0 \text{ and } x \in \mathbb{T},$$

where we recall $\mathbb{T}$, as a set, is identified with the interval $[-1\,,1]$.

Define

$$D_s := \mathrm{e}^{-M_s - \frac12\langle M\rangle_s} \qquad \text{for all } s > 0,$$

where $\{M_s\}_{s\geqslant 0}$ is the continuous local martingale defined by

$$M_s := \frac{1}{\lambda t_0}\int_{(0,s)\times\mathbb{T}} \left\{\psi_0(y) - \frac{A_0}{2}\right\} \frac{\mathbf{1}_{[\delta/2,A+1]}(\psi(r\,,y))}{\sigma(\psi(r\,,y))}\, W(\mathrm{d}r\,\mathrm{d}y).$$

Because $\mathrm{L}_\sigma > 0$ [see (2.1)], the quadratic variation of M satisfies

$$\begin{aligned}
\langle M\rangle_s &= \frac{1}{\lambda^2 t_0^2}\int_{(0,s)\times\mathbb{T}} \left\{\psi_0(y) - \frac{A_0}{2}\right\}^2 \frac{\mathbf{1}_{[\delta/2,A+1]}(\psi(r\,,y))}{\sigma^2(\psi(r\,,y))}\,\mathrm{d}r\,\mathrm{d}y\\
&\leqslant \frac{1}{\lambda^2 t_0^2 \mathrm{L}_\sigma^2}\left\{A + \frac{A_0}{2}\right\}^2 \int_{(0,s)\times\mathbb{T}} \frac{\mathbf{1}_{[\delta/2,A+1]}(\psi(r\,,y))}{|\psi(r\,,y)|^2}\,\mathrm{d}r\,\mathrm{d}y\\
&\leqslant \frac{4}{\lambda^2 t_0^2 \mathrm{L}_\sigma^2\delta^2}\left\{A + \frac{A_0}{2}\right\}^2 \int_{(0,s)\times\mathbb{T}} \mathrm{d}r\,\mathrm{d}y =: Cs \quad \text{for every } s > 0.
\end{aligned}$$

This inequality shows that the exponential local martingale $\{D_s\}_{s\geqslant 0}$ is in fact a continuous $L^2(\mathrm{P})$-martingale. The DDS martingale representation theorem ensures the existence of a Brownian motion B such that $M_s = B(\langle M\rangle_s)$ for all $s \geqslant 0$, whence we learn from the Cauchy-Schwarz inequality and the reflection principle that, for every $s > 0$,

$$\mathrm{E}(D_s^2) \leqslant \mathrm{E}\left(\mathrm{e}^{2M_s}\right) = \mathrm{E}\left(\mathrm{e}^{2B(\langle M\rangle_s)}\right) \leqslant \mathrm{E}\left[\exp\left\{2\sup_{r\in[0,Cs]} B(r)\right\}\right] \leqslant 2\mathrm{e}^{2Cs}.$$

Define

$$\underline{W}(s\,,x) := W(s\,,x) + \frac{1}{\lambda t_0}\int_{(0,s)\times\mathbb{T}} \left\{\psi_0(y) - \frac{A_0}{2}\right\} \frac{\mathbf{1}_{[\delta/2,A+1]}(\psi(r\,,y))}{\sigma(\psi(r\,,y))}\,\mathrm{d}r\,\mathrm{d}y,$$

for all $s > 0$ and $x \in \mathbb{T}$. Girsanov's theorem ensures that $\underline{W}$ is a Brownian sheet under the measure Q defined via

$$D_s := \left.\frac{\mathrm{dQ}}{\mathrm{dP}}\right|_{\mathscr{F}_s} \qquad \text{for all } s > 0;$$

see Allouba (1998) for the precise version of the Girsanov theorem that we need, and Chapter 10 of Da Prato and Zabczyk (2014) for the general theory. Among other things, the Cauchy-Schwarz inequality implies that

$$\mathrm{Q}(\Lambda) \leqslant \|D_t\|_2\sqrt{\mathrm{P}(\Lambda)} \leqslant \mathrm{e}^{3Ct_0/2}\sqrt{2\mathrm{P}(\Lambda)} \qquad \text{for all } \Lambda \in \mathscr{F}_{t_0}. \tag{6.7}$$

And a similar estimate holds that bounds $\mathrm{P}(\Lambda)$ by a [large] multiple of $\sqrt{Q(\Lambda)}$ for every $\Lambda \in \mathscr{F}_{t_0}$. In particular, it follows that Q and P are mutually absolutely continuous probability measures on the sigma algebra $\mathscr{F}_{t_0}$.

The above application of Girsanov's theorem ensures that ψ solves the following SPDE driven by $\underline{\dot W}$: Q-almost surely,

$$\partial_t \psi(s\,,x) = \partial_x^2 \psi(s\,,x) + V(\psi(s\,,x)) - \frac{1}{t_0}\left(\psi_0(x) - \frac{A_0}{2}\right)\mathbf{1}_{[\delta/2,A+1]}(\psi(s\,,x)) + \lambda\sigma(\psi(s\,,x))\,\underline{\dot W}(s\,,x).$$

We can write this in mild form [see (2.2)] in order to see that with probability one [Q],

$$\psi(s\,,x) = (\mathcal{P}_s\psi_0)(x) + J_1(s\,,x) - J_2(s\,,x) + \lambda\underline{\mathcal{I}}(s\,,x), \tag{6.8}$$

where

$$\begin{aligned} J_1(s\,,x) &:= \int_{(0,s)\times\mathbb{T}} p_{s-r}(x\,,y) V(\psi(r\,,y))\,\mathrm{d}r\,\mathrm{d}y,\\ J_2(s\,,x) &:= \frac{1}{t_0}\int_{(0,s)\times\mathbb{T}} p_{s-r}(x\,,y)\left\{\psi_0(y) - \frac{A_0}{2}\right\}\mathbf{1}_{[\delta/2,A+1]}(\psi(r\,,y))\,\mathrm{d}r\,\mathrm{d}y, \end{aligned}$$

for every $s > 0$ and $x \in \mathbb{T}$, and $\underline{\mathcal{I}}$ is defined exactly as was $\mathcal{I}$, but with W replaced by $\underline{W}$.

Next, consider the events,

$$\underline{\boldsymbol{\mathcal{G}}}_s := \left\{ \sup_{r\in(0,s)} \|\underline{\mathcal{I}}(r)\|_{C^{\alpha/2}(\mathbb{T})} \leqslant \frac{\delta}{10\lambda}\right\}.$$

That is, $\underline{\boldsymbol{\mathcal{G}}}_s$ is defined exactly as was $\boldsymbol{\mathcal{G}}_s$, but with $\mathcal{I}$ replaced by $\underline{\mathcal{I}}$. Recall from Lemma 2.1 that

$$K = \sup_{w\in\mathbb{R}} V(w) < \infty,$$

and since $\psi \geqslant 0$, observe that $\sup_{x\in\mathbb{T}} J_1(s\,,x) \leqslant 2Ks \leqslant 2Kt_0$ for every $s \in (0\,,t_0]$ a.s. Because $J_2(s\,,x) \geqslant 0$ a.s., we combine these statements with (6.6) and (6.8) in order to see that

$$\sup_{s\in(0,t_0]}\sup_{x\in\mathbb{T}} \psi(s\,,x) \leqslant A + 2Kt_0 + \frac{\delta}{10} < A + \frac{1}{2} \qquad \text{Q-a.s. on } \underline{\boldsymbol{\mathcal{G}}}_{t_0}. \tag{6.9}$$

Now, (6.9) and (6.6) together imply that, for all $s \in (0\,,t_0]$ and $x \in \mathbb{T}$,

$$|J_1(s\,,x)| \leqslant \sup_{|w|\leqslant A+1} |V(w)| \int_{(0,s)\times\mathbb{T}} p_{s-r}(x\,,y)\,\mathrm{d}r\,\mathrm{d}y \leqslant \frac{\delta}{10} \qquad \text{Q-a.s. on } \underline{\boldsymbol{\mathcal{G}}}_{t_0}. \tag{6.10}$$

Next, we observe that

$$\begin{aligned} (\mathcal{P}_s\psi_0)(x) - J_2(s\,,x) &\geqslant (\mathcal{P}_s\psi_0)(x) - \frac{1}{t_0}\int_{(0,s)\times\mathbb{T}} p_{s-r}(x\,,y)\left\{\psi_0(y) - \frac{A_0}{2}\right\}\mathrm{d}r\,\mathrm{d}y\\ &= (\mathcal{P}_s\psi_0)(x) - \frac{1}{t_0}\int_0^s (\mathcal{P}_r\psi_0)(x)\,\mathrm{d}r + \frac{s}{t_0}\frac{A_0}{2}, \end{aligned}$$

for all $s \in (0\,,t_0]$ and $x \in \mathbb{T}$. Let $\{\beta(s)\}_{s\geqslant 0}$ denote a Brownian motion on $\mathbb{T}$ and observe that

$$
\begin{aligned}
\left|\left(\frac{s}{t_0}\right)(\mathcal{P}_s\psi_0)(x) - \frac{1}{t_0}\int_0^s (\mathcal{P}_r\psi_0)(x)\,\mathrm{d}r\right| &= \frac{1}{t_0}\left|\int_0^s \left\{\mathrm{E}\left[\psi_0(\beta(s)+x)\right] - \mathrm{E}\left[\psi_0(\beta(r)+x)\right]\right\}\mathrm{d}r\right| \\
&\leqslant \frac{\|\psi_0\|_{C^\alpha(\mathbb{T})}}{t_0}\int_0^s \mathrm{E}\left(|\beta(s)-\beta(r)|^\alpha\right)\mathrm{d}r \\
&\leqslant \frac{A\mathscr{C}_\alpha}{t_0}\int_0^s r^{\alpha/2}\,\mathrm{d}r \qquad\qquad [\text{see } (6.2)] \\
&< A\mathscr{C}_\alpha s^{\alpha/2} \qquad \text{for all } s \in (0\,,t_0] \text{ and } x \in \mathbb{T}.
\end{aligned}
\tag{6.11}
$$

Since $A\mathscr{C}_\alpha s^{\alpha/2} \leqslant A\mathscr{C}_\alpha t_0^{\alpha/2} \leqslant \delta/10$, it follows from the preceding and from (6.1) that

$$
(\mathcal{P}_s\psi_0)(x) - \frac{1}{t_0}\int_0^s (\mathcal{P}_r\psi_0)(x)\,\mathrm{d}r \geqslant \left(1 - \frac{s}{t_0}\right)\frac{A_0}{2} - \frac{\delta}{10} \qquad \text{for all } s \in (0\,,t_0] \text{ and } x \in \mathbb{T}.
$$

and hence

$$
(\mathcal{P}_s\psi_0)(x) - J_2(s\,,x) \geqslant \frac{A_0}{2} - \frac{\delta}{10}.
$$

Thus, (6.6), (6.8), and (6.10) together ensure that

$$
\inf_{s\in(0,t_0]}\inf_{x\in\mathbb{T}} \psi(s\,,x) \geqslant \frac{A_0}{2} - \frac{3\delta}{10} > \frac{\delta}{2} \qquad \text{Q-a.s. on } \underline{\boldsymbol{\mathcal{G}}}_{t_0}, \tag{6.12}
$$

and (6.9) and (6.12) together yield the following Q-a.s. on $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$: For all $x \in \mathbb{T}$,

$$
\begin{aligned}
(\mathcal{P}_{t_0}\psi_0)(x) - J_2(t_0\,,x) &= (\mathcal{P}_{t_0}\psi_0)(x) - \frac{1}{t_0}\int_{(0,t_0)\times\mathbb{T}} \left\{\psi_0(y) - \frac{A_0}{2}\right\} p_{t_0-r}(x\,,y)\,\mathrm{d}r\,\mathrm{d}y \\
&= A_0 + \mathcal{P}_{t_0}\psi_0(x) - \frac{1}{t_0}\int_0^{t_0} (\mathcal{P}_r\psi_0)(x)\,\mathrm{d}r.
\end{aligned}
\tag{6.13}
$$

Thus, it follows from (6.8), (6.10), (6.11) and the definition of the event $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$ that, Q-a.s. on $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$,

$$
\sup_{x\in\mathbb{T}} |\psi(t_0\,,x) - A_0| \leqslant \|(\mathcal{P}_s\psi_0 - J_2(s)\|_{C(\mathbb{T})} + \|J_1(t_0)\|_{C(\mathbb{T})} + \lambda\,\|\underline{\mathcal{I}}(t_0)\|_{C(\mathbb{T})} \leqslant \frac{3\delta}{10} < \delta. \tag{6.14}
$$

Also, (6.13) tells us that Q-a.s. on $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$,

$$
\begin{aligned}
|\psi(t_0\,,x) - \psi(t_0\,,y)| &\leqslant |J_1(t_0\,,x) - J_1(t_0\,,z)| + \lambda|\underline{\mathcal{I}}(t_0\,,x) - \underline{\mathcal{I}}(t_0\,,z)| + 2\|(\mathcal{P}_s\psi_0 - J_2(s)\|_{C(\mathbb{T})} \\
&\leqslant |J_1(t_0\,,x) - J_1(t_0\,,z)| + \frac{\delta}{10}|x-z|^{\alpha/2} + \frac{\delta}{5},
\end{aligned}
\tag{6.15}
$$

simultaneously for all $x, z \in \mathbb{T}$. We estimate the remaining term as follows: Because of (6.9), the following holds Q-a.s. on $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$:

$$
|J_1(t_0\,,x) - J_1(t_0\,,z)| \leqslant \sup_{w\leqslant A+1}|V(w)|\int_{(0,t_0)\times\mathbb{T}} |p_r(x\,,y) - p_r(z\,,y)|\,\mathrm{d}r\,\mathrm{d}y,
$$

simultaneously for all $x, z \in \mathbb{T}$. Now we apply (4.2) to see that Q-a.s. on $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$,

$$
|J_1(t_0\,,x) - J_1(t_0\,,z)| \leqslant 2\sqrt{2}\sup_{w\leqslant A+1}|V(w)|\sum_{k=1}^\infty (|x-z|k \wedge 1)\int_0^{t_0} \mathrm{e}^{-\pi^2k^2r}\,\mathrm{d}r,
$$

simultaneously for every $x, z \in \mathbb{T}$. Since $(|a| \wedge 1) \leqslant |a|^{\alpha/2}$ for all $a \in \mathbb{R}$, it follows that Q-a.s. on $\underline{\boldsymbol{\mathcal{G}}}_{t_0}$,

$$
\begin{aligned}
|J_1(t_0\,, x) - J_1(t_0\,, z)| &\leqslant 2\sqrt{2} \sup_{w \leqslant A+1} |V(w)| |x-z|^{\alpha/2} \sum_{k=1}^{\infty} k^{\alpha/2} \left(\frac{1-\mathrm{e}^{-\pi^2 k^2 t_0}}{\pi^2 k^2} \right) \\
&\leqslant \frac{2\sqrt{2}}{\pi^2} \sup_{w \leqslant A+1} |V(w)| |x-z|^{\alpha/2} \sum_{k=1}^{\infty} \frac{k^2 t_0 \wedge 1}{k^{(4-\alpha)/2}} \\
&\leqslant \frac{\sqrt{2}}{20} |x-z|^{\alpha/2} \qquad \text{for all } x, z \in \mathbb{T};
\end{aligned}
$$

see (6.6). This and (6.15) together yield

$$
\sup_{\substack{x,z \in \mathbb{T} \\ x \neq z}} \frac{|\psi(t_0\,, x) - \psi(t_0\,, z)|}{|x-z|^{\alpha/2}} \leqslant \frac{\sqrt{2}}{20} + \frac{3\delta}{10} \qquad \text{Q-a.s. on } \underline{\boldsymbol{\mathcal{G}}}_{t_0}.
$$

Thus, we may deduce from (6.9) that

$$
\|\psi(t_0)\|_{C^\alpha(\mathbb{T})} \leqslant A + \frac{1}{2} + \frac{\sqrt{2}}{20} + \frac{3\delta}{10} < A + 1 \qquad \text{Q-a.s. on } \underline{\boldsymbol{\mathcal{G}}}_{t_0}. \tag{6.16}
$$

Thanks to (6.6) and Girsanov's theorem, $\mathrm{Q}(\underline{\boldsymbol{\mathcal{G}}}_{t_0}) = \mathrm{P}(\boldsymbol{\mathcal{G}}_{t_0}) \geqslant 1/2$. Therefore, (6.7), (6.14), and (6.16) together imply that

$$
\begin{aligned}
&\mathrm{P}\left\{ \|\psi(t_0) - A_0\|_{C(\mathbb{T})} \leqslant \delta\,, \|\psi(t_0)\|_{C^{\alpha/2}(\mathbb{T})} \leqslant A+1 \right\} \\
&\geqslant \mathrm{e}^{-3Ct_0} \left| \mathrm{Q}\left\{ \|\psi(t_0) - A_0\|_{C(\mathbb{T})} \leqslant \delta\,, \|\psi(t_0)\|_{C^{\alpha/2}(\mathbb{T})} \leqslant A+1 \right\} \right|^2 \geqslant \mathrm{e}^{-3Ct_0} \left| \mathrm{Q}(\underline{\boldsymbol{\mathcal{G}}}_{t_0}) \right|^2 \geqslant \frac{1}{4} \mathrm{e}^{-3Ct_0}.
\end{aligned}
$$

This has the desired result. □

7 Natural, Independent, and AM/PM Couplings

The principal aim of this section is proof of the statement that if $\mathrm{L}_\sigma > 0$ and λ is small, then there can only exist one probability measure μ_+ on $C_+(\mathbb{T})$ such that $\mu_+\{\mathbb{0}\} = 0$ and μ_+ is invariant for the SPDE (1.1). We have demonstrated already in Proposition 4.16 that if $\mathrm{L}_\sigma > 0$ and λ is sufficiently small, then at least one invariant measure μ_+ exists such that $\mu_+\{\mathbb{0}\} = 0$. The main point of this section is that μ_+ is the only invariant measure of the type outlined. In order to do this, we build on coupling ideas of Mueller (1993). Let ψ_1 and ψ_2 denote the solutions to the SPDE (1.1) starting respectively, given respective initial data $\psi_{1,0}, \psi_{2,0} \in C_{>0}(\mathbb{T})$. We will not assume that they are driven by the same noise, or even are defined on the same probability space. With this in mind, recall that a *coupling* of $(\psi_1\,, \psi_2)$ is a construction of $(\psi_1\,, \psi_2)$ jointly on the same probability space such that ψ_1 and ψ_2 have the correct respective marginals. In other words, a coupling of $(\psi_1\,, \psi_2)$ involves the construction of two space-time white noises $\dot{\mathcal{W}}_1$ and $\dot{\mathcal{W}}_2$ such that the following stochastic integral equations

$$
\begin{aligned}
\psi_j(t\,, x) = (\mathcal{P}_t \psi_{j,0})(x) &+ \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,, y) V(\psi_j(s\,, y))\, \mathrm{d}s\, \mathrm{d}y \\
&+ \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,, y) \sigma(\psi_j(s\,, y))\, \mathcal{W}_j(\mathrm{d}s\, \mathrm{d}y)
\end{aligned} \tag{7.1}
$$

are valid for all $t > 0$, $x \in \mathbb{T}$, and $j \in \{1\,,2\}$. The novelty here can be in the fact that $\mathcal{W}_1$ and $\mathcal{W}_2$ might be correlated with one another, and even constructed *a priori* using the solution $(\psi_1\,,\psi_2)$ to the above. This is of course a pairwise coupling. One can imagine also a more general N-wise coupling of $N \geqslant 2$ solutions to (1.1), etc.

Next we devote some time to describe four notions of coupling, all of which are used in this paper. We call them *natural*, *independent*, *pairwise monotone* (PM, for short), and *anchored monotone* (AM, for short) couplings for the sake of comparison and ease of later reference. The first two coupling methods are standard; the more subtle PM and the AM couplings of this paper were introduced in Mueller (1993).

(i) Natural coupling. By a *natural coupling* of ψ_1 and ψ_2 we simply mean the construction of ψ_1 and ψ_2 using the same underlying white noise. This is the coupling that we have tacitly used so far in the paper. The natural coupling has a number of obvious advantages. For example, if $\psi_{1,0} \leqslant \psi_{2,0}$, then $\psi_1 \leqslant \psi_2$ a.s.; see Lemma 3.3. Another attractive feature of natural couplings is that they are not limited to pairwise couplings, or even N-wise couplings. One can in fact solve (1.1) simultaneously for every non-random initial profile $\psi_0 \in C_+(\mathbb{T})$.

(ii) Independent coupling. By an *independent* coupling of ψ_1 and ψ_2 we simply mean that the underlying noises $\dot{\mathcal{W}}_1$ and $\dot{\mathcal{W}}_2$ in (7.1) are independent from one another. This is the most naive form of coupling, but as we shall see has its uses.

(iii) Pairwise monotone* (PM) *coupling. *PM coupling* refers to the first step of a two-step coupling method that was introduced in Mueller (1993). In order to recall that method, and adapt it to the present setting, let us first define $\dot{\mathcal{W}}_1$ and $\dot{\mathcal{W}}_2$ to be two independent space-time white noises. Also, consider the real-valued functions

$$f(y) := \sqrt{|y| \wedge 1} \quad \text{and} \quad g(y) := \sqrt{1 - |f(y)|^2} = \sqrt{1 - (|y| \wedge 1)} \qquad \text{for } y \in \mathbb{R}. \tag{7.2}$$

Now, we first let ψ_1 solve (1.1), driven by $\dot{W}_1$; that is,

$$\begin{aligned} \psi_1(t\,,x) = (\mathcal{P}_t\psi_{1,0})(x) &+ \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y) V(\psi_1(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \\ &+ \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_1(s\,,y))\,\mathcal{W}_1(\mathrm{d}s\,\mathrm{d}y), \end{aligned} \tag{7.3}$$

for every $t > 0$ and $x \in \mathbb{T}$. Next, we let ψ_2 define the solution to the coupled SPDE,

$$\begin{aligned} \psi_2(t\,,x) = (\mathcal{P}_t\psi_{2,0})(x) &+ \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y) V(\psi_2(s\,,y))\,\mathrm{d}s\,\mathrm{d}y \\ &+ \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_2(s\,,y)) g\left(\psi_1(s\,,y)) - \psi_2(s\,,y)\right) \mathcal{W}_1(\mathrm{d}s\,\mathrm{d}y) \\ &+ \lambda \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_2(s\,,y)) f\left(\psi_1(s\,,y)) - \psi_2(s\,,y)\right) \mathcal{W}_2(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{7.4}$$

Soon, we will elaborate on the existence of the PM coupling briefly, following the work of Mueller (1993), and adapting that work to the present setting. For now, let us make a few remarks:

- As was mentioned by Mueller (1993) in a similar setting, we do not make a statement about the pathwise uniqueness of the solution to the SPDE system that defines $(\psi_1\,,\psi_2)$ in the PM coupling. Nor does pathwise uniqueness affect us. We care only about weak existence and uniqueness (in the probabilistic sense), which we shall establish soon.

- We can write the PM coupling of $(\psi_1\,,\psi_2)$, in differential notation, as the following interacting pair of SPDEs:

$$
\begin{aligned}
&① \begin{bmatrix} \partial_t\psi_1 = \partial_x^2\psi_1 + V(\psi_1) + \lambda\sigma(\psi_1)\dot{\mathcal{W}}_1 & \text{on } (0\,,\infty)\times\mathbb{T}, \\ \text{subject to } \psi_1(0) = \psi_{1,0} & \text{on } \mathbb{T}, \end{bmatrix} \\
&② \begin{bmatrix} \partial_t\psi_2 = \partial_x^2\psi_2 + V(\psi_2) + \lambda\sigma(\psi_2)\left[g\left(\psi_1-\psi_2\right)\dot{\mathcal{W}}_1 + f\left(\psi_1-\psi_2\right)\dot{\mathcal{W}}_2\right] & \text{on } (0\,,\infty)\times\mathbb{T}, \\ \text{subject to } \psi_2(0) = \psi_{2,0} & \text{on } \mathbb{T}. \end{bmatrix}.
\end{aligned}
$$

  As long as the solution $(\psi_1\,,\psi_2)$ exists as a 2-D predictable random field in the sense of Walsh (1986), and because $f^2+g^2=1$, the random distribution $g(\psi_1-\psi_2)\dot{\mathcal{W}}_1 + f(\psi_1-\psi_2)\dot{\mathcal{W}}_2$ is *a priori* a space-time white noise; see Corollary A.7 of the appendix. This proves that if ① and ② jointly have a random field solution $(\psi_1\,,\psi_2)$, then that solution is *a fortiori* a coupling of ψ_1 and ψ_2.
- If $|\psi_1-\psi_2| \ll 1$, then $g(\psi_1-\psi_2)\approx 1$ and $f(\psi_1-\psi_2)\approx 0$, and if $|\psi_1-\psi_2|\gg 1$, then $g(\psi_1-\psi_2)\approx 0$ and $f(\psi_1-\psi_2)\approx 1$. This suggests somewhat informally that the PM coupling of $(\psi_1\,,\psi_2)$ ought to behave roughly as follows:

$$
\partial_t\psi_2 \approx \begin{cases} \partial_x^2\psi_2 + V(\psi_2) + \lambda\sigma(\psi_2)\dot{\mathcal{W}}_1 & \text{when } |\psi_1-\psi_2|\ll 1, \\ \partial_x^2\psi_2 + V(\psi_2) + \lambda\sigma(\psi_2)\dot{\mathcal{W}}_2 & \text{when } |\psi_1-\psi_2|\gg 1. \end{cases}
$$

  Of course, these remarks are not rigorous, in part because SPDEs are not local equations. Still, the preceding serves as a reasonable heuristic to suggest that the PM coupling of $(\psi_1\,,\psi_2)$ ought to behave as independent coupling when ψ_1 and ψ_2 are far apart, and it works as natural coupling when ψ_1 and ψ_2 are close.

Before we go on to describe AM coupling, let us pause and state and prove an existence result [Proposition 7.1], and a "successful coupling" result [Lemma 7.2], for PM couplings. In particular, part 2 of the following proposition justifies the terminology "pairwise monotone," or "PM."

Proposition 7.1. *Choose and fix two non-random functions $\psi_{1,0},\psi_{2,0}\in C_+^\alpha(\mathbb{T})$ for some $\alpha\in(0\,,1/2)$. After possibly enlarging the underlying probability space, one can contruct a pair $(\dot{\mathcal{W}}_1\,,\dot{\mathcal{W}}_2)$ of two independent space-time white noises for which (7.3) and (7.4) have random-field solutions $(\psi_1\,,\psi_2)$. Moreover:*

1. *For every $i\in\{1\,,2\}$, the law of ψ_i is the same as the law of (1.1) started at $\psi_{i,0}$;*
2. *If, in addition, $\psi_{1,0}\geqslant\psi_{2,0}$, then*

$$
\mathrm{P}\{\psi_1(t\,,x)\geqslant\psi_2(t\,,x) \textit{ for all } t\geqslant 0 \textit{ and } x\in\mathbb{T}\} = 1; \;\; \textit{and}
$$

3. *$\{(\psi_1(t)\,,\psi_2(t))\}_{t\geqslant 0}$ is a Feller process with values in the space $C(\mathbb{T}\,;\mathbb{R}^2)$.*

Proof. If the functions $x\mapsto V(x)=x-F(x)$ and σ were replaced by bounded, globally Lipschitz functions, then parts 1 and 3 of this proposition reduce to the construction of Mueller (1993) with our $(\psi_1\,,\psi_2)$ being replaced by $(u\,,v)$ of Mueller (*ibid.*). We adapt Mueller's arguments, and fill in some additional details to cover the present setting.

Let us start with two independent space-time white noises $\dot{W}_1$ and $\dot{W}_2$. Theorem 2.3 ensures that the process ψ_1 of (7.3) is well defined on any probability space that supports a space-time white noise $\dot{W}_1$. However, the non-Lipschitz behavior of f and g at the origin prevent us from using standard SPDE machinary to produce a strong solution ψ_2. We overcome this, as in Mueller (1993), by producing instead a weak solution (in the sense of probability).

We follow Mueller (1993) and define, for every $n \in \mathbb{N}$ and $y \in \mathbb{R}$,

$$f_n(y) := \left(\left[|y| + \frac{1}{n} \right] \wedge 1 \right)^{1/2} - \left(\frac{1}{n} \right)^{1/2} \quad \text{and} \quad g_n(y) := \sqrt{1 - |f_n(x)|^2}.$$

Then every f_n and g_n is a Lipschitz continuous function, and $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$, both limits holding uniformly on $\mathbb{R}$.

Recall that ψ_1 has already been constructed via (7.3) using the space-time white noise $\dot{W}_1$. Theorem 2.3 of course justifies the existence and uniqueness of this construction.

For every $n, N \in \mathbb{N}$, let $\psi_{2,n,N}$ denote the solution to the SPDE,

$$\begin{cases} \partial_t \psi_{2,n,N} = \partial_x^2 \psi_{2,n,N} + V_N(\psi_{2,n,N}) + \lambda\sigma(\psi_{2,n,N}) \left\{ g_n\left(\psi_1 - \psi_{2,n,N}\right) \dot{W}_1 + f_n\left(\psi_1 - \psi_{2,n,N}\right) \dot{W}_2 \right\}, \\ \hfill \text{on } (0\,,\infty) \times \mathbb{T}, \\ \text{subject to} \quad \psi_{2,n,N}(0) = \psi_{2,0}, \end{cases}$$

where V_N denotes our existing truncation of V from (3.1). This is a standard SPDE with Lipschitz-continuous coefficient as in Walsh (1986), and hence has a strong solution on any probability space that supports two independent copies $\dot{W}_1$ and $\dot{W}_2$ of a space-time white noise. Because $f_n^2 + g_n^2 = 1$,

$$g_n\left(\psi_1 - \psi_{2,n,N}\right) \dot{W}_1 + f_n\left(\psi_1 - \psi_{2,n,N}\right) \dot{W}_2$$

defines a space-time white noise; see Corollary A.7 of the appendix. Therefore, $\psi_{2,n,N}$ has the same law as $\psi_{2,N}$, started at $\psi_{2,0}$, where $\psi_{2,N}$ denotes the solution to (1.1) with V replaced by V_N. The proof of Theorem 2.3 shows that there exist stopping times $T_1, T_2, \ldots$ (depending on n) such that $\lim_{N\to\infty} T_N = \infty$ a.s. and $\psi_{2,n,N}(t) = \psi_{2,n,N+1}(t)$ for all $t \in [0\,, T_N]$. In this way, we obtain a predictable random field $\psi_{2,n}$ such that $\psi_{2,n}(t) = \psi_{2,n,N}(t)$ for all $t \in [0\,, T_N]$, and $\psi_{2,n}$ is the strong solution to the SPDE,

$$\begin{cases} \partial_t \psi_{2,n} = \partial_x^2 \psi_{2,n} + V(\psi_{2,n}) + \lambda\sigma(\psi_{2,n}) \left\{ g_n\left(\psi_1 - \psi_{2,n}\right) \dot{W}_1 + f_n\left(\psi_1 - \psi_{2,n}\right) \dot{W}_2 \right\} \text{ on } (0\,,\infty) \times \mathbb{T}, \\ \text{subject to} \quad \psi_{2,n}(0) = \psi_{2,0}. \end{cases}$$

Once again, Theorem 2.3 ensures that this SPDE can be solved on any probability space that supports $(\dot{W}_1\,, \dot{W}_2)$.

Since

$$\dot{w}_n := g_n\left(\psi_1 - \psi_{2,n}\right) \dot{W}_1 + f_n\left(\psi_1 - \psi_{2,n}\right) \dot{W}_2 \tag{7.5}$$

is a space-time white noise [Corollary A.7], Theorem 2.3 ensures that the law of $\psi_{2,n}$ is the same as the law of ψ_2, any solution to (1.1) started at $\psi_{2,0}$, and is in particular does not depend on $n \in \mathbb{N}$.

Next, we use Proposition 4.6 to see that

$$\sup_{t \geqslant 0} \mathrm{E}\left(\|\psi_1(t)\|_{C^\alpha(\mathbb{T})}^k \right) < \infty \qquad \text{for every } k \geqslant 2.$$

Propositions 4.6 and 4.8, and a chaining argument together imply that for every $t_0 > 0$,

$$\mathrm{E}\left(\sup_{t \in (0, t_0)} \|\psi_1(t)\|_{C^\alpha(\mathbb{T})}^k \right) < \infty \qquad \text{for every } k \geqslant 2.$$

Let ψ_2 denote any solution to (1.1) starting at $\psi_{2,0}$. Since every $\psi_{2,n}$ has the same law as ψ_2, an appeal to the Arzelà-Ascoli theorem (see the proof of Proposition 4.6 for details) shows that the random fields $[0\,,t_0] \ni t \mapsto \psi_{2,n}(t)$ – as n varies in $\mathbb{N}$ – are tight in the space $C([0\,,t_0]\times\mathbb{T})$. Therefore, the laws of the vector-valued random fields

$$[0\,,t_0] \ni t \mapsto (\psi_1(t)\,,\psi_{2,n}(t)\,,\mathcal{S}(t)\,,\mathcal{T}_n(t)) \tag{7.6}$$

are tight in $\mathscr{C} := C([0\,,t_0];\,C(\mathbb{T}\,;\mathbb{R}^4))$ as n roams over $\mathbb{N}$, where $\dot{w}_n$ is the white noise of (7.5), and

$$\mathcal{S}(t\,,x) := \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_1(s\,,y))\,\dot{W}_1(\mathrm{d}s\,\mathrm{d}y),$$
$$\mathcal{T}_n(t\,,x) := \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_{2,n}(s\,,y))\,\dot{w}_n(\mathrm{d}s\,\mathrm{d}y),$$

are the stochastic integrals used to define ψ_1 and $\psi_{2,n}$ in their mild form. Because tight probability laws on $\mathscr{C}$ have weak subsequential limits, (7.6) has a subsequence (as $n\to\infty$) that converges weakly to a vector-value random field[6]

$$[0\,,t_0] \ni t \mapsto (\psi_1(t)\,,\psi_2(t)\,,\mathcal{S}(t)\,,\mathcal{T}(t))\,,$$

and

$$\mathcal{S}(t\,,x) := \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_1(s\,,y))\,\dot{W}_1(\mathrm{d}s\,\mathrm{d}y),$$
$$\mathcal{T}(t\,,x) := \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,,y)\sigma(\psi_2(s\,,y))\,\dot{w}(\mathrm{d}s\,\mathrm{d}y),$$

for a space-time white noise w. And we obtain, additionally, that this new pair $(\psi_1\,,\psi_2)$ solve respectively (7.3) and (7.4) for $t\in(0\,,t_0)$. This proves the existence assertion of the proposition.

The above construction also readily yields part 1 of the proposition since any subsequential limit of $\psi_{2,n}$'s, as $n\to\infty$ has the same law as ψ_2 because each $\psi_{2,n}$ does.

In the case that V were replaced by a Lipschitz-continuous and bounded function, Mueller (1993, Lemma 3.1) includes part 2. In other words, the latter result shows that $\psi_1 \geqslant \psi_{2,n,N}$. Let N and n tend to infinity – as we did previously – in order to deduce part 2.

Finally, we observe that $(\psi_1\,,\psi_{2,n,N})$ is a Feller process for the very same reasons that ψ_1 and ψ_2 are individually Feller. Moreover, the estimates required for the Feller property can all be made to hold uniformly in $(n\,,N)$; see the proof of Proposition 4.13. Let $n,N\to\infty$ as above to deduce part 3 and hence the proposition. □

The following is the second, and final, result of this section about PM couplings. After this, we shall move on to describe the fourth [and final] example of couplings for SPDEs, which is our AM coupling.

Lemma 7.2. *Choose and fix non-random numbers $C_0 > c_0 > 0$ and $\alpha,\varepsilon\in(0\,,1/2)$. Also consider two functions $\psi_{1,0},\psi_{2,0}\in C^\alpha_+(\mathbb{T})$ such that $\psi_{2,0}\leqslant\psi_{1,0}$, $\max_{i\in\{1,2\}}\|\psi_{i,0}\|_{C^\alpha(\mathbb{T})}\leqslant C_0$, and $\inf_{x\in\mathbb{T}}\psi_{2,0}(x)\geqslant c_0$. Let $(\psi_1\,,\psi_2)$ denote a PM coupling of two solutions to (1.1) with respective initial profiles $\psi_{1,0}$ and $\psi_{2,0}$, and consider the stopping time,*

$$\tau := \inf\{s>0:\,\psi_1(s)=\psi_2(s)\} \qquad [\inf\varnothing:=\infty].$$

Then there exists non-random numbers $t_1,\delta_1\in(0\,,1)$ – depending only on $(c_0\,,C_0\,,\alpha)$ – such that

$$\mathrm{P}\left\{\psi_1(\tau+s)=\psi_2(\tau+s)\ \textit{for all}\ s\geqslant 0\ \textit{and}\ \tau\leqslant t_1\right\}\geqslant 1-\varepsilon,$$

provided that $\|\psi_{1,0}-\psi_{2,0}\|_{L^1(\mathbb{T})}\leqslant\delta_1$.

[6]The notation is deliberately slightly inconsistent, as the recently derived random field ψ_1 is not defined on the same probability space that the earlier-defined ψ_1 was. It does, however, have the same law of course.

Let us make two brief remarks first. We will prove the lemma afterward.

Remark 7.3. In the context of PM couplings, when we say that τ is a stopping time we mean that τ is a stopping time with respect to the filtration $\{\mathscr{G}_t\}_{t\geqslant 0}$ generated by the underlying two noise $\dot{\mathcal{W}}_1$ and $\dot{\mathcal{W}}_2$ used in the PM coupling. That is, for every $t \geqslant 0$, we first let $\mathscr{G}_t$ define the sigma algebra generated by all random variables of the form $\int_{(0,t)\times\mathbb{T}} \phi(s\,,x)\,\mathcal{W}_i(\mathrm{d}s\,\mathrm{d}x)$ as i ranges over $\{1\,,2\}$ and ϕ roams over non-random elements of $L^2([0\,,t]\times\mathbb{T})$. This forms a filtration $\{\mathscr{G}_t\}_{t\geqslant 0}$, which we then augment by making it right continuous, and then by completing the sigma algebra $\mathscr{G}_0$ with respect to the measure P, as is done in martingale theory.

Definition. Let $(\psi_1\,,\psi_2)$ denote a given coupling, using any coupling method, of the solutions to (1.1) with respective initial profiles $\psi_{1,0}$ and $\psi_{2,0}$. Also, choose and fix some number $t > 0$. We say that the coupling $(\psi_1\,,\psi_2)$ is *successful by time t* if $\psi_1(t+s) = \psi_2(t+s)$ for all $s \geqslant 0$.

We are ready to prove Lemma 7.2.

Proof. Define

$$\Delta(t\,,x) := \psi_1(t\,,x) - \psi_2(t\,,x) \qquad \text{for all } t \geqslant 0 \text{ and } x \in \mathbb{T}.$$

The same reasoning that led to eq. (3.4) of Mueller (1993) leads us to the assertion that Δ solves the SPDE

$$\begin{bmatrix} \partial_t\Delta = \partial_x^2\Delta + V(\psi_1) - V(\psi_2) + \lambda\left[|\sigma(\psi_1)-\sigma(\psi_2)|^2 + 2\sigma(\psi_1)\sigma(\psi_2)\dfrac{f^2(\Delta)}{1+g(\Delta)}\right]^{1/2}\dot{F}, \\ \text{subject to } \Delta(0) = \psi_{1,0} - \psi_{2,0}, \end{bmatrix}$$

where $\dot{F}$ is a space-time white noise, and f and g were defined in (7.2). Since $\psi_{1,0} \geqslant \psi_{2,0}$, part 2 of Proposition 7.1 ensures that $\Delta \geqslant 0$ a.s., and hence $\|\psi_1(t) - \psi_2(t)\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} \Delta(t\,,x)\,\mathrm{d}x$ for all $t \geqslant 0$. The main portion of the proof is to demonstrate that

$$\mathrm{P}\{\tau > t\} = \mathrm{P}\left\{\inf_{s\in(0,t)} X(s) > 0\right\} < 1 \quad \text{for all sufficiently small } (t\,,\delta) \in (0\,,\infty)^2, \tag{7.7}$$

where

$$X(t) := \int_{\mathbb{T}} \Delta(t\,,x)\,\mathrm{d}x \qquad [t \geqslant 0].$$

Since ψ_1 and ψ_2 are continuous random fields, this proves that $\mathrm{P}\{\tau > t\} < 1$ for $(t\,,\delta)$ small, which is the more challenging portion of this proof. Once we prove this, we will easily complete the proof of the remainder of the proposition at the end.

Lemma 3.2 of Mueller (1993) [with our X playing the role of that lemma's U] implies that

$$X(t) = \delta + \int_0^t C(s)\,\mathrm{d}s + M(t) \qquad \text{for all } t > 0, \tag{7.8}$$

where $\delta = \|\psi_{1,0} - \psi_{2,0}\|_{L^1(\mathbb{T})} \in (0\,,1)$,

$$C(t) = \int_{\mathbb{T}} [V(\psi_1(t\,,x)) - V(\psi_2(t\,,x))]\,\mathrm{d}x = X(t) - \int_{\mathbb{T}} [F(\psi_1(t\,,x)) - F(\psi_2(t\,,x))]\,\mathrm{d}x,$$

and $M = \{M(t)\}_{t\geqslant 0}$ is a continuous $L^2(P)$-martingale with quadratic variation,

$$\langle M\rangle(t) = \lambda^2\int_0^t \mathrm{d}s\int_{\mathbb{T}}\mathrm{d}x\left[(\sigma(\psi_1(s\,,x)) - \sigma(\psi_2(s\,,x)))^2 + 2\sigma(\psi_1(s\,,x))\sigma(\psi_2(s\,,x))\frac{f^2(\Delta(s\,,x))}{1+g(\Delta(s\,,x))}\right],$$

for every $t > 0$. In particular, we use the facts that: (a) $g(z) \leqslant 1$ for all $z \in \mathbb{R}$; and (b) $\psi_2 \leqslant \psi_1$ [Proposition 7.1] in order to see that a.s. for all $t > 0$,

$$\begin{aligned}\frac{\mathrm{d}\langle M\rangle(t)}{\mathrm{d}t} &\geqslant 2\lambda^2 \mathrm{L}_\sigma^2 \inf_{r\in(0,t)} \inf_{y\in\mathbb{T}} |\psi_2(r\,,y)|^2 \int_{\mathbb{T}} \frac{f^2(\Delta(s\,,x))}{1+g(\Delta(s\,,x))}\,\mathrm{d}x \\ &\geqslant \lambda^2 \mathrm{L}_\sigma^2 \inf_{r\in(0,t)} \inf_{y\in\mathbb{T}} |\psi_2(r\,,y)|^2 \int_{\mathbb{T}} \min\{\Delta(s\,,x)\,,1\}\,\mathrm{d}x.\end{aligned} \tag{7.9}$$

Because of **(F1)**, F is non decreasing. Therefore, we can infer that $C(t) \leqslant X(t)$ a.s. for all $t \geqslant 0$, and hence (7.8) ensures that X satisfies the stochastic differential inequality,

$$\mathrm{d}X(t) \leqslant X(t)\,\mathrm{d}t + \mathrm{d}M(t).$$

Next, we choose and fix some $\varepsilon \in (0\,, \frac{1}{2}\{c_0 \wedge (1-\delta)\})$, and consider the stopping time,

$$\mathcal{H} := \min_{i\in\{1,2\}} \inf\left\{ s > 0 : \sup_{x\in\mathbb{T}} |\psi_i(s\,,x) - \psi_{i,0}(x)| \geqslant \varepsilon \right\},$$

where $\inf \varnothing := \infty$.

For every $t > 0$, the following holds almost surely on $\{\mathcal{H} > t\}$:

$$\sup_{s\in(0,t)} \sup_{x\in\mathbb{T}} \Delta(s\,,x) < \delta + 2\varepsilon < 1 \text{ and } \inf_{r\in(0,t)} \inf_{y\in\mathbb{T}} \psi_2(r\,,y) > c_0 - \varepsilon > \frac{c_0}{2} > 0.$$

Therefore, (7.9) implies that

$$\frac{\mathrm{d}\langle X\rangle(t)}{\mathrm{d}t} = \frac{\mathrm{d}\langle M\rangle(t)}{\mathrm{d}t} \geqslant \left(\frac{\lambda \mathrm{L}_\sigma c_0}{4}\right)^2 X(t) \qquad \text{a.s. on } \{\mathcal{H} > t\},$$

for every non-random real number $t > 0$. In particular,

$$\int_0^t \mathrm{e}^{-s}\,\frac{\mathrm{d}\langle X\rangle(s)}{X(s)} \geqslant \left(\frac{\lambda \mathrm{L}_\sigma c_0 \sqrt{1-\mathrm{e}^{-t}}}{4}\right)^2 \qquad \text{a.s. on } \{\mathcal{H} > t\}.$$

Combine these facts together with Proposition A.4, and recall that $X(0) = \delta$, in order to see that

$$\begin{aligned}\mathrm{P}\{\tau > t\,, \mathcal{H} > t\} &\leqslant \mathrm{P}\left\{ \inf_{s\in(0,t)} X(s) > 0\,, \int_0^t \mathrm{e}^{-s}\,\frac{\mathrm{d}\langle X\rangle(s)}{X(s)} \geqslant \left(\frac{\lambda \mathrm{L}_\sigma c_0 t\sqrt{1-\mathrm{e}^{-t}}}{4}\right)^2 \right\} \\ &\leqslant 2\mathrm{P}\left\{ |\mathcal{Z}| < \frac{8\sqrt{\delta}}{\lambda \mathrm{L}_\sigma c_0 \sqrt{1-\mathrm{e}^{-t}}} \right\} \leqslant \frac{16\sqrt{\delta}}{\lambda \mathrm{L}_\sigma c_0 \sqrt{t}},\end{aligned}$$

where $\mathcal{Z}$ has a standard normal distribution. We have appealed to the simple bound $1-\exp(-t) \geqslant t$ and the fact that the probability density function of $\mathcal{Z}$ is at most $(2\pi)^{-1/2} < 1/2$ for the last inequality.

Next, we observe that

$$\mathrm{P}\{\mathcal{H} \leqslant t\} \leqslant \sum_{i=1}^2 \mathrm{P}\left\{ \sup_{s\in(0,t)} \sup_{x\in\mathbb{T}} |\psi_i(s\,,x) - \psi_{i,0}(x)| \geqslant \varepsilon \right\}.$$

Recall that, for a standard 1-D Brownian motion β, $i \in \{1\,,2\}$, and $s > 0$,

$$\|\mathcal{P}_s\psi_{i,0} - \psi_{i,0}\|_{C(\mathbb{T})} \leqslant \sup_{x\in\mathbb{T}} \mathrm{E}\,|\psi_{i,0}(\beta(s)+x) - \psi_{i,0}(x)| \leqslant \|\psi_{i,0}\|_{C^\alpha(\mathbb{T})}\mathrm{E}\,(|\beta(s)|^\alpha) \leqslant KC_0 s^{\alpha/2},$$

where $K = \mathrm{E}(|\beta(1)|^\alpha) = 2^{(1+\alpha)/2}\pi^{-1/2}\Gamma((2+\alpha)/2)$. Therefore, Proposition 4.8 and Chebyshev's inequality together imply that for every $k \geqslant 2$ there exists a real number $K_1 = K_1(k\,,\alpha\,,A) > 0$ such that $\mathrm{P}\{\mathcal{H} \leqslant t\} \leqslant K_1 t^{\alpha k/2}$ for all $t \in [0\,,1]$. Choose k large enough to ensure that $\mathrm{P}\{\mathcal{H} \leqslant t\} \leqslant K_1\sqrt{t}$ and hence

$$\inf_{t\in(0,1)} \mathrm{P}\left\{\tau > t\right\} \leqslant K_2 \inf_{t\in(0,1)} \left(\sqrt{\frac{\delta}{t}} + \sqrt{t}\right) \leqslant 2K_2\delta^{1/4},$$

where $K_2 > 0$ does not depend on $\delta \in (0\,,1)$, even though it might depend on $(c_0\,,C_0\,,\alpha)$. This implies (7.7). In order to complete the proof, it suffices to show that

$$\psi_1(\tau+s\,,x) = \psi_2(\tau+s\,,x) \text{ for all } s > 0 \text{ and } x \in \mathbb{T}, \text{ almost surely on } \{\tau < \infty\}.$$

Equivalently, it remains to prove that

$$X(\tau+t) = 0 \text{ for all } t > 0, \text{ almost surely on } \{\tau < \infty\}.$$

Define $Y(t) := \exp(-t)X(t)$ and apply Itô's formula to (7.8) in order to see that Y is a continuous, non-negative supermartingale. Since τ denotes the first time Y hits zero, it follows from a classical exercise in elementary martingale theory that

$$Y(\tau+t) = 0 \text{ for all } t > 0, \text{ almost surely on } \{\tau < \infty\}.$$

Because $Y(s) = 0$ iff $X(s) = 0$ for any and every $s \geqslant 0$, this has the desired effect. □

(iv) Anchored monotone* (AM) *coupling. The AM coupling is a more attractive variation of the PM coupling, where the qualifier "attractive" is used in the same vein as it is used in particle systems.

Before we describe the AM coupling, let us mention the following "attractive" property of the AM coupling.

Lemma 7.4. *Choose and fix non-random numbers $C_0 > c_0 > 0$ and $\alpha, \varepsilon \in (0\,,1/2)$, and consider $\psi_{1,0}, \psi_{2,0} \in C^\alpha_+(\mathbb{T})$ such that $\max_{i\in\{1,2\}} \|\psi_{i,0}\|_{C^\alpha(\mathbb{T})} \leqslant C_0$, and $\min_{i\in\{1,2\}} \inf_{x\in\mathbb{T}} \psi_{i,0}(x) \geqslant c_0$. Let $(\psi_1\,,\psi_2)$ denote an AM coupling of two solutions to (1.1) with respective initial profiles $\psi_{1,0}$ and $\psi_{2,0}$, and consider the stopping time, $\tau := \{s > 0 :\ \psi_1(s) = \psi_2(s)\}$, where $\inf\varnothing := \infty$. Then, for the same numbers $t_1, \delta_1 \in (0\,,1)$ that arise in Lemma 7.2,*

$$\mathrm{P}\left\{\psi_1(\tau+s) = \psi_2(\tau+s) \textit{ for all } s \geqslant 0 \textit{ and } \tau \leqslant t_1\right\} \geqslant 1 - 2\varepsilon,$$

provided that $\|\psi_{1,0} - \psi_{2,0}\|_{L^1(\mathbb{T})} \leqslant \delta_1/2$.

This is exactly the same assertion as the one in Lemma 7.2, except we no longer need to assume that $\psi_{1,0} \leqslant \psi_{2,0}$. Next, we describe the AM coupling which accomplishes this generalization. Lemma 7.4 will be an immediate consequence of that description.

Suppose $\psi_{1,0}, \psi_{2,0} \in C_+(\mathbb{T})$ are fixed non-random initial profiles. Let us introduce three independent space-time white noises $\dot{\mathcal{W}}$, $\dot{\mathcal{W}}_1$, and $\dot{\mathcal{W}}_2$, and let ψ denote the solution to the SPDE,

$$\partial_t\psi = \partial_x^2\psi + V(\psi) + \sigma(\psi)\dot{\mathcal{W}}, \quad \text{subject to} \quad \psi(0) = \psi_0 := \psi_{1,0} \vee \psi_{2,0}.$$

Then, we use $\dot{\mathcal{W}}$ and $\dot{\mathcal{W}}_i$ $[i = 1, 2]$ to construct a PM coupling $(\psi\,,\psi_i)$, where the initial distribution of ψ_i is $\psi_{i,0}$. We refer to this construction of $(\psi_1\,,\psi_2)$ as an *AM coupling* of the solutions to (1.1) with respect initial data $(\psi_{1,0}\,,\psi_{2,0})$, and to ψ as the *anchor* process for ψ_1 and ψ_2. The following is a ready consequence of the proof of Proposition 4.13.

Lemma 7.5. *Let $(\psi_1\,,\psi_2)$ denote an AM coupling of the solutions to the SPDE (1.1) with respective initial profiles $\psi_{1,0},\psi_{2,0}\in\cup_{\alpha\in(0,1/2)}C^\alpha_+(\mathbb{T})$, and let ψ denote the associated process. Then, $\{(\psi(t)\,,\psi_1(t)\,,\psi_2(t))\}_{t\geqslant 0}$ is a Feller process with values in $C(\mathbb{T}\,;\mathbb{R}^3)$.*

In order to be guaranteed that we can perform this construction, we might of course have to enlarge the underlying probability space; see Proposition 7.1. It follows immediately from Proposition 7.1 that the marginal laws of ψ_i is the same as the law of the solution to the SPDE (1.1) starting at $\psi_{i,0}$. So this produces a coupling indeed. And Lemma 7.5 follows from the method of proof of Proposition 4.13. We skip the details of the proof, and merely refer to the comments made about the proof of part 3 of Proposition 7.1.

Now we prove Lemma 7.4.

Proof of Lemma 7.4. This is basically the argument that appears at the very last portion of the paper by Mueller (1993); see the paragraphs surrounding eq. therein (*ibid.*). We repeat the proof here for the convenience of the reader.

Proposition 7.1 insures that $\psi\geqslant\max\{\psi_1\,,\psi_2\}$ a.s., and Lemma 7.2 ensures that the coupling $(\psi\,,\psi_i)$ is successful for either choice of $i\in\{1\,,2\}$, with probability $\geqslant 1-\varepsilon$, by the same time $t_1\in(0\,,1)$ as was given in Lemma 7.2, provided that the condition $\|\psi_0-\psi_{i,0}\|_{L^1(\mathbb{T})}\leqslant\delta_1$ is met for either $i\in\{1\,,2\}$. Because $\|\psi_{1,0}-\psi_{2,0}\|_{L^1(\mathbb{T})}\leqslant\delta_1/2$, we find that $\|\psi_0-\psi_{i,0}\|_{L^1(\mathbb{T})}\leqslant\delta_1$ for both $i=1,2$, in fact. Thus, it follows that

$$\mathrm{P}\left\{\text{the PM coupling of }(\psi\,,\psi_i)\text{ is successful by time }t_1\right\}\geqslant 1-\varepsilon,$$

for both $i=1,2$. If the coupling of $(\psi\,,\psi_1)$ is successful by time t_1 and the coupling of $(\psi\,,\psi_2)$ is successful by time t_1, then certainly the AM coupling of $(\psi_1\,,\psi_2)$ is successful by time t_1. Therefore,

$$\begin{aligned}&\mathrm{P}\left\{\text{the AM coupling of }(\psi_1\,,\psi_2)\text{ is not successful by time }t_1\right\}\\&\qquad\leqslant\sum_{i=1}^2\mathrm{P}\left\{\text{the PM coupling of }(\psi\,,\psi_i)\text{ is not successful by time }t_1\right\}\leqslant 2\varepsilon.\end{aligned}$$

Thanks to the strong Markov property of $(\psi\,,\psi_1\,,\psi_2)$ [Lemma 7.5] if the above couplings are successful then the first time to succeed is a stopping time. This completes the proof. □

8 Uniqueness of a non-trivial invariant measure via coupling

The main result of this section is the following uniqueness result.

Theorem 8.1. *If $\mathrm{L}_\sigma>0$ and λ is small enough to ensure that the conclusion of Proposition 4.16 is valid, then there is at most one invariant measure μ for (1.1) that satisfies $\mu\{\mathbb{0}\}=0$. That measure is μ_+ of Proposition 4.16.*

We shall combine coupling ideas from the previous section in order to prove Theorem 8.1. As first step in that proof, we offer the following technical result, valid for any coupling method, including those that are possibly not mentioned in this paper.

Lemma 8.2. *Choose and fix two non-random functions $\psi_{1,0},\psi_{2,0}\in C_{>0}(\mathbb{T})$, and let $(\psi_1\,,\psi_2)$ denote any coupling of two solutions to (1.1) with respective initial profiles $\psi_{1,0}$ and $\psi_{2,0}$. Choose an arbtirary non-random number $q>0$, and let $\mathcal{T}_0$ denote any a.s.-finite stopping time with respect to* *the underlying noises of $(\psi_1\,,\psi_2)$. Then, there exist non-random numbers $C_0 > c_0 > 0$ such that the stopping times*

$$\mathcal{T}_n := \inf\left\{ s > \mathcal{T}_{n-1} + q : \max_{i\in\{1,2\}} \|\psi_i(s)\|_{C^\alpha(\mathbb{T})} \leqslant C_0 \quad or \quad \min_{i\in\{1,2\}} \inf_{x\in\mathbb{T}} \psi_i(s\,,x) \geqslant c_0 \right\}$$

are a.s. finite for every $n \in \mathbb{N}$. The constants c_0 and C_0 do not depend on the particular coupling method used.

Proof. Since ψ_1 and ψ_2 are continuous random fields [see Theorem 2.3], the random mappings $t \mapsto \min_{i\in\{1,2\}} \psi_i(t\,,x)$ and $t \mapsto \max_{i\in\{1,2\}} \|\psi_i(t)\|_{C^\alpha(\mathbb{T})}$ define continuous and adapted processes. This proves that every $\mathcal{T}_n$ is indeed a stopping time. We now prove the more interesting statement that these $\mathcal{T}_n$'s are a.s. finite for suitable non-random choices of $C_0 \gg 1$ and $c_0 \ll 1$ that do not depend on the particular details of the coupling.

Recall that our proof of (4.12) hinged on proving that if $\psi_0 = \mathbb{1}$ and ψ solves (1.1) starting from ψ_0 then

$$\lim_{c\downarrow 0} \liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\{c \leqslant \inf_{x\in\mathbb{T}} \psi(t,x) \leqslant \|\psi(t)\|_{C^\alpha(\mathbb{T})} \leqslant 1/c\}}\,\mathrm{d}t = 1 \qquad \text{a.s.}$$

See the paragraphs that follow Proposition 4.16, as well as (4.13). A brief inspection of the random walk argument shows that the same fact holds for every $\psi_0 \in C_{>0}(\mathbb{T})$ [not just $\psi_0 \equiv 1$].[7] In particular, we can find non-random numbers $0 < c_i < C_i$ such that

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\{c_i \leqslant \inf_{x\in\mathbb{T}} \psi_i(t,x) \leqslant \|\psi_i(t)\|_{C^\alpha(\mathbb{T})} \leqslant C_i\}}\,\mathrm{d}t \geqslant \frac{3}{4} \qquad \text{a.s. for } i \in \{1\,,2\}.$$

Clearly, $(c_1\,,c_2\,,C_1\,,C_2)$ depend only on the marginal laws of ψ_1 and ψ_2. Therefore, $(c_1\,,c_2\,,C_1\,,C_2)$ does not depend on how ψ_1 and ψ_2 are coupled.

Define

$$\mathcal{E}_i := \left\{ t \geqslant 0 :\ c_i \leqslant \inf_{x\in\mathbb{T}} \psi_i(t,x) \leqslant \|\psi_i(t)\|_{C^\alpha(\mathbb{T})} \leqslant C_i \right\},$$

and let m_T denote the measure defined by

$$m_T(F) := \frac{1}{T} \int_0^T \mathbf{1}_F(t)\,\mathrm{d}t \qquad \text{for all } T > 0 \text{ and Borel sets } F \subset \mathbb{R}_+.$$

Since $m_T([\mathcal{E}_1 \cap \mathcal{E}_2]^c) \leqslant m_T(\mathcal{E}_1^c) + m_T(\mathcal{E}_2^c)$, it follows that $\liminf_{T\to\infty} m_T(\mathcal{E}_1 \cap \mathcal{E}_2) \geqslant 1/2 > 0$ whence it follows that $\mathcal{E}_1 \cap \mathcal{E}_2$ is unbounded with probability one. The a.s.-finiteness of the $\mathcal{T}_n$'s is now immediate. $\square$

[7] Indeed, the fact that Proposition 4.6 holds equally well for non-constant initial data implies that the same proof that follows Proposition 4.16 goes to show that, in the present setting,

$$\lim_{c\downarrow 0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\{\|\psi(t)\|_{C^\alpha(\mathbb{T})} > 1/c\}}\,\mathrm{d}t = 0 \qquad \text{a.s.}$$

It therefore remains to prove that

$$\lim_{c\downarrow 0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{1}_{\{\inf_{x\in\mathbb{T}} \psi(t) < c\}}\,\mathrm{d}t = 0 \qquad \text{a.s.} \tag{8.1}$$

Let $a := \inf_{x\in\mathbb{T}} \psi_0(x)$. Since $a > 0$, we can appeal to Lemma 3.3 and compare ψ to the solution of (1.1) in order to reduce the problem to proving the above in the case that $\psi_0 = a\mathbb{1}$. Now, $a^{-1}\psi$ solves (1.1) but with $\Theta := (\sigma\,,F)$ replaced by $\Theta_a := (a^{-1}\sigma(a\cdot)\,,a^{-1}F(a\cdot))$, starting from $\mathbb{1}$. This proves (8.1) since Θ_a has the same analytic properties that we required of Θ.

We now use Lemma 8.2 as a "regeneration result," in order to prove the main step in the proof of Theorem 8.1. In anticipation of future potential applications, we record that regeneration result as the following theorem.

Theorem 8.3. *Choose and fix two non-random functions $\psi_{1,0}, \psi_{2,0} \in C_+(\mathbb{T}) \setminus \{\mathbb{0}\}$. Then there exists a successful coupling $(\psi_1\,,\psi_2)$ of solutions to (1.1) with respective initial profiles $(\psi_{1,0}\,,\psi_{2,0})$.*

Proof. We first prove the theorem in the case that $\psi_{1,0}, \psi_{2,0} \in C_{>0}(\mathbb{T})$, a condition which we assume until further notice. Throughout, we choose and fix some $\alpha, \varepsilon \in (0\,,1/2)$; to be concrete,

$$\alpha = \varepsilon = \frac{1}{4}.$$

We build a hybrid coupling that makes appeals to natural coupling, independent coupling, and AM coupling. Our coupling is performed inductively, and in stages.

Throughout, let us choose and fix the non-random numbers $0 < c_0 < C_0$ – depending on $(\psi_{1,0}\,,\psi_{2,0})$ – whose existence (and properties) are guaranteed by Lemma 8.2. It should be clear from Lemma 8.2 that the assertion of that lemma continues to remain valid if instead of C_0 we choose a larger number. Therefore, we increase C_0 once and for all, if need be, in order to ensure additionally that

$$C_0 > 3c_0. \tag{8.2}$$

We begin with the natural coupling of $(\psi_1\,,\psi_2)$ until stopping time,

$$\mathcal{T}_1 := \inf\left\{ s > 0 : \max_{1\in\{1,2\}} \|\psi_i(s)\|_{C^\alpha(\mathbb{T})} \leqslant C_0 \quad \text{or} \quad \min_{i\in\{1,2\}} \inf_{x\in\mathbb{T}} \psi_i(s\,,x) \geqslant c_0 \right\}.$$

Lemma 8.2 ensures that $\mathcal{T}_1 < \infty$ a.s. This yields a coupling of the sort that we want, but only until time $\mathcal{T}_1$.

To extend our coupling beyond time $\mathcal{T}_1$ we first apply Proposition 6.1 with $A := C_0$ and $A_0 := 2c_0$ in order to obtain a non-random strictly positive number t_0. Then, starting from $(\psi_1(\mathcal{T}_1)\,,\psi_2(\mathcal{T}_1))$ we run an independent coupling for t_0 units of time. By the strong Markov property, this yields a coupling of the sort that we want until stopping time $\mathcal{T}_1 + t_0$.

In order to continue our construction beyond time $\mathcal{T}_1 + t_0$, we first let t_1 denote the number that was defined in Lemma 7.2. Then, conditionally independently from the construction so far, we run an AM coupling starting from $(\psi_1(\mathcal{T}_1 + t_0)\,,\psi_2(\mathcal{T}_1 + t_0))$ for t_1 units of time. By the strong Markov property, this yields a coupling that we want until stopping time $\mathcal{T}_1 + t_0 + t_1$. Thanks to Lemma 7.4 if the two processes have merged some time in $(\mathcal{T}_1 + t_0\,,\mathcal{T}_1 + t_0 + t_1)$, then from that time until time $\mathcal{T}_1 + t_0 + t_1$ they are equal. In this case, we just continue running our AM coupling to see that we have a successful coupling, as desired. If the two processes have not merged by time $\mathcal{T}_1 + t_0 + t_1$, then we continue our AM coupling until time

$$\mathcal{T}_2 := \inf\left\{ s > \mathcal{T}_1 + t_0 + t_1 : \max_{1\in\{1,2\}} \|\psi_i(s)\|_{C^\alpha(\mathbb{T})} \leqslant C_0 \quad \text{or} \quad \min_{i\in\{1,2\}} \inf_{x\in\mathbb{T}} \psi_i(s\,,x) \geqslant c_0 \right\}.$$

Then, run an independent coupling for t_0 units of time, and then an AM coupling for another t_1 units of time [all conditionally independently of the past in order to maintain the strong Markov property]. This yields a coupling up to time $\mathcal{T}_2 + t_0 + t_1$. If the two processes have merged some time between $\mathcal{T}_2 + t_0$ and $\mathcal{T}_2 + t_0 + t_1$ then continue running the final AM coupling *ad infinitum.* Lemma 7.4 ensures that this is the desired successful coupling. Else, we continue inductively.

Choose and fix some $n \in \mathbb{N}$, and let δ_1 by the number given by Lemma 7.2. Thanks to Lemma 7.2, we may (and will) assume without loss of generality that

$$\delta_1 < 1 \wedge \frac{c_0}{10}. \tag{8.3}$$

By the strong Markov property, Proposition 6.1 ensures that, almost surely on the event that the coupling is not successful by time $\mathcal{T}_n$, the conditional probability of the event

$$\mathcal{E}_n := \left\{ \|\psi_1(\mathcal{T}_n + t_0) - \psi_2(\mathcal{T}_n + t_0)\|_{C(\mathbb{T})} \leqslant \frac{\delta_1}{2} \right\} \cap \bigcap_{i=1}^{2} \left\{ \|\psi_i\|_{C^{\alpha/2}(\mathbb{T})} \leqslant C_0 + 1 \right\}$$

is at least $[\mathfrak{p}_{C_0,2c_0}(t_0\,, \alpha\,, \delta_1)]^2 > 0$. In order for this assertion to be true, we need to know additionally that $[2c_0 - \delta_1\,, 2c_0 + \delta_1] \subset [c_0\,, C_0]$; this is so because of (8.2) and (8.3).

We emphasize that δ_1 is deterministic (as it did not depend on the initial condition in Proposition 6.1) as well as independent of n. And Lemma 7.4 [with α replaced by $\alpha/2$] ensures that, a.s. on $\mathcal{E}_n$ the conditional probability that the coupling has succeeded by time $\mathcal{T}_n + t_0 + t_1$ given everything by time $\mathcal{T}_n + t_0$ is at least 3/4. Thus,

$$\mathrm{P}\left(\text{the coupling succeeds by time } \mathcal{T}_n + t_0 + t_1 \mid \text{no success by time } \mathcal{T}_n\right) \geqslant \frac{3\left[\mathfrak{p}_{C_0,2c_0}(t_0\,, \alpha\,, \delta_1)\right]^2}{4}.$$

Since the right-hand side does not depend on n, the tower property of conditional probabilities yield the following for every $n \in \mathbb{N}$:

$$\mathrm{P}\left(\text{the coupling does not succeed by time } \mathcal{T}_n + t_0 + t_1\right) \leqslant \left(1 - \frac{3\left[\mathfrak{p}_{C_0,2c_0}(t_0\,, \alpha\,, \delta_1)\right]^2}{4}\right)^{n-1}.$$

Since $\mathcal{T}_{n+1} - \mathcal{T}_n > t_0 + t_1$ a.s. for every $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \mathcal{T}_n = \infty$ a.s. Thus, we let $n \to \infty$ to conclude the proof, from preceding display, in the case that $\psi_{1,0}, \psi_{2,0} \in C_{>0}(\mathbb{T})$.

In order to prove the general result, we first run our natural coupling for one unit of time, starting from $(\psi_{1,0}\,, \psi_{2,0})$. Theorem 2.3 assures us that with probability one, $\psi_i(1\,, x) > 0$ for every $x \in \mathbb{T}$ and $i \in \{1\,, 2\}$. Condition on everything by time one, and run our hybrid coupling from then on, conditionally independently of the first one time unit, starting from $(\psi_1(1)\,, \psi_2(1))$. Apply the strong Markov property and the first portion of the proof to finish. □

We are in position to prove Theorem 8.1.

Proof of Theorem 8.1. Let $\phi \in C_+(\mathbb{T}) \setminus \{\mathbb{0}\}$ be non-random. Theorem 8.3 ensures that, after we possibly enlarging the underlying probability space, we can construct a successful coupling $(\psi_0\,, \psi_1)$ of solutions to (1.1) that start respectively from $(\phi\,, \mathbb{1})$. In particular,

$$\lim_{T\to\infty} \sup_{\substack{\Gamma\subset C(\mathbb{T})\\ \Gamma \text{ Borel}}} \left| \frac{1}{T}\int_0^T \mathbf{1}_\Gamma(\psi_0(t))\,\mathrm{d}t - \frac{1}{T}\int_0^T \mathbf{1}_\Gamma(\psi_1(t))\,\mathrm{d}t \right| = 0 \qquad \text{a.s.}$$

Take expectations and appeal to the bounded convergence theorem in order to deduce that

$$\lim_{T\to\infty} \sup_{\substack{\Gamma\subset C(\mathbb{T})\\ \Gamma \text{ Borel}}} \left| \mathrm{E}\left(\frac{1}{T}\int_0^T \mathbf{1}_\Gamma(\psi_0(t))\,\mathrm{d}t\right) - \mathrm{E}\left(\frac{1}{T}\int_0^T \mathbf{1}_\Gamma(\psi_1(t))\,\mathrm{d}t\right) \right| = 0.$$

Notice that the preceding is a statement about probability laws and does not depend on the coupling construction that was devised in order to prove it. Therefore, it is convenient to set $F := \mathbf{1}_\Gamma$ and revert to the notation of Markov process theory (see §4.3 and especially §4.4), and rewrite the above in terms of the Feller semigroup $\{P_t\}_{t\geqslant 0}$ associated to (1.1) as follows:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (\delta_\phi P_t)\,\mathrm{d}t = \mu_+ \qquad \text{in total variation.}$$

Let μ denote any probability measure on $C_+(\mathbb{T})$ that satisfies $\mu\{\mathbb{0}\} = 0$ and is invariant for (1.1). Proposition (4.16) ensures that there is at least one such measure μ_+ and that, in fact, μ_+ concentrates on $C_{>0}(\mathbb{T})$. Because μ is invariant, Tonelli's theorem ensures that

$$\int_{C(\mathbb{T})} \left(\frac{1}{T}\int_0^T (\delta_\phi P_t)(\Gamma)\,\mathrm{d}t\right)\mu(\mathrm{d}\phi) = \frac{1}{T}\int_0^T \left(\int_{C(\mathbb{T})} (\delta_\phi P_t)(\Gamma)\mu(\mathrm{d}\phi)\right)\mathrm{d}t = \mu(\Gamma),$$

for all $T > 0$ and Borel sets $\Gamma \subset C(\mathbb{T})$. Therefore, the bounded convergence theorem yields

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (\delta_{\mathbb{1}} P_t)\,\mathrm{d}t = \mu \qquad \text{in total variation.}$$

Our proof of Proposition 4.16 consisted of proving that μ_+ is a subsequential weak limit of the probability measures $\{T^{-1}\int_0^T (\delta_{\mathbb{1}} P_t)\,\mathrm{d}t\}_{T>0}$. Therefore, the above shows that $\mu = \mu_+$, which completes the uniqueness of μ_+. □

Let us conclude this section with an ergodic theorem for the solution to (1.1).

Corollary 8.4 (Ergodic theorem)**.** *Suppose* $\mathrm{L}_\sigma > 0$ *and* λ *is small enough to ensure that the conclusion of Proposition 4.16 is valid. Then, for every probability measure* ν *on* $C_+(\mathbb{T})$ *that satisfies* $\nu\{\mathbb{0}\} = 0$*,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (\nu P_t)\,\mathrm{d}t = \mu_+ \quad \text{in total variation,}$$

where μ_+ *is the invariant measure produced by Proposition 4.16.*

Proof. The proof of Theorem 8.1 readily implies that for every $\phi \in C_+(\mathbb{T}) \setminus \{\mathbb{0}\}$,

$$\frac{1}{T}\int_0^T (\delta_\phi P_t)\,\mathrm{d}t - \frac{1}{T}\int_0^T (\delta_{\mathbb{1}} P_t)\,\mathrm{d}t \quad \text{in total variation, as } T\to\infty.$$

In particular, we use the above twice [once for ϕ_1, and once for ϕ_2, in place of ϕ] in order to see that for every $\phi_1, \phi_2 \in C_+(\mathbb{T}) \setminus \{\mathbb{0}\}$,

$$\frac{1}{T}\int_0^T (\delta_{\phi_1} P_t)\,\mathrm{d}t - \frac{1}{T}\int_0^T (\delta_{\phi_2} P_t)\,\mathrm{d}t \quad \text{in total variation, as } T\to\infty.$$

Integrate over all such ϕ_1 [$\mathrm{d}\nu$] and all such ϕ_2 [$\mathrm{d}\mu_+$] to deduce the corollary from the bounded convergence theorem and the invariance of μ_+. □

9 Proofs of Theorem 2.4 and Remark 2.5

Parts 1(a) and 1(d) of Theorem 2.4 were proved respectively in Theorem 8.1 and Corollary 8.4.

Part 1(b) of Theorem 2.4 is now easy to prove. Indeed, any probability measure μ on $C_+(\mathbb{T})$ is a linear combination of $\delta_{\mathbb{0}}$ and some other probability measure μ_1 on $C_+(\mathbb{T}) \setminus \{\mathbb{0}\}$. Since $\delta_{\mathbb{0}}$ is always invariant, it follows immediately that so is μ_1. Part 1(a) now shows that $\mu = \mu_+$. This proves part 1(b) because the converse is obvious, as every element of $\mathscr{IM}$ is manifestly invariant.

Thanks to part 1(d) of Theorem 2.4 – which we have already verified – and a standard approximation theorem,

$$\int F\, \mathrm{d}\mu_+ \leqslant \limsup_{T\to\infty} \frac{1}{T}\int_0^T F(\psi(t))\, \mathrm{d}t,$$

for every lower semicontinuous function $F : C_+(\mathbb{T}) \to \mathbb{R}_+$. This is basically a restatement of Fatou's theorem of classical integration theory. Since $F(\omega) := \|\omega\|^k_{C^\alpha(\mathbb{T})}$ defines a lower semicontinuous function on $C(\mathbb{T})$, it follows that

$$\int \|\omega\|^k_{C^\alpha(\mathbb{T})}\mu_+(\mathrm{d}\omega) \leqslant \sup_{t\geqslant 1} \mathrm{E}\left(\|\psi(t)\|^k_{C^\alpha(\mathbb{T})}\right). \tag{9.1}$$

This and Proposition 4.6 yield 1(c) of Theorem 2.4; see (2.4).

Next we prove part 2 of Theorem 2.4 and conclude its proof. That is, we plan to show that $\delta_{\mathbb{0}}$ is the only invariant measure for (1.1) when λ is large.

Because $V(w) \leqslant w$ for all $w \geqslant 0$, the comparison theorem for SPDEs [Lemma 3.3] shows that $\psi(t\,,x) \leqslant u(t\,,x)$ for all $t \geqslant 0$ and $x \in \mathbb{T}$, where u solves the SPDE,

$$\partial_t u(t\,,x) = \partial_x^2 u(t\,,x) + u(t\,,x) + \lambda\sigma(u(t\,,x))\dot{W}(t\,,x) \qquad \text{for } (t\,,x) \in (0\,,\infty)\times\mathbb{T},$$

subject to $u(0) = \psi_0$. Define $v(t\,,x) := \exp(-t)u(t\,,x)$, and observe that v is the solution to the SPDE,

$$\partial_t v(t\,,x) = \partial_x^2 v(t\,,x) + \lambda\sigma(t\,,v(t\,,x))\dot{W}(t\,,x) \qquad \text{for } (t\,,x) \in (0\,,\infty)\times\mathbb{T},$$

subject to $v(0) = \psi_0$, where

$$\sigma(t\,,w) := \mathrm{e}^{-t}\sigma\left(\mathrm{e}^t w\right) \qquad \text{for all } t \geqslant 0 \text{ and } w \in \mathbb{R}.$$

Evidently, $\sigma(t)$ is Lipschitz continuous, uniformly for all $t \geqslant 0$. In fact,

$$\sup_{t\geqslant 0} |\sigma(t\,,w) - \sigma(t\,,z)| \leqslant \mathrm{Lip}_\sigma |w-z| \qquad \text{for all } w, z \in \mathbb{R}.$$

Therefore, the proof of Theorem 1.2 of Khoshnevisan et al. (2020) [with σ replaced everywhere by $\sigma(t)$] works verbatim to imply the existence of a real number $c > 0$ – independent of λ – such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|v(t)\|_{C(\mathbb{T})} \leqslant -c\lambda^2 \qquad \text{a.s.}$$

The above statements together show that, with probability one,

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in\mathbb{T}}\psi(t\,,x) \leqslant \limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in\mathbb{T}}u(t\,,x) = 1 + \limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in\mathbb{T}}v(t\,,x) \leqslant 1 - c\lambda^2,$$

which is < 0 when λ is sufficiently large. Since ψ is positive, this proves that $\psi(t) \to \mathbb{0}$ as $t \to \infty$ a.s. when $1 - c\lambda^2 < 0$, when convergence takes place in $C(\mathbb{T})$. Therefore, it remains to prove that

$\delta_{\mathbb{0}}$ is the only invariant measure for (1.1) when $1 - c\lambda^2 < 0$. Thankfully, this is easy to do as all of the harder work is done by now. Indeed, suppose μ is invariant for (1.1) and $1 - c\lambda^2 < 0$ so that $\lim_{t\to\infty} \psi(t) = \mathbb{0}$ a.s.

Choose and fix $\varepsilon \in (0\,,1)$, and consider the following relatively open subsets of $C(\mathbb{T})$:

$$S_r := \left\{\omega \in C_+(\mathbb{T}) :\, \|\omega\|_{C(\mathbb{T})} > r\right\}, \quad L_R := \left\{\omega \in C_+(\mathbb{T}) :\, \|\omega\|_{C(\mathbb{T})} < R\right\} \qquad \text{for all } r, R > 0.$$

We may select $R > 0$ large enough to insure that $\mu(L_R^c) \leqslant \varepsilon$. Next, we use the notation of Markov process theory to write, for every $r, t > 0$,

$$\mu(S_r) = \int \mathrm{P}_{\psi_0}\{\psi(t) \in S_r\}\, \mu(\mathrm{d}\psi_0) \leqslant \varepsilon + \int_{L_R} \mathrm{P}_{\psi_0}\{\psi(t) \in S_r\}\, \mu(\mathrm{d}\psi_0).$$

Let ϕ be the solution to (1.1) starting from $\psi(0) = R\mathbb{1}$, using the same noise that was used for ψ. Lemma 3.3 implies that $\phi(t\,,x) \geqslant \psi(t\,,x)$ for all $t \geqslant 0$ and $x \in \mathbb{T}$, P_{ψ_0}-a.s. for every $\psi_0 \in L_R$. In other words, continuing to write using Markov process theory notation, we have

$$\mu(S_r) \leqslant \varepsilon + \mu(L_R)\mathrm{P}_{R\mathbb{1}}\{\psi(t) \in S_r\} \leqslant \varepsilon + \mathrm{P}_{R\mathbb{1}}\{\psi(t) \in S_r\},$$

for all $r, t > 0$. Since $R\mathbb{1} \in C_{>0}(\mathbb{T})$, the portion of part 2 that we proved already tells us that $\lim_{t\to\infty} \psi(t) = \mathbb{0}$, $\mathrm{P}_{R\mathbb{1}}$-a.s. In particular, $\mathrm{P}_{R\mathbb{1}}\{\psi(t) \in S_r\} \to 0$ as $t \to \infty$. Because $\mu(S_r)$ does not depend on $(t\,,\varepsilon)$, it follows that $\mu(S_r) = 0$ for every $r > 0$. This proves that $\mu = \delta_{\mathbb{0}}$, and completes the proof of Theorem 2.4. □

We conclude this section with a proof of Remark 2.5. Thanks to (9.1) and Proposition 4.6,

$$\int \|\omega\|_{C^\alpha(\mathbb{T})}^k\, \mu_+(\mathrm{d}\omega) \leqslant L_1^k \left(\sqrt{k}\,\mathcal{R}(k) + [\mathcal{R}(m_0 k)]^{m_0}\right)^k,$$

for all $k \geqslant 2$, where $\mathcal{R}$ is the function defined in Lemma 4.3, L_1 is described in Proposition 4.6, and m_0 comes from hypothesis **(F3)** from the beginning portions of this paper. Now we appeal to Example 4.7 to see that, in the context of Remark 2.5, there exists a constant $L > 0$ such that

$$\int \|\omega\|_{C^\alpha(\mathbb{T})}^k\, \mu_+(\mathrm{d}\omega) \leqslant L^k k^{2(1+\nu)k/\nu}, \quad \text{uniformly for all } k \geqslant 2.$$

This and Stirling's formula together imply Remark 2.5. □

10 On the support of μ_+

For the remainder of the paper we assume that $\lambda \in (0\,,\lambda_0)$, so that Theorem 2.4 ensures the existence of a non-trivial invariant measure μ_+. We have seen already that μ_+ has finite moments on $C^\alpha(\mathbb{T})$. In this section we derive a few additional properties of μ_+.

Throughout this section, we write $f \lesssim g$ for two nonnegative, real-valued functions f and g when there exists a number $c > 0$ such that $f(x) \leqslant cg(x)$ uniformly for all x in the common domain of definition of f and g.

We have seen already that μ_+ lives on the strictly positive functions in $C(\mathbb{T})$. Our first result is a quantitative bound that complements this fact.

Proposition 10.1. $\mu_+\{\omega \in C_+(\mathbb{T}) :\, \inf_{x\in\mathbb{T}} \omega(x) \leqslant \varepsilon\} \lesssim \varepsilon^{1/4}$ *for all* $\varepsilon \in (0\,,1)$.

It is in fact possible to adapt the proof to see that for every $\theta \in (0\,,1)$ there exists λ_θ such that if $\lambda \in (0\,,\lambda_\theta)$, then

$$\mu_+\left\{\omega \in C_+(\mathbb{T}) :\ \inf_{x\in\mathbb{T}} \omega(x) \leqslant \varepsilon\right\} \lesssim \varepsilon^\theta \qquad \text{for all } \varepsilon \in (0\,,1).$$

We skip the details and prove only Proposition 10.1.

Proof. We plan to prove that $\mu_+(\Gamma_\varepsilon) \lesssim \varepsilon^{1/4}$ uniformly for all $\varepsilon \in (0\,,1)$, where

$$\Gamma_\varepsilon := \left\{\omega \in C_+(\mathbb{T}) :\ \inf_{x\in\mathbb{T}} \omega(x) \leqslant \varepsilon\right\}.$$

According to the random walk argument [see (5.15)], with probability one,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{\Gamma_\varepsilon}(\psi(t))\,\mathrm{d}t \lesssim \sqrt{\limsup_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m-1} \mathbf{1}_{\{X_{j+1}\leqslant -|\log_2(8\varepsilon)|\}}}.$$

Apply (5.16) to deduce the result. □

From now on, we assume additionally that

$$F'(x) = O(x^{m_0-1}) \qquad \text{as } x \to \infty. \tag{10.1}$$

This condition implies **(F3)**, and is only a little stronger than **(F3)** for many examples. Recall that $\mu_+(C^\alpha(\mathbb{T})) = 1$ for every $\alpha \in (0\,,1/2)$. The following shows that μ_+ does not charge the critical case.

Theorem 10.2. *For μ_+-almost all $\omega \in C(\mathbb{T})$,*

$$\begin{aligned}
&\limsup_{r\downarrow 0} \frac{1}{\sqrt{r\log(1/r)}} \sup_{|y|<r}\sup_{x\in\mathbb{T}} |\omega(x+y)-\omega(x)| = \lambda \sup_{x\in\mathbb{T}} |\sigma(\omega(x))|,\\
&\liminf_{r\downarrow 0} \sqrt{\frac{16\log(1/r)}{\pi^2 r}} \inf_{|y|<r}\sup_{x\in\mathbb{T}} |\omega(x+y)-\omega(x)| = \lambda \sup_{x\in\mathbb{T}} |\sigma(\omega(x))|.
\end{aligned} \tag{10.2}$$

In particular, $\mu_+(C^{1/2}(\mathbb{T})) = 0$.

Before we prove Theorem 10.2 let us state a result about the fractal nature of the functions in the support of μ_+. Recall that the *Hausdorff dimension* of a Borel set $G \subset \mathbb{T}$ is

$$\dim_{\mathrm{H}}(G) = \sup\left\{s > 0 :\ I_s(m) < \infty \text{ for some probability measure } m \text{ on } G\right\},$$

where $I_s(m)$ denotes the s-dimensional "energy integral,"

$$I_s(m) := \iint \frac{m(\mathrm{d}x)\,m(\mathrm{d}y)}{|x-y|^s}.$$

Then, we have the following property.

Theorem 10.3. *Choose and fix an arbitrary Borel set $G \subset \mathbb{T}$. Then,*

$$\dim_{\mathrm{H}} \omega(G) = 1 \wedge 2\dim_{\mathrm{H}}(G) \qquad \textit{for } \mu_+\textit{-almost all } \omega \in C(\mathbb{T}).$$

Theorems 10.2 and 10.3 suggest that the functions in the support of ω "look" like Brownian paths. Of course, this cannot be interpretted too strongly as μ_+ is singular with respect to Wiener measure $\mathbb{W}$ on $C(\mathbb{T})$; in fact, $\mathbb{W}(C_+(\mathbb{T})) = 0$, yet $\mu_+(C_+(\mathbb{T})) = 1$ by Theorem 2.4.

We begin the proofs of Theorems 10.2 and 10.3. As a simple first step we offer the following real-variable consequence of (10.1).

Lemma 10.4. *There exists $c > 0$ such that*

$$|V(x) - V(y)| \leqslant c(1 \vee x \vee y)^{m_0 - 1}|x - y| \qquad \text{for all } x, y \geqslant 0.$$

Proof. Without loss of generality, $x \geqslant y \geqslant 0$, in which case,

$$0 \leqslant |V(x) - V(y)| \leqslant x - y + \int_y^x |F'(w)|\,\mathrm{d}w \leqslant \left[1 + \sup_{w \leqslant x} |F'(w)|\right](x - y).$$

This yields the result. □

From now on we consider (1.1) with initial value $\psi(0) = \mathbb{1}$. Also, let $\mathcal{J}$ denote the solution to the following linearized version of (1.1) with $\lambda = 1$,

$$\partial_t \mathcal{J} = \partial_x^2 \mathcal{J} + \dot{W},$$

subject to $\mathcal{J}(0) = \mathbb{0}$. That is,

$$\mathcal{J}(t\,, x) = \int_{(0,t)\times\mathbb{T}} p_{t-s}(x\,, y)\, W(\mathrm{d}s\,\mathrm{d}y) \tag{10.3}$$

We have the following second-order regularity result. Related results have been found by Foondun et al (2015) and Hairer and Pardoux (2015).

Lemma 10.5. *For every $\varepsilon \in (0\,, 3/4)$ and $k \geqslant 2$,*

$$\sup_{t \geqslant 2} \mathrm{E}\left(\sup_{\substack{x,z\in\mathbb{T}\\ x\neq z}} \frac{|\psi(t\,,x) - \psi(t\,,z) - \lambda\sigma(\psi(t\,,z))\{\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)\}|^k}{|x - z|^{\left(\frac34 - \varepsilon\right)k}}\right) < \infty.$$

Proof. Throughout, let us choose and fix $t \geqslant 2$ and $k \geqslant 2$. Thanks to (2.2),

$$\begin{aligned} &\psi(t\,,x) - \psi(t\,,z) \\ &\qquad = \int_{(0,t)\times\mathbb{T}} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, V(\psi(s\,,y))\,\mathrm{d}s\,\mathrm{d}y + \lambda\,\{\mathcal{I}(t\,,x) - \mathcal{I}(t\,,z)\}\,, \end{aligned} \tag{10.4}$$

almost surely for every $x, z \in \mathbb{T}$. We estimate the two quantities on the right-hand side of (10.4) separately and in order.

First of all, note that the $L^k(\Omega)$-norm of the first term on the right-hand side of (10.4) can be written as

$$\begin{aligned} &\left\|\int_{(0,t)\times\mathbb{T}} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, V(\psi(s\,,y))\,\mathrm{d}s\,\mathrm{d}y - \int_{(0,t)\times\mathbb{T}} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, V(\psi(t\,,x))\,\mathrm{d}s\,\mathrm{d}y\right\|_k \\ &\leqslant \int_{(0,t)\times\mathbb{T}} |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|\, \|V(\psi(s\,,y)) - V(\psi(t\,,x))\|_k\,\mathrm{d}s\,\mathrm{d}y \\ &\lesssim \int_{(0,t)\times\mathbb{T}} |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)| \left\|\{\psi(s\,,y) - \psi(t\,,x)\}\,(1 + |\psi(s\,,y)| + |\psi(t\,,x)|)^{m_0-1}\right\|_k \mathrm{d}s\,\mathrm{d}y; \end{aligned}$$

see Lemma 10.4 for the last line. We emphasize that the implied constant does not depend on the choice of $(t\,,x\,,z) \in [2\,,\infty) \times \mathbb{T}^2$. In any case, it follows readily from Lemma 4.5 and Proposition 4.8 that for any $\alpha \in (0\,,1/2)$ and $\beta \in (0\,,1/4)$,

$$\begin{aligned}\left\| \int_{(0,t)\times\mathbb{T}} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, V(\psi(s\,,y))\, \mathrm{d}s\, \mathrm{d}y \right\|_k \\ \lesssim \int_{(0,t)\times\mathbb{T}} |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)| \left((t-s)^\beta + |x-y|^\alpha\right) \mathrm{d}s\, \mathrm{d}y,\end{aligned} \tag{10.5}$$

where once again the implied constant does not depend on the choice of $(t\,,x\,,z) \in [2\,,\infty) \times \mathbb{T}^2$. By (4.2),

$$|p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|(t-s)^\beta \lesssim (t-s)^\beta \sum_{k=1}^\infty \mathrm{e}^{-\pi^2 k^2 (t-s)} \left(|x-z|k \wedge 1\right),$$

for similar constant dependencies as above. Therefore,

$$\int_{(0,t)\times\mathbb{T}} |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|(t-s)^\beta\, \mathrm{d}s\, \mathrm{d}y \lesssim \int_0^t s^\beta\, \mathrm{d}s \sum_{k=1}^\infty \left(|x-z|k \wedge 1\right) \mathrm{e}^{-\pi^2 k^2 s},$$

valid uniformly for all $(t\,,x\,,z) \in [2\,,\infty) \times \mathbb{T}^2$. We split the above sum in two parts according to whether or not $k \leqslant 1/|x-z|$. First,

$$\begin{aligned}\int_0^t s^\beta\, \mathrm{d}s \sum_{\substack{k\in\mathbb{N}:\\ k\leqslant 1/|x-z|}} \left(|x-z|k \wedge 1\right) \mathrm{e}^{-\pi^2 k^2 s} &= |x-z| \int_0^t s^\beta\, \mathrm{d}s \sum_{k=1}^\infty \mathrm{e}^{-\pi^2 k^2 s} \\ &= \frac{|x-z|}{\pi^{2\beta+2}} \sum_{k=1}^\infty k^{-2-2\beta} \int_0^{\pi^2 k^2 t} r^\beta \mathrm{e}^{-r}\, \mathrm{d}r \lesssim |x-z|.\end{aligned}$$

Next, we observe that

$$\begin{aligned}\int_0^t s^\beta\, \mathrm{d}s \sum_{\substack{k\in\mathbb{N}:\\ k> 1/|x-z|}} \left(|x-z|k \wedge 1\right) \mathrm{e}^{-\pi^2 k^2 s} &= \int_0^t s^\beta\, \mathrm{d}s \sum_{\substack{k\in\mathbb{N}:\\ k> 1/|x-z|}} \mathrm{e}^{-\pi^2 k^2 s} \leqslant \int_0^t s^\beta\, \mathrm{d}s \int_{1/|x-z|}^\infty \mathrm{d}w\ \mathrm{e}^{-\pi^2 w^2 s} \\ &\lesssim \int_0^t s^{\beta-1/2} \exp\left(-\frac{\pi^2 s}{|x-z|^2}\right) \mathrm{d}s \lesssim |x-z|^{1+2\beta} \lesssim |x-z|.\end{aligned}$$

Combine to deduce the uniform estimate

$$\int_{(0,t)\times\mathbb{T}} |p_{t-s}(x\,,y) - p_{t-s}(z\,,y)|(t-s)^\beta\, \mathrm{d}s\, \mathrm{d}y \lesssim |x-z|,$$

for all $\beta \in (0, 1/4)$. Therefore, Lemma 4.1 and (10.5) together yield

$$\left\| \int_{(0,t)\times\mathbb{T}} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, V(\psi(s\,,y))\, \mathrm{d}s\, \mathrm{d}y \right\|_k \lesssim |x-z| \log_+(1/|x-z|), \tag{10.6}$$

where the implied constant does not depend on the choice of $(t\,,x\,,z) \in [2\,,\infty) \times \mathbb{T}^2$.

We now define

$$\begin{aligned}
A_1 &:= (0, t - |x-z|) \times \mathbb{T}, \\
A_2 &:= (t - |x-z|, t) \times [z - 2|x-z|^\gamma, z + 2|x-z|^\gamma]^c, \\
A_3 &:= (t - |x-z|, t) \times [z - 2|x-z|^\gamma, z + 2|x-z|^\gamma],
\end{aligned}$$

where $\gamma \in (0\,,1/2)$ is a fixed constant. Here, we assume that $|x-z|$ is small enough so that $(z - 2|x-z|^\gamma, z + 2|x-z|^\gamma) \subset (-1\,,1)$. If $|x-z|$ is not small – i.e., if $|x-z| \geqslant c$ for some constant $c > 0$ – then we may use Lemma 4.5 (more precisely, the proof of of Lemma 4.5) to prove the lemma. Since we assume $t \geqslant 2$, we have $0 < t - |x-z| < t$. Therefore, we may define

$$\mathcal{I}(t\,,x) - \mathcal{I}(t\,,z) - \sigma(\psi(t\,,z))\{\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)\} := \sum_{i=1}^{5} Q_i,$$

where

$$\begin{aligned}
Q_1 &:= \int_{A_1} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, \sigma(\psi(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y) \\
Q_2 &:= \int_{A_2} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, \sigma(\psi(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y) \\
Q_3 &:= \int_{A_3} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\, [\sigma(\psi(s\,,y)) - \sigma(\psi(t - |x-z|\,,z)]\; W(\mathrm{d}s\,\mathrm{d}y) \\
Q_4 &:= [\sigma(\psi(t - |x-z|\,,z) - \sigma(\psi(t\,,z))] \int_{A_3} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\; W(\mathrm{d}s\,\mathrm{d}y) \\
Q_5 &:= \sigma(\psi(t\,,z)) \int_{A_1 \cup A_2} [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]\; W(\mathrm{d}s\,\mathrm{d}y).
\end{aligned}$$

We repeatedly use $|\sigma(a)| \leqslant \mathrm{Lip}_\sigma |a|$ and the boundedness of the moments of ψ. First consider Q_1. We can appeal to the Burkholder-Davis-Gundy inequality (see the proof of Theorem 2.3) and (4.3) to see that

$$\begin{aligned}
\|Q_1\|_k^2 &\lesssim \int_0^{t-|x-z|} \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\;\; [p_{t-s}(x\,,y) - p_{t-s}(z\,,y)]^2 \\
&\propto \sum_{n=1}^{\infty} [1 - \cos(\pi n|x-z|)] \int_{|x-z|}^{t} \mathrm{e}^{-2\pi^2 n^2 s}\, \mathrm{d}s \\
&\lesssim \sum_{n=1}^{\infty} \left(|x-z|^2 n^2 \wedge 1\right) \left(\frac{\mathrm{e}^{-2\pi^2 n^2 |x-z|}}{n^2}\right) \\
&= \sum_{n=1}^{\infty} \left(|x-z|^2 \wedge \frac{1}{n^2}\right) \mathrm{e}^{-2\pi^2 n^2 |x-z|} \\
&= |x-z|^2 \sum_{n \leqslant 1/|x-z|} \mathrm{e}^{-2\pi^2 n^2 |x-z|} + \sum_{n > 1/|x-z|} n^{-2} \mathrm{e}^{-2\pi^2 n^2 |x-z|} \\
&\lesssim |x-z|^{3/2},
\end{aligned}$$

uniformly for all $t \geqslant 2$.

Next, consider Q_2. A similar appeal to the BDG inequality yields

$$\begin{aligned}
\|Q_2\|_k^2 &\lesssim \int_{t-|x-z|}^t \mathrm{d}s \int_{[z-2|x-z|^\gamma, z+2|x-z|^\gamma]^c} \mathrm{d}y\ \ [p_{t-s}(x\,,y)-p_{t-s}(z\,,y)]^2\\
&\lesssim \int_0^{|x-z|} \mathrm{d}s \int_{[z-2|x-z|^\gamma, z+2|x-z|^\gamma]^c} \mathrm{d}y\ \ \left([p_s(x\,,y)]^2+[p_s(z\,,y)]^2\right)\\
&\lesssim \int_0^{|x-z|} \mathrm{d}s \int_{[z-2|x-z|^\gamma, z+2|x-z|^\gamma]^c} \mathrm{d}y\ \ \left(\frac{1}{4\pi s}\exp\left\{-\frac{(y-z)^2}{2s}\right\}+\frac{1}{4\pi s}\exp\left\{-\frac{(y-x)^2}{2s}\right\}\right),
\end{aligned}$$

where the last inequality comes from the simple fact that

$$p_t(x\,,y) \lesssim \frac{1}{\sqrt{t}}\exp\left\{-\frac{(y-z)^2}{4t}\right\},$$

for all $x, y \in \mathbb{T}$ and for all $t \leqslant 1$. Since $\min\{|y-z|, |y-x|\} \geqslant |x-z|$ when $y \in [z-2|x-z|^\gamma, z+2|x-z|^\gamma]^c$, it follows that

$$\begin{aligned}
\|Q_2\|_k^2 &\lesssim \int_0^{|x-z|} \mathrm{d}s\ s^{-1/2} \int_{|y-z|\geqslant|x-z|} \mathrm{d}y\ \ \left(\frac{1}{\sqrt{s}}\exp\left\{-\frac{(y-z)^2}{2s}\right\}\right)\\
&\lesssim |x-z|^{1/2}\exp\left(-|x-z|^{2\gamma-1}\right)\\
&\lesssim |x-z|^{1+\gamma},
\end{aligned}$$

uniformly for all $t \geqslant 2$. We used the facts that $\gamma < 1/2$ and $|x-z| \ll 1$ in order to guarantee the last inequality above.

Now we consider Q_3. Apply the BDG inequality once more to see that

$$\begin{aligned}
\|Q_3\|_k^2 &:= \int_{t-|x-z|}^t \mathrm{d}s \int_{|y-z|\leqslant|x-z|^\gamma} \mathrm{d}y\,[p_{t-s}(x\,,y)-p_{t-s}(z\,,y)]^2\,\|\sigma(\psi(s\,,y))-\sigma(\psi(t-|x-z|\,,z)\|_k^2\\
&\lesssim \int_{t-|x-z|}^t \mathrm{d}s \int_{|y-z|\leqslant|x-z|^\gamma} \mathrm{d}y\,[p_{t-s}(x\,,y)-p_{t-s}(z\,,y)]^2\left\{(s-t+|x-z|)^{1/2}+|y-z|\right\}\\
&\lesssim \left\{|x-z|^{1/2}+|x-z|^\gamma\right\}\int_0^{|x-z|}\mathrm{d}s\int_{\mathbb{T}}\mathrm{d}y\,[p_s(x\,,y)-p_s(z\,,y)]^2\\
&\lesssim \left\{|x-z|^{1/2}+|x-z|^\gamma\right\}\left\{\sum_{n=1}^\infty\left(\frac{1-\mathrm{e}^{-2\pi^2n^2|x-z|}}{n^2}\right)[1-\cos(\pi|x-z|n)]\right\} \quad \text{(see (4.3))}\\
&\lesssim \left\{|x-z|^{1/2}+|x-z|^\gamma\right\}\left\{\sum_{n=1}^\infty\left(\frac{1-\mathrm{e}^{-2\pi^2n^2|x-z|}}{n^2}\right)\left[1\wedge(|x-z|n)^2\right]\right\}\\
&\lesssim |x-z|^{3/2}+|x-z|^{1+\gamma},
\end{aligned}$$

uniformly for all $t \geqslant 2$.

Next, we consider Q_4. Hölder's inequality, the BDG inequality, and the fact that σ is Lipchitz together imply that

$$\|Q_4\|_k^2 \lesssim \|\psi(t-|x-z|\,,z)-\psi(t\,,z)\|_{2k}^2\int_{t-|x-z|}^t\mathrm{d}s\int_{|y-z|\leqslant|x-z|^\gamma}\mathrm{d}y\,[p_{t-s}(x\,,y)-p_{t-s}(z\,,y)]^2\,.$$

We now apply Proposition 4.8 and use a similar calculation as for Q_3 to get that

$$\|Q_4\|_k^2 \lesssim |x-z|^{3/2},$$

uniformly for all $t \geqslant 2$.

Lastly, we consider Q_5. Since the moments of $\psi(t\,,x)$ are uniformly bounded, we may use the fact that $|\sigma(a)| \leqslant |a|$ and follow the calculations for Q_1 and Q_2 to see that

$$\|Q_5\|_k^2 \lesssim |x-z|^{3/2} + |x-z|^{1+\gamma},$$

uniformly for all $t \geqslant 2$. Combine the preceding estimates to find that for every $\gamma \in (0\,,1/2)$

$$\|\mathcal{I}(t\,,x) - \mathcal{I}(t\,,z) - \sigma(\psi(t\,,z))\{\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)\}\|_k \lesssim |x-z|^{(1+\gamma)/2},$$

uniformly for all $t \geqslant 2$. This, (10.6), and (10.4) together imply that for every $\gamma \in (0\,,1/2)$

$$\|\psi(t\,,x) - \psi(t\,,z) - \lambda\sigma(\psi(t\,,z))\{\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)\}\|_k \lesssim |x-z|^{(1+\gamma)/2},$$

uniformly for all $t \geqslant 2$. The remainder of the result follows from the above and a standard chaining argument that uses Lemma 4.5 and Proposition 4.8; we skip the details as they are routine. ☐

Proof of Theorem 10.2. We plan to prove only (10.2). The first assertion of (10.2) immediately also implies that $\mu_+(C^{1/2}(\mathbb{T})) = 0$.

Recall (10.3). We study the Gaussian process $\mathcal{J}$ by studying its incremental variance using (4.3) as follows: For all $t > 0$ and $x, z \in \mathbb{T}$,

$$\begin{aligned} \mathrm{E}\left(|\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)|^2\right) &= \int_0^t \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \ [p_s(x\,,y) - p_s(z\,,y)]^2 \\ &= 2\int_0^t \mathrm{d}s \sum_{n=1}^\infty \mathrm{e}^{-2\pi^2 n^2 s}\,[1-\cos(\pi|x-z|n)] \\ &= \frac{1}{\pi^2}\sum_{n=1}^\infty \left(\frac{1-\mathrm{e}^{-2\pi^2 n^2 t}}{n^2}\right)[1-\cos(\pi|x-z|n)] \end{aligned} \tag{10.7}$$

Let $\hat{W}$ denote an independent space-time Brownian sheet and define a new, independent, Gaussian process $\mathcal{K}$ as follows:

$$\mathcal{K}(x) := \int_{(0,\infty)\times\mathbb{T}} [p_{t+s}(x\,,y) - p_{t+s}(0\,,y)]\,\hat{W}(\mathrm{d}s\,\mathrm{d}y).$$

The above is well defined, as

$$\begin{aligned} \mathrm{E}\left(|\mathcal{K}(x)|^2\right) &= \int_t^\infty \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \ [p_s(x\,,y) - p_s(z\,,y)]^2 \\ &\leqslant 2\int_t^\infty \mathrm{d}s \sum_{n=1}^\infty \mathrm{e}^{-2\pi^2 n^2 s} = \frac{1}{\pi^2}\sum_{n=1}^\infty \frac{\mathrm{e}^{-2\pi^2 n^2 t}}{n^2}; \end{aligned}$$

see (4.3). Moreover, we apply (4.3) yet again to see that

$$\begin{aligned} \mathrm{E}\left(|\mathcal{K}(x) - \mathcal{K}(z)|^2\right) &= \int_t^\infty \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \ [p_s(x\,,y) - p_s(z\,,y)]^2 \\ &= 2\int_t^\infty \mathrm{d}s \sum_{n=1}^\infty \mathrm{e}^{-2\pi^2 n^2 s}\,[1-\cos(\pi|x-z|n)] \\ &= \frac{1}{\pi^2}\sum_{n=1}^\infty \left(\frac{\mathrm{e}^{-2\pi^2 n^2 t}}{n^2}\right)[1-\cos(\pi|x-z|n)]\,. \end{aligned} \tag{10.8}$$

Now, define

$$\beta(x) := \sqrt{2}\,[\mathcal{J}(t\,,x) - \mathcal{J}(t\,,0) + \mathcal{K}(x) - \mathcal{K}(0)] + \frac{x}{\sqrt{2}} \qquad \text{for all } x \in \mathbb{T}. \tag{10.9}$$

Then, (10.7) and (10.8) together yield

$$\mathrm{E}\left(|\beta(x) - \beta(z)|^2\right) = |x - z| \qquad \text{for all } x, z \in \mathbb{T}.$$

Since $\beta(0) = 0$, it follows that β is a two-sided Brownian motion indexed by $\mathbb{T} \simeq [-1\,,1]$. Also, we apply (4.1) yet another time to find that

$$\begin{aligned}\mathrm{Cov}\left(\mathcal{K}(x)\,,\mathcal{K}(z)\right) &= \int_t^\infty \mathrm{d}s \int_{\mathbb{T}} \mathrm{d}y\ \left[p_s(x\,,y) - p_s(0\,,y)\right]\left[p_s(z\,,y) - p_s(0\,,y)\right] \\ &= \frac{1}{4\pi^2}\sum_{n=1}^\infty \left(\frac{1 - \mathrm{e}^{-i\pi xn}}{n}\right)\left(\frac{1 - \mathrm{e}^{i\pi zn}}{n}\right)\mathrm{e}^{-2\pi^2 n^2 t},\end{aligned}$$

for every $x, z \in \mathbb{T}$. Since $t \geqslant 16\pi^2 > 0$, the preceding is a C^∞ function of x and z. Therefore, a standard fact about Gaussian random fields implies that $\mathcal{K}$ is a.s. C^∞. In light of (10.9), we have proved the following version of an observation of Walsh (1986, Exercise 3.10, page 326):

$$\mathcal{J}(t\,,x) - \mathcal{J}(t\,,0) = \frac{\beta(x)}{\sqrt{2}} + \text{ a } C^\infty \text{ Gaussian process.}$$

Because β has the following "modulus of non-differentiability" (see Csörgő and Révész, 1981, Theorem 1.6.1),

$$\liminf_{r\downarrow 0} \sqrt{\frac{8\log(1/r)}{\pi^2 r}} \inf_{|y|<r} \sup_{x\in\mathbb{T}} |\beta(x+y) - \beta(x)| = 1 \qquad \text{a.s.,}$$

it follows from (10.9) that a.s.,

$$\liminf_{r\downarrow 0} \sqrt{\frac{16\log(1/r)}{\pi^2 r}} \inf_{|y|<r} \sup_{x\in\mathbb{T}} |\mathcal{J}(t\,,x+y) - \mathcal{J}(t\,,x)| = 1.$$

Therefore, we can deduce from Lemma 10.5 that

$$\liminf_{r\downarrow 0} \sqrt{\frac{16\log(1/r)}{\pi^2 r}} \inf_{|y|<r} \sup_{x\in\mathbb{T}} |\psi(t\,,x+y) - \psi(t\,,x)| = \lambda \sup_{x\in\mathbb{T}} |\sigma(\psi(t\,,x))|.$$

In other words, if $t \geqslant 2$, then

$$\mathrm{P}\{\psi(t) \in \Lambda\} = 1, \quad \text{where}$$

$$\Lambda := \left\{\omega \in C(\mathbb{T}) :\ \liminf_{r\downarrow 0} \sqrt{\frac{\log(1/r)}{r}} \inf_{|y|<r} \sup_{x\in\mathbb{T}} |\omega(x+y) - \omega(x)| = \frac{\pi\lambda}{4} \sup_{x\in\mathbb{T}} |\sigma(\omega(x))|\right\}.$$

According to the ergodic theorem [1(d) of Theorem 2.4],

$$\mu_+(\Lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{P}\{\psi(t) \in \Lambda\}\,\mathrm{d}t = 1.$$

This proves the second assertion of (10.2). The first assertion is proved using the same kind of arguments, except we appeal to the following

$$\limsup_{r\downarrow 0} \frac{1}{\sqrt{2r\log(1/r)}} \sup_{|y|<r} \sup_{x\in\mathbb{T}} |\beta(x+y) - \beta(x)| = 1 \qquad \text{a.s.,}$$

which is the celebrated modulus of continuity of Brownian motion; see Lévy (1938). □

Proof of Theorem 10.3. If $\omega \in C^\alpha(\mathbb{T})$ for some $\alpha \in (0\,,1]$, then a standard covering argument shows that

$$\dim_{\rm H} \omega(G) \leqslant 1 \wedge \alpha^{-1} \dim_{\rm H}(G); \tag{10.10}$$

see for example McKean (1955). Since $\mu_+(C^\alpha(\mathbb{T})) = 1$ for all $\alpha \in (0\,,1/2)$ [1(c) of Theorem 2.4], it follows that (10.10) holds for μ_+-almost all $\omega \in C(\mathbb{T})$. Let $\alpha \uparrow 1/2$ to deduce from (10.10) that

$$\dim_{\rm H} \omega(G) \leqslant 1 \wedge 2\dim_{\rm H}(G) \qquad \text{for } \mu_+\text{-almost all } \omega \in C(\mathbb{T}). \tag{10.11}$$

Next we derive a matching lower bound.

Choose and fix an arbitrary non-random number $s \in (0\,, 1 \wedge 2\dim_{\rm H}(G))$. Frostman's theorem ensures that there exists a probability measure m on G such that

$$I_{s/2}(m) < \infty. \tag{10.12}$$

See, for example, Theorem 2 of Kahane (1985, page 133). Choose and fix not only the s, but also the probability measure m.

Choose and fix an arbitrary non-random number

$$t \geqslant 2. \tag{10.13}$$

According to Lemma 10.5,

$$X := \sup_{\substack{x,z\in\mathbb{T}\\ x\neq z}} \frac{|\psi(t\,,x) - \psi(t\,,z) - \lambda\sigma(\psi(t\,,z))\{\mathcal{J}(t\,,x) - \mathcal{J}(t\,,x)\}|}{|x-z|^{3/5}} \in \bigcap_{k\geqslant 2} L^k(\Omega).$$

Let

$$Y := \lambda \inf_{x\in\mathbb{T}} \sigma(\psi(t\,,x)).$$

It is immediately clear from (2.4) that $Y \geqslant {\rm L}_\sigma \inf_{x\in\mathbb{T}} \psi(t\,,x) > 0$ a.s. Thus, we can write

$$\begin{aligned} |\psi(t\,,x) - \psi(t\,,z)| &\geqslant \left[Y|\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)| - X|x-z|^{3/5}\right]_+ \\ &\geqslant Y\left[|\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)| - \left(\frac{X}{Y}\right)|x-z|^{3/5}\right]_+, \end{aligned}$$

where $\mathcal{J}$ was defined in (10.3).

Define m_ψ to be the push forward of m by $\psi(t)$. More precisely,

$$\int g\,{\rm d}m_\psi := \int g(\psi(t\,,x))\,m({\rm d}x) \qquad \text{for all } g \in C_+(\mathbb{R}).$$

Clearly m_ψ is a random probability measure on the random set

$$\psi(t\,,G) := \{\psi(t\,,x) :\, x \in G\}.$$

Now,

$$I_s(m_\psi) = \iint \frac{m({\rm d}x)\,m({\rm d}z)}{|\psi(t\,,x) - \psi(t\,,z)|^s} \leqslant Y^{-s}\tilde{I}_s(m\,;X/Y),$$

where

$$\tilde{I}_s(m_\psi\,;N) := \iint \frac{m({\rm d}x)\,m({\rm d}z)}{\left[|\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)| - N|x-z|^{3/5}\right]_+^s} \qquad \text{for all } N > 0.$$

The first portion of the proof is concerned with proving that

$$\tilde{I}_s(m_\psi\,;N) < \infty \quad \text{almost surely for every } N > 0. \tag{10.14}$$

In this way we will see that $\tilde{I}_s(m_\psi\,;X/Y) < \infty$ almost surely on $\{X \leqslant NY\}$, and in particular,

$$\mathrm{P}\left\{\tilde{I}_s(m_\psi\,;X/Y) < \infty\right\} \geqslant \lim_{N\to\infty} \mathrm{P}\{X \leqslant NY\} = 1.$$

With this aim in mind, we first recall the incremental variance of $\mathcal{K}$ from (10.8): For all $x, z \in \mathbb{T}$,

$$\begin{aligned}
\mathrm{E}\left(|\mathcal{K}(x) - \mathcal{K}(z)|^2\right) &= \frac{1}{\pi^2}\sum_{n=1}^\infty \left(\frac{\mathrm{e}^{-2\pi^2 n^2 t}}{n^2}\right)[1 - \cos(\pi|x-z|n)]\\
&\leqslant |x-z|^2 \sum_{n=1}^\infty \mathrm{e}^{-2\pi^2 n^2 t} \leqslant |x-z|^2 \int_0^\infty \mathrm{e}^{-2\pi^2 w^2 t}\,\mathrm{d}w\\
&= \frac{|x-z|^2}{2\sqrt{2\pi t}}.
\end{aligned}$$

Since $t \geqslant 2$ [see (10.13)], the above quantity is $\leqslant \frac{1}{4}|x-z|^2$, and hence

$$\mathrm{E}\left(|\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)|^2\right) = \frac{\mathrm{E}\left(|\beta(x) - \beta(z)|^2\right)}{2} + \frac{(x-z)^2}{4} - \mathrm{E}\left(|\mathcal{K}(x) - \mathcal{K}(z)|^2\right) \geqslant \frac{|x-z|}{2}. \tag{10.15}$$

Thus, we find that

$$\begin{aligned}
\sup_{N>0} \mathrm{E}\left[\tilde{I}_s(m_\psi\,;N)\right] &\leqslant \frac{\Gamma\left(\frac{1-s}{2}\right)}{2^{s/2}\sqrt{\pi}} \iint \frac{m_\psi(\mathrm{d}x)\, m_\psi(\mathrm{d}z)}{[\mathrm{Var}\left(\mathcal{J}(t\,,x) - \mathcal{J}(t\,,z)\right)]^{s/2}} && \text{[see Lemma A.1]}\\
&\leqslant \frac{\Gamma\left(\frac{1-s}{2}\right)}{\sqrt{\pi}} I_{s/2}(m_\psi) && \text{[by (10.15)].}
\end{aligned}$$

Therefore, the above and (10.12) together imply (10.14). In turn, this and Frostman's theorem together imply that, $\dim_{\mathrm{H}} \psi(t\,,G) \geqslant s$ a.s. This is valid for every $s \in (0\,,1 \wedge 2\dim_{\mathrm{H}}(G))$. Therefore, we let s converge upward to $1 \wedge 2\dim_{\mathrm{H}}(G)$ in order to deduce the following: For every $t \geqslant 2$, $\dim_{\mathrm{H}} \psi(t\,,G) \geqslant 1 \wedge 2\dim_{\mathrm{H}}(G)$ a.s. In other words, if we define

$$\Theta := \{\omega \in C(\mathbb{T}) : \dim_{\mathrm{H}} \omega(G) \geqslant 1 \wedge 2\dim_{\mathrm{H}}(G)\},$$

then $\mathrm{P}\{\psi(t) \in \Theta\} = 1$ for all $t \geqslant 2$, whence

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{P}\{\psi(t) \in \Theta\}\,\mathrm{d}t = 1.$$

Apply part 1(d) of Theorem 2.4 to see that $\mu_+(\Theta) = 1$. This and (10.11) together imply the theorem. □

A Appendix: Some technical results

A.1 A Gaussian integral

Lemma A.1. *Let X have a non-degenerate centered normal distribution. Then,*

$$\sup_{a\geqslant 0} \mathrm{E}\left[\frac{1}{(|X| - a)_+^s}\right] = \frac{\Gamma((1-s)/2)}{[2\mathrm{Var}(X)]^{s/2}\sqrt{\pi}} \qquad \textit{for all } s \in (0\,,1).$$

Proof. Let $v^2 := \mathrm{Var}(X)$. Clearly,

$$\mathrm{E}\left[\frac{1}{(|X|-a)_+^s}\right] = \frac{1}{v\sqrt{\pi/2}}\int_a^\infty \frac{\mathrm{e}^{-x^2/(2v^2)}}{(x-a)^s}\,\mathrm{d}x = \frac{1}{v\sqrt{\pi/2}}\int_0^\infty \frac{\mathrm{e}^{-(x+a)^2/(2v^2)}}{x^s}\,\mathrm{d}x$$
$$\leqslant \frac{1}{v\sqrt{\pi/2}}\int_0^\infty \frac{\mathrm{e}^{-x^2/(2v^2)}}{x^s}\,\mathrm{d}x,$$

with identity in place of "$\leqslant$" when $a = 0$. Now compute. □

A.2 An elementary coupling

Let us document the following very well known elementary fact about couplings of random variables.

Lemma A.2. *Let X be a random variable such that $G(x) := \mathrm{P}\{X \leqslant x\} \geqslant H(x)$ for all $x \in \mathbb{R}$, where $H : \mathbb{R} \to [0\,,1]$ is a cumulative distribution function on $\mathbb{R}$. Then, one can construct a random variable Y whose cumulative distribution function is H, and satisfies $\mathrm{P}\{X \leqslant Y\} = 1$.*

We include a proof for the sake of completeness, also as it is so short.

Proof. Let H^{-1} denote the right-continuous inverse of H and recall that the distribution function of $Y := H^{-1}(G(X))$ is H. The lemma follows from the facts that $G \geqslant H$ and H^{-1} is monotone. □

A.3 A random walk inequality

We mention the following simple inequality about the expected number of large negative excursions of a simple random walk on $\mathbb{Z}$ with positive upward drift.

Lemma A.3. *Let $\{Z_n\}_{n=1}^\infty$ be i.i.d. with $p := \mathrm{P}\{Z_1 = 1\} > 1/2$ and $q := \mathrm{P}\{Z_1 = -1\} = 1 - p$. Then,*

$$\mathrm{E}\left[\sum_{n=1}^\infty \mathbf{1}_{\{Z_1+\cdots+Z_n\leqslant -k\}}\right] \leqslant \frac{\sqrt{4pq}}{1-\sqrt{4pq}}\left(\frac{q}{p}\right)^{k/2} \qquad \text{for all } k \in \mathbb{R}_+.$$

One usually verifies results of this type by appealing to excursion theory. We include a simpler proof instead.

Proof. We use Markov's [Chernoff's] inequality to see that for all $n \in \mathbb{N}$, $\theta > 0$, and $k \in \mathbb{R}_+$,

$$\mathrm{P}\{Z_1 + \cdots + Z_n \leqslant -k\} = \mathrm{P}\left\{\mathrm{e}^{-\theta(Z_1+\cdots+Z_n)} \geqslant \mathrm{e}^{\theta k}\right\} \leqslant \mathrm{e}^{-\theta k}\left[p\mathrm{e}^{-\theta} + q\mathrm{e}^{\theta}\right]^n.$$

Therefore, we set $\theta := \frac{1}{2}\log(p/q)$ to minimize the quantity in square brackets and see that

$$\mathrm{P}\{Z_1 + \cdots + Z_n \leqslant -k\} \leqslant \left(\frac{q}{p}\right)^{k/2}[4pq]^{n/2} \quad \text{for all } n \in \mathbb{N} \text{ and } k \in \mathbb{R}_+.$$

Because $4pq < 1$, we may sum over $n \in \mathbb{N}$ to deduce the announced result. □

A.4 On a class of stochastic differential inequalities

The primary purpose of this portion of the appendix is to prove a bound on the hitting probability of a small number by a certain non-negative Itô process.

Proposition A.4. *Suppose $X = \{X_t\}_{t\geqslant 0}$ is a non-negative, continuous $L^2(\mathrm{P})$-martingale that starts from $X_0 = a^2$ for a non-random number $a > 0$, and solves the stochastic differential inequality,*

$$\mathrm{d}X_t \leqslant X_t\,\mathrm{d}t + \mathrm{d}M_t \qquad \textit{for all } t > 0,$$

where $\{M_t\}_{t\geqslant 0}$ is a mean-zero, continuous $L^2(\mathrm{P})$-martingale. More precisely $\mathrm{d}X_t = C_t\,\mathrm{d}t + \mathrm{d}M_t$ where $C_t \leqslant X_t$ and $\{C_t\}_{t\geqslant 0}$ is a.s. of bounded variation. Then,

$$\mathrm{P}\left\{\inf_{s\in(0,t)} X_s > \varepsilon^2\,,\,\int_0^t \mathrm{e}^{-s}\,\frac{\mathrm{d}\langle X\rangle_s}{X_s} \geqslant b^2\right\} \leqslant \sqrt{\frac{2}{\pi}}\int_{|x|<2(a-\varepsilon\mathrm{e}^{-t/2})/b} \mathrm{e}^{-x^2/2}\,\mathrm{d}x,$$

uniformly for all $b, t > 0$ and $\varepsilon \in (0\,, a\mathrm{e}^{t/2})$.

The proof of Proposition A.4 hinges on a simple small-ball estimate for continuous $L^2(\mathrm{P})$-martingales, which we include next.

Lemma A.5. *Let $\{M_s\}_{s\geqslant 0}$ be a mean-zero, continuous $L^2(\mathrm{P})$-martingale. Then,*

$$\mathrm{P}\left\{\inf_{0<s<t} M_s \geqslant -\varepsilon\ ,\ \langle M\rangle_t \geqslant A\right\} \leqslant \sqrt{\frac{2}{\pi}}\int_{|x|<\varepsilon/\sqrt{A}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \qquad \textit{for all } \varepsilon, A, t > 0, \tag{A.1}$$

The right-hand side of (A.1) can be expressed more compactly in terms of $\mathrm{erf}(\varepsilon\sqrt{2/A})$, but we prefer the above "probabilistic" description.

Proof. By the Dubins, Dambis-Schwarz Brownian representation of continuous martingales (see Revuz and Yor, 1999, Theorem 1.6) there exists a linear Brownian motion $\{B(s)\}_{s\geqslant 0}$ such that $B(0) = 0$ and $M_s = B(\langle M\rangle_s)$ for all $s \geqslant 0$. Therefore,

$$\mathrm{P}\left\{\inf_{0<s<t} M_s \geqslant -\varepsilon\ ,\ \langle M\rangle_t \geqslant A\right\} = \mathrm{P}\left\{\inf_{0<s<\langle M\rangle_t} B(s) \geqslant -\varepsilon\ ,\ \langle M\rangle_t \geqslant A\right\} \leqslant \mathrm{P}\left\{\inf_{0<s<A} B(s) \geqslant -\varepsilon\right\},$$

which yields the result, thanks to Brownian scaling and the reflection principle. □

Proof of Proposition A.4. Apply Itô's formula to see that $Y_t := X_t^{1/2}$ solves $Y_0 = a$ subject to

$$\mathrm{d}Y_t \leqslant \frac{1}{2}Y_t\,\mathrm{d}t + \mathrm{d}N_t, \tag{A.2}$$

where $\{N_t\}_{t\geqslant 0}$ is a mean-zero continuous $L^2(\mathrm{P})$-martingale with quadratic variation,

$$\langle N\rangle_t = \frac{1}{4}\int_0^t \frac{\mathrm{d}\langle X\rangle_s}{X_s}.$$

We remove the drift in (A.2) in a standard way by setting $Z_t := \mathrm{e}^{-t/2}Y_t$ for all $t \geqslant 0$. Then, $Z_0 = a$ and Z satisfies the stochastic differential inequality, $\mathrm{d}Z_t = \mathrm{e}^{-t/2}\,\mathrm{d}Y_t - \frac{1}{2}\mathrm{e}^{-t/2}Y_t\,\mathrm{d}t \leqslant \mathrm{e}^{-t/2}\,\mathrm{d}N_t$. In other words,

$$Z_t \leqslant a + \int_0^t \mathrm{e}^{-s/2}\,\mathrm{d}N_s := a + \mathcal{M}_t \qquad \text{for all } t > 0,$$

where $\{\mathcal{M}_t\}_{t\geqslant 0}$ is a mean-zero continuous $L^2(\mathrm{P})$ martingale whose quadratic variation is given by

$$\mathrm{d}\langle \mathcal{M}\rangle_t = \mathrm{e}^{-t}\,\mathrm{d}\langle N\rangle_t = \frac{\mathrm{e}^{-t}}{4}\frac{\mathrm{d}\langle X\rangle_t}{X_t}.$$

Since

$$\left\{\inf_{s\in(0,t)} X_s > \varepsilon^2\,,\ \int_0^t \mathrm{e}^{-s}\,\frac{\mathrm{d}\langle X\rangle_s}{X_s} \geqslant b^2\right\} \subset \left\{\inf_{s\in(0,t)} \mathcal{M}_s > \varepsilon\mathrm{e}^{-t/2} - a\,,\ \langle\mathcal{M}\rangle_t \geqslant \frac{b^2}{4}\right\},$$

Proposition A.4 follows from the above and Lemma A.5. □

A.5 On martingale measures

Consider a continuous-in-time $L^2(\mathrm{P})$-martingale measure $M := \{M_t(A)\}_{t\geqslant 0, A\in\mathcal{B}(\mathbb{T})}$, in the sense of Walsh (1986), where $\mathcal{B}(\mathbb{T})$ denotes the collection of Borel subsets of $\mathbb{T}$. The following is a ready consequence of Itô calculus.

Lemma A.6. *If $\langle M(A)\,, M(B)\rangle_t = t|A\cap B|$ for all $t\geqslant 0$ and $A, B\in\mathcal{B}(\mathbb{T})$, then we can realize M as*

$$M_t(A) = \int_{(0,t)\times A} w(\mathrm{d}s\,\mathrm{d}y) \qquad \text{for all } t\geqslant 0 \text{ and } A\in\mathcal{B}(\mathbb{T}),$$

where $\dot{w} = \{\dot{w}(t\,,x)\}_{t\geqslant 0, x\in\mathbb{T}}$ is a space-time white noise.

Proof. We prove that $M := \{M_t(A)\}_{t\geqslant 0, A\in\mathcal{B}(\mathbb{T})}$ is a Gaussian process; the remainder of the lemma follows from this and a simple covariance computation which we skip.

If A_1 and A_2 are two Borel subsets of $\mathbb{T}$ such that the Haar measures of A_1 and A_2 are 1 and $A_1\cap A_2$ has zero Haar measure, then $\langle M(A_i)\,, M(A_j)\rangle_t = \delta_{i,j}t$ for all $t\geqslant 0$. Thus, Lévy's characterization theorem (see Revuz and Yor, 1999, Theorem 3.6) asserts that $\{M_t(A_1), M_t(A_2)\}$ is a 2-dimensional Brownian motion, whence M is a Gaussian process by induction.This completes the proof. □

Lemma A.6 immediately implies the following result, which will play a key role in our coupling construction in Section 7.

Corollary A.7. *Let $\dot{V}_1$ and $\dot{V}_2$ denote two independent space-time white noises, and suppose $\{\phi_1(t\,,x)\}_{t\geqslant 0, x\in\mathbb{T}}$ and $\{\phi_2(t\,,x)\}_{t\geqslant 0, x\in\mathbb{T}}$ are continuous predictable random fields, in the filtration defined by $\dot{V}_1$ and $\dot{V}_2$, that takes values in $[0\,,1]$ and satisfy*

$$\phi_1^2(t\,,x) + \phi_2^2(t\,,x) = 1 \quad \text{for all } t>0 \text{ and } x\in\mathbb{T} \text{ a.s.}$$

Define $\dot{M}(t\,,x) = \phi_1(t\,,x)\dot{V}_1(t\,,x) + \phi_2(t\,,x)\dot{V}_2(t\,,x)$; more formally,

$$M_t(A) := \int_{(0,t)\times A} \phi_1(s\,,y)\,V_1(\mathrm{d}s\,\mathrm{d}y) + \int_{(0,t)\times A} \phi_2(s\,,y)\,V_2(\mathrm{d}s\,\mathrm{d}y),$$

for $t>0$ and $A\in\mathcal{B}(\mathbb{T})$. Then, $\dot{M}$ is space-time white noise; more formally, there exists a space-time white noise $\dot{w}$ such that $M_t(A) = \int_{(0,t)\times A} w(\mathrm{d}s\,\mathrm{d}y)$ a.s. for every $t>0$ and $A\in\mathcal{B}(\mathbb{T})$.

Proof. Since ϕ_1 and ϕ_2 are bounded, the stochastic integrals are Walsh integrals that define a martingale measure. It remains to verify that $\langle M(A)\,, M(B)\rangle_t = t|A\cap B|$ for all $t>0$ and $A, B\in\mathcal{B}(\mathbb{T})$; see Lemma A.6. But this mutual variation formula is an immediate consequence of the construction of Walsh integrals. □

B Appendix: Sketch of proof of Lemma 5.2

There have been many results similar to Lemma 5.2, as we mention below. We will give an outline of the argument.

By (5.6) and the definition of σ_n in (5.7), we have that

$$|\sigma_n(t\,,x)| \leqslant 4\mathrm{Lip}_\sigma L_{\tau_n}(v_n).$$

Consulting definition (5.8), we see that $\mathcal{I}_n^{\mathrm{tr}}(t\,,x)$ is a white noise integral of $\lambda p_{t-s}(x-y)\sigma_n(s\,,y)$ against a time-shifted space-time white noise.

Now Lemma (5.2) would follow from Dalang et al. (2009, Theorem 4.2, page 126), except in that reference $p_t(x\,,y)$ is replaced by the heat kernel on $\mathbb{R}$, not $\mathbb{T}$. However, we can easily modify the proof in Dalang et al. (2009) to cover our case. Using the expansion (2.3), we find that Lemma 4.3 on page 126 of Dalang et al. (2009) still holds, except that $|x-y|$ is replaced by the distance from x to y on $\mathbb{T}$. The rest of the argument goes through as before, giving us Lemma 5.2. $\square$

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