Polarity of almost all points for systems of non-linear stochastic heat equations in the critical dimension

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Abstract

We study vector-valued solutions $u(t,x) \in \mathbb{R}^d$ to systems of nonlinear stochastic heat equations with multiplicative noise:

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \sigma(u(t,x))\dot{W}(t,x).$$

Here $t \geq 0$, $x \in \mathbb{R}$ and $\dot{W}(t,x)$ is an $\mathbb{R}^d$-valued space-time white noise. We say that a point $z \in \mathbb{R}^d$ is polar if

$$P\{u(t,x) = z \text{ for some } t > 0 \text{ and } x \in \mathbb{R}\} = 0.$$

We show that in the critical dimension $d = 6$, almost all points in $\mathbb{R}^d$ are polar.

1 Introduction

We say that a vector-valued stochastic process $(X_t,\, t \in I)$ hits a set B if

$$P\{X_t \in B \text{ for some } t \in I\} > 0.$$

Hitting properties constitute one of the most intensively studied topics in probability theory. For many Markov processes, probabilistic potential theory gives a powerful set of tools for answering such questions [1, 12, 13]. However, for processes taking values in infinite

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dimensional spaces, potential theoretic calculations are usually intractable and we must fall back on more basic methods, such as covering arguments.

We also note that such hitting questions are always the most difficult in the critical dimension, and we expect that hitting does not occur in the critical dimension. For example, if a family of vector-valued processes $(X_t^{(d)})$ can be defined so that for each $d \geq 1$, $X_t^{(d)}$ takes values in $\mathbb{R}^d$, and if $B = \{z\}$ is a one-point set in $\mathbb{R}^d$, we say that d_c is the critical dimension if hitting of B occurs for $d < d_c$ but not for $d > d_c$ (often, the superscript d is omitted from the notation, as in (1.1) and (2.1)). For many natural families of such processes, we can often identify the critical dimension d_c even if we usually cannot prove that hitting of points fails to occur in that dimension.

In this paper, we deal with vector-valued solutions $u(t,x)$ to the stochastic heat equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}$, where $\dot{W}$ is a vector of d independent space-time white noises, and where σ is matrix-valued. We give more precise conditions in the next section.

It would be possible to consider the solution $u(t,\cdot)$ as a stochastic process parameterized by t taking values in function space. In view of the difficulties mentioned above, we restrict ourselves to the question of hitting points. We say that $(u(t,\cdot),\, t \in \mathbb{R}_+)$ hits the point $z \in \mathbb{R}^d$ if

$$P\left\{u(t,x) = z \text{ for some } t > 0 \text{ and } x \in \mathbb{R}\right\} > 0.$$

So we are asking whether $(u(t,\cdot),\, t \in \mathbb{R}_+)$ can hit the set B of continuous functions $f(x)$ such that $f(x) = z$ for some value of $x \in \mathbb{R}$.

Defined in this way, the question of hitting probabilities for stochastic partial differential equations (SPDE) has been studied by a number of authors, see [16, 10, 7, 8, 11]. They obtain results for a broad class of sets B, but only [16, 10] deal with the critical dimension. There are also some earlier papers about the question of whether random fields can hit points or other sets. For the vector-valued Brownian sheet, Orey and Pruitt [18] gave a necessary and sufficient condition for hitting points, and Khoshnevisan and Shi [15] developed a complete potential theory which answers the hitting question for any set. Both groups of authors used special properties of the Brownian sheet. For fractional Brownian fields, Talagrand [20, 21] answered the question of whether the process can hit points, including in the critical dimension. He also dealt with multiple points.

Building on Talagrand's methods, the article [9] proved that for a broad class of Gaussian random fields, the process does not hit points in the critical dimension. This paper also provided a general framework for this type of problem. Some follow-up papers also deal with the question of multiple points for Gaussian random fields [5, 4].

In this paper, we deal with the nonlinear stochastic heat equation (1.1) and show that in the critical dimension, which was known to be $d_c = 6$ (see [17, 8]), almost every point is polar. We will give a more precise statement in the next section. Because of the multiplicative noise term, the equation is nonlinear and in most cases $u(t,x)$ will not be a Gaussian process. It is usually difficult to carry over results about Gaussian processes to more general processes. However, it is well-known that on small scales $u(t,x)$ resembles a Gaussian process. By freezing in particular the coefficient $\sigma(u)$, it becomes possible to carry over many of the arguments from [9]. However, we are still unable to prove that all points are polar in the critical dimension. As part of our proof, we show that in dimensions $d \geq 6$, the 6-dimensional Hausdorff measure of the range of u is 0.

For the linear heat equation, where $\sigma \equiv 1$ and the solution of (1.1) is a Gaussian random field, the extra step that allows to go from "almost all points are polar" to "all points are polar" involves taking the conditional expectation of the random field given its value at a specific point (see [9, Section 5]). In the Gaussian case, conditional expectations can be computed explicitly, but in the nonlinear SPDE where $\sigma \not\equiv 1$, this is no longer true and a new argument seems to be needed.

We should also mention that because we can only show that almost every point is polar in the critical dimension, we do not expect for the moment to be able to extend the results of this paper to the question of existence of multiple points and show that there are no multiple points in the critical dimensions (except in the cases handled by [5, 4] where $\sigma(u)$ is constant and so $u(t,x)$ is a Gaussian process).

2 Setup and main theorem

Let $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ be a matrix function. We are dealing with solutions $u(t,x)$ to the system of d equations

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \qquad t > 0,\ x \in \mathbb{R}, \tag{2.1}$$

where $\dot{W}(t,x) = (\dot{W}_1(t,x), \ldots, \dot{W}_d(t,x))$ is a d-dimensional space-time white noise (see [14]) defined on a probability space $(\Omega, \mathcal{F}, P)$, with i.i.d. components, subject to the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{2.2}$$

where $u_0 : \mathbb{R} \to \mathbb{R}^d$. We associate to the white noise its natural filtration $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$, where $\mathcal{F}_t$ is the σ-field generated by the white noise on $[0,t] \times \mathbb{R}$ (and completed with P-null sets).

For an element $z \in \mathbb{R}^d$, $|z|$ denotes the Euclidean norm of z. We use the same notation for a matrix $\sigma_0 \in \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{d^2}$.

Assumption 2.1. *(a) The function $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is Lipschitz continuous with Lipschitz constant L: for all $v_1, v_2 \in \mathbb{R}^d$,*

$$|\sigma(v_1) - \sigma(v_2)| \leq L|v_1 - v_2|.$$

(b) There is a finite constant $\sigma_1 \in \mathbb{R}$ such that for all $v \in \mathbb{R}^d$,

$$|\sigma(v)| \leq \sigma_1.$$

(c) The initial function $u_0(x)$ is bounded: there is $K_0 \in \mathbb{R}_+$ such that, for all $x \in \mathbb{R}$,

$$|u_0(x)| \leq K_0.$$

Finally, we note that (2.1) has a rigorous formulation in terms of the mild form, see [6]: $(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ is a jointly measurable and $(\mathcal{F}_t)$-adapted process such that, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$u(t,x) = \int_{-\infty}^{\infty} G(t, x-y) u_0(y) dy + \int_0^t \int_{-\infty}^{\infty} G(t-s, x-y) \sigma(u(s,y)) W(dy, ds),$$

where

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

is the heat kernel on $\mathbb{R}$. Existence and uniqueness is proved in [22, Chapter 3] in the case $d = 1$, and this proof extends directly to $d \geq 1$ (see [8, Section 2]). The random field $(u(t,x))$ has a continuous version on $]0,\infty[\,\times\mathbb{R}$ (see [22]), and if the initial condition u_0 is continuous (which we do not assume here), then this version of $(u(t,x))$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ [2, Theorem 3.1]. We will work only with this continuous version.

The main result of this paper is the following. For the definition of Hausdorff measure, see [13, Appendix C].

Theorem 2.2. *Assume that $d \geq 6$. Almost surely, the range of $u = (u(t,x),\, (t,x) \in\,]0,\infty[\times\mathbb{R})$ has 6-dimensional Hausdorff-measure 0. In particular, if $d \geq 6$, then almost all points in $\mathbb{R}^d$ are polar for u.*

This theorem is proved at the end of Section 7.

Remark 2.3. *For linear systems of stochastic heat equations (σ constant), according to [16] and [9], $d = 6$ is the critical dimension for hitting points and points are polar when $d = 6$. According to [8, Corollary 1.5], when the matrix function σ is smooth and uniformly elliptic and $d > 6$, then the Hausdorff dimension of the range of u is precisely 6. This implies of course that for $d > 6$, almost all points in $\mathbb{R}^d$ are polar. Therefore, the conclusions of Theorem 2.2 are most interesting in the critical dimension $d = 6$.*

3 Local decomposition

In this section, our goal is to study the range of $(t,x) \mapsto u(t,x)$ when (t,x) belongs to a small rectangle with center $(t_0, x_0) \in R_0 := [1,2] \times [0,1]$, where t_0 and x_0 are fixed. Throughout most of the paper, we will be working on subrectangles of R_0.

For $\rho \in\,]0, \frac{1}{2}]$, define

$$R_\rho = R_\rho(t_0, x_0) := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R} : |t - t_0| < \rho^4,\ |x - x_0| < \rho^2\}. \tag{3.1}$$

This rectangle has side-lengths that are compatible with the metric

$$d((t,x);(s,y)) = \Delta(t-s, x-y) := \max(|t-s|^{1/4}, |x-y|^{1/2}).$$

We often write p for a couple $(t,x) \in \mathbb{R}^2$. Finally, we define the oscillation of an $\mathbb{R}^d$-valued function f on a rectangle $R \subset \mathbb{R}^2$ as follows:

$$\operatorname{osc}_R(f) = \sup_{p_1, p_2 \in R} |f(p_1) - f(p_2)|.$$

In a first stage, we would like to replace $(t,x) \mapsto u(t,x)$ by a modified process obtained by freezing coefficients at stopping times, so that certain regularity and growth conditions are satisfied. We will also do this for an associated Gaussian process. For this, we define a first stopping time $\tau_{K,1}$ that will help with Hölder-continuity properties of the solution, then a stopping time $\tau_{K,2}$ that will deal with growth as $x \to \pm\infty$, and a third stopping time $\tau_{K,3}$ that will help with an associated Gaussian process.

First stopping time $\tau_{K,1}$

Fix $T_0 > 3$ and a large constant $K > 0$. From [2, Theorem 3.1], we know that $u(t,x)$ is locally $(1-\delta)/4$-Hölder continuous in t and $(1-\delta)/2$-Hölder continuous in x on $]0,\infty[\,\times\mathbb{R}$. More precisely, for each $\delta \in]0,1[$, there is an almost surely finite positive random variable Z such that for all $s,t \in [\frac{1}{2}, T_0]$ and $x,y \in [-2,2]$,

$$|u(t,x) - u(s,y)| \leq Z\Delta(t-s, x-y)^{1-\delta}. \tag{3.2}$$

Now we define the stopping time $\tau_{K,1}$ to be the first time $t \in [\frac{1}{2}, T_0]$ such that there exist $x,y \in [-2,2]$ with

$$|u(t,x) - u(s,y)| \geq K\Delta(t-s, x-y)^{1-\delta};$$

if there is no such time t, let $\tau_{K,1} = T_0$.

Now (3.2) shows that

$$\lim_{K\to\infty} P\{\tau_{K,1} < T_0\} = 0. \tag{3.3}$$

Also note that $u(t \wedge \tau_{K,1}, x)$ satisfies

$$|u(t_1 \wedge \tau_{K,1}, x_1) - u(t_2 \wedge \tau_{K,1}, x_2)| \leq K\Delta\big(t_1 - t_2, x_1 - x_2\big)^{1-\delta} \tag{3.4}$$

for $(t_i, x_i) \in [\frac{1}{2}, T_0] \times [-2,2]$.

Modified solution $\tilde{u}$

We will modify the random field u using $\tau_{K,1}$. We define $\tilde{u}(t,x) = \tilde{u}_K(t,x)$ as the (continuous version on $]0,\infty[\,\times\mathbb{R}$ of the) solution of

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{u}(t,x) &= \frac{\partial^2}{\partial x^2}\tilde{u}(t,x) + \sigma(u(t\wedge\tau_{K,1},x))\dot{W}(t,x), \qquad t>0,\ x\in\mathbb{R},\\ \tilde{u}(0,x) &= u_0(x), \qquad x\in\mathbb{R}.\end{aligned}$$

Note that on the right-hand side of the equation for $\tilde{u}$, σ is evaluated at u, not at $\tilde{u}$. In terms of the mild form,

$$\begin{aligned}\tilde{u}(t,x) = &\int_{-\infty}^{\infty} G(t, x-y)u_0(y)dy\\ &+ \int_0^t \int_{-\infty}^{\infty} G(t-s, x-y)\sigma(u(s\wedge\tau_{K,1}, y))W(dy,ds).\end{aligned} \tag{3.5}$$

Finally, note that on $\{\tau_{K,1} = T_0\}$, we have that $u(t,x) = \tilde{u}(t,x)$ for all $(t,x) \in [0,T_0] \times \mathbb{R}$. Thus,

$$\lim_{K\to\infty} P\left\{u(t,x) = \tilde{u}_K(t,x) \text{ for all } (t,x) \in [0,T_0] \times \mathbb{R}\right\} = 1. \tag{3.6}$$

For the time being, we will work with $\tilde{u}$.

Second stopping time $\tau_{K,2}$

We also want to control the growth of our solution $\tilde{u}$ as $x \to \pm\infty$. Let $\tau_{K,2}$ be the first time $t \in [0,T_0]$ such that there exists $x \in \mathbb{R}$ with

$$|\tilde{u}(t,x)| \geq K(1+|x|).$$

If there is no such time t, let $\tau_{K,2} = T_0$.

Since we are assuming that our initial function $u_0(x)$ is bounded, it is a consequence of Lemma 6.8 below (taking $\phi(r,z) = \sigma(u(r \wedge \tau_{K,1}, z))$ in (6.4) and $\phi_1 = \sigma_1$ in (6.5)) that

$$\lim_{K\to\infty} P\left\{\tau_{K,2} < T_0\right\} = 0. \tag{3.7}$$

Third stopping time $\tau_{K,3}$

We also work with the (continuous version on $]0,\infty[\, \times \mathbb{R}$ of the) following linear system of stochastic heat equations with additive noise:

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x) &= \frac{\partial^2}{\partial x^2} v(t,x) + \dot{W}(t,x), \qquad t>0,\ x \in \mathbb{R}, \\ v(0,x) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{3.8}$$

Now we define $\tau_{K,3}$ in the same way as $\tau_{K,2}$, but with respect to v rather than $\tilde{u}$:

$$\tau_{K,3} = T_0 \wedge \inf\{t \in [0,T_0] : \exists x \in \mathbb{R} \text{ with } |v(t,x)| \geq K(1+|x|)\}.$$

As with the stopping time $\tau_{K,2}$, since we are assuming that our initial function $u_0(x)$ is bounded, it is a consequence of Lemma 6.8 (taking $\phi(r,z) \equiv 1$ in (6.4) and $\phi_1 = 1$ in (6.5)) below that

$$\lim_{K\to\infty} P\left\{\tau_{K,3} < T_0\right\} = 0.$$

4 Local decomposition of the solution

Fix

$$\alpha \in \left]\tfrac{1}{2}, \tfrac{2}{3}\right[, \qquad \beta \in \left]\alpha, \tfrac{2}{3}\right[. \tag{4.1}$$

Consider the rectangle $R_\rho(t_0,x_0)$ defined in (3.1). In order to study the behavior of $\tilde{u}$ in this rectangle, we are going to use a decomposition based on a time prior to $t_0 - \rho^4$, namely, we define

$$t_0^- = t_0^-(\rho) = t_0 - \rho^4 - \rho^{4(1-\alpha)}$$

(notice that since $t_0 \geq 1$, $\rho \in]0, \frac{1}{2}]$ and $\alpha < \frac{2}{3}$, we have $t_0^- > \frac{1}{2}$). We also set

$$L_1 = L_1(\rho) = \rho^2 + \rho^{2(1-\beta)}.$$

The rectangle

$$R^+ = [t_0^-(\rho), t_0 + \rho^4] \times [x_0 - L_1, x_0 + L_1].$$

is an enlargement of $R_\rho(t_0, x_0)$.

Note that for small $\rho > 0$,

$$\rho^{4(1-\alpha)} \ll \left[\rho^{2(1-\beta)}\right]^2;$$

the idea is that in the x-direction, R^+ is larger than parabolic scaling would indicate. Indeed, we would have exact parabolic scaling if β were equal to α.

4.1 Isolating the dominant term

We use the Markov property [3, Chapter 9] to start $\tilde{u}$ afresh at time t_0^-, so that for $(t, x) \in R_\rho(t_0, x_0)$, we have

$$\tilde{u}(t, x) = \tilde{u}_{t_0,x_0,\rho}(t, x) + N_{t_0,x_0,\rho}(t, x),$$

where, for $t \geq t_0^-$ and $x \in \mathbb{R}$,

$$\tilde{u}_{t_0,x_0,\rho}(t, x) = \int_{-\infty}^{\infty} G(t - t_0^-, x - y)\tilde{u}(t_0^-, y)dy \tag{4.2}$$

$$N_{t_0,x_0,\rho}(t, x) = \int_{t_0^-}^{t} \int_{-\infty}^{\infty} G(t - s, x - y)\sigma(u(s \wedge \tau_{K,1}, y))W(dy, ds).$$

Note that $\tilde{u}_{t_0,x_0,\rho}(t, x)$ is the solution of the heat equation started at time t_0^- with initial function $\tilde{u}(t_0^-, \cdot)$.

We further decompose $N_{t_0,x_0,\rho}(t, x)$ as follows. Let

$$N^{(0)}(t, x) = \int_0^t \int_{-\infty}^{\infty} G(t - s, x - y)W(dy, ds),$$

$$N^{(1)}_{t_0,x_0,\rho}(t, x) = \int_{t_0^-}^{t} \int_{x_0-L_1}^{x_0+L_1} G(t - s, x - y)\big[\sigma(u(s \wedge \tau_{K,1}, y)) - \sigma(u(t_0^- \wedge \tau_{K,1}, x_0))\big]W(dy, ds),$$

$$N^{(2)}_{t_0,x_0,\rho}(t, x) = \int_{t_0^-}^{t} \int_{[x_0-L_1,x_0+L_1]^c} G(t - s, x - y)\big[\sigma(u(s \wedge \tau_{K,1}, y)) - \sigma(u(t_0^- \wedge \tau_{K,1}, x_0))\big]W(dy, ds),$$

$$\begin{aligned} v^{(1)}_{t_0,x_0,\rho}(t, x) &= \int_0^{t_0^-} \int_{-\infty}^{\infty} G(t - s, x - y)W(dy, ds) \\ &= \int_{-\infty}^{\infty} G(t - t_0^-, x - y)N^{(0)}(t_0^-, y)dy. \end{aligned} \tag{4.3}$$

In the last line above, we have used semigroup property of G and the stochastic Fubini theorem, see [22, Theorem 2.6]; notice that the dependence of these processes on K is omitted from the notation. Note also that $v^{(1)}_{t_0,x_0,\rho}$ is very similar to $\tilde u_{t_0,x_0,\rho}$. We see that for $t \geq t_0^-$ and $x \in \mathbb{R}$,

$$N_{t_0,x_0,\rho}(t,x) = \sigma(u(t_0^- \wedge \tau_{K,1},0))N^{(0)}(t,x) + N^{(1)}_{t_0,x_0,\rho}(t,x) + N^{(2)}_{t_0,x_0,\rho}(t,x) \\ - \sigma(u(t_0^- \wedge \tau_{K,1},0))v^{(1)}_{t_0,x_0,\rho}(t,x).$$

With the notation in (4.2) and (4.3) above, we have

$$\begin{aligned} \tilde u(t,x) = \sigma(u(t_0^- \wedge \tau_{K,1},0))N^{(0)}(t,x) \\ + \tilde u_{t_0,x_0,\rho}(t,x) + N^{(1)}_{t_0,x_0,\rho}(t,x) + N^{(2)}_{t_0,x_0,\rho}(t,x) \\ - \sigma(u(t_0^- \wedge \tau_{K,1},0))v^{(1)}_{t_0,x_0,\rho}(t,x). \end{aligned}$$

We also impose some growth conditions on $x \mapsto \tilde u_{t_0,x_0,\rho}(t,x)$ and $x \mapsto v^{(1)}_{t_0,x_0,\rho}(t,x)$ by defining, for $t \geq t_0^-$ and $x \in \mathbb{R}$,

$$\hat u_{t_0,x_0,\rho}(t,x) = \begin{cases} \tilde u_{t_0,x_0,\rho}(t,x) & \text{if } t_0^- \leq \tau_{K,2}, \\ 0 & \text{if } t_0^- > \tau_{K,2}, \end{cases}$$

$$\hat v^{(1)}_{t_0,x_0,\rho}(t,x) = \begin{cases} v^{(1)}_{t_0,x_0,\rho}(t,x) & \text{if } t_0^- \leq \tau_{K,3}, \\ 0 & \text{if } t_0^- > \tau_{K,3}. \end{cases} \tag{4.4}$$

Finally, for $t \geq t_0^-$ and $x \in \mathbb{R}$, we define a new process $(w_{t_0,x_0,\rho}(t,x))$, which is related to the solution $u(t,x)$ but with frozen coefficients and controlled growth, by

$$w_{t_0,x_0,\rho}(t,x) = \sigma(u(t_0^- \wedge \tau_{K,1},0))N^{(0)}(t,x) + E_{t_0,x_0,\rho}(t,x), \tag{4.5}$$

where

$$E_{t_0,x_0,\rho}(t,x) = N^{(1)}_{t_0,x_0,\rho}(t,x) + N^{(2)}_{t_0,x_0,\rho}(t,x) - \sigma(u(t_0^- \wedge \tau_{K,1},0))\hat v^{(1)}_{t_0,x_0,\rho}(t,x) + \hat u_{t_0,x_0,\rho}(t,x). \tag{4.6}$$

Observe that if $\tau_{K,2} \wedge \tau_{K,3} = T_0$, then $\hat u_{t_0,x_0,\rho} \equiv \tilde u_{t_0,x_0,\rho}$ and $\hat v^{(1)}_{t_0,x_0,\rho} \equiv v^{(1)}_{t_0,x_0,\rho}(t,x)$. Thus, if $\tau_{K,2} \wedge \tau_{K,3} = T_0$, then for $t \geq t_0^-$ and $x \in \mathbb{R}$,

$$\tilde u(t,x) = w_{t_0,x_0,\rho}(t,x). \tag{4.7}$$

We wish to show that the oscillation of $\tilde u$ on the rectangle $R_\rho(t_0,x_0)$ is comparable to the oscillation of $N^{(0)}$ on $R_\rho(t_0,x_0)$. By (4.7), it suffices to study the oscillation of $w_{t_0,x_0,\rho}$ on $R_\rho(t_0,x_0)$. The oscillation of $w_{t_0,x_0,\rho}$ on $R_\rho(t_0,x_0)$ consists of those of $N^{(0)}$ and $E_{t_0,x_0,\rho}$. The oscillation of $E_{t_0,x_0,\rho}$ comes from those of $N^{(1)}$, $N^{(2)}$, $\hat v^{(1)}$ and $\hat u$. Roughly speaking, the term in the square brackets in the definitions of $N^{(1)}$ is small, and so the oscillation of $N^{(1)}$ is small compared to the oscillation of $N^{(0)}$. Also, in the definition of $N^{(2)}$, the heat kernel G is small on the region of integration. The oscillations of $\hat v^{(1)}$ and $\hat u$ are small because t_0^- is chosen far enough in the past of t_0 so that the heat kernel has the time to smooth the initial condition at time t_0^-, thanks to the growth bound $1 + |x|$ related to the stopping times $\tau_{K,2}$ and $\tau_{K,3}$. So altogether, we will see that $N^{(0)}$ is the term with dominant oscillation, and since it is Gaussian, we have precise estimates for it (see Proposition 5.1).

4.2 Oscillations of $N^{(0)}(t,x)$, $N^{(1)}_{t_0,x_0,\rho}(t,x)$ and $N^{(2)}_{t_0,x_0,\rho}(t,x)$

The random field $N^{(0)}(t,x)$ is a Gaussian process whose canonical metric is bounded by the metric Δ defined at the beginning of Section 3. Thus Talagrand's analysis [20, 21] will apply to this case. To obtain a modulus of continuity for $N^{(0)}(t,x)$, we can use Corollary 6.7 below with $\phi \equiv 1$, $\phi_1 = 1$, $S_0 = 0$, $S_1 = t_0 - \rho^4$, $T = t_0 + \rho^4$, to obtain constants C_0 and C_1 such that for all $\lambda > 0$,

$$P\left(\operatorname{osc}_{R_\rho}(N^{(0)}) > \lambda\rho\right) \leq C_0 \exp\left(-C_1\lambda^2\right).$$

Oscillations of $N^{(1)}_{t_0,x_0,\rho}(t,x)$

Lemma 4.1. *Let β be defined in (4.1). There are constants $C_0, C_1 \in \mathbb{R}_+$ (which may depend on K and the Lipschitz constant L) such that, for all $(t_0,x_0) \in [1,2]\times[-1,1]$, $\rho \in]0,1[$ and $\lambda > 0$,*

$$P\left\{\operatorname{osc}_{R_\rho(t_0,x_0)}\left(N^{(1)}_{t_0,x_0,\rho}\right) > \lambda\right\} \leq C_0 \exp\left(-C_1\lambda^2\rho^{-2(1-\beta)(1-\delta)-2}\right).$$

Proof. Because σ is Lipschitz with Lipschitz constant L (by Assumption 2.1(a)), and since, for $(t,x) \in R_\rho(t_0,x_0)$, in the stochastic integral that defines $N^{(1)}_{t_0,x_0,\rho}(t,x)$, we have $t_0^- \leq s \leq t_0 + \rho^4$ and $|y - x_0| \leq L_1$, we know from (3.4) that

$$\begin{aligned}\left|\sigma(u(s\wedge\tau_{K,1},y)) - \sigma(u(t_0^-\wedge\tau_{K,1},x_0))\right| &\leq L|u(s\wedge\tau_{K,1},y) - u(t_0^-\wedge\tau_{K,1},x_0)| \\ &\leq LK\Delta(t_0+\rho^4-t_0^-, L_1)^{1-\delta}. \\ &\leq LK\rho^{(1-\beta)(1-\delta)}.\end{aligned} \tag{4.8}$$

It therefore follows from Corollary 6.7 below, with

$$\phi(r,z) = \sigma(u(r\wedge\tau_{K,1},z)) - \sigma(u(t_0^-\wedge\tau_{K,1},x_0)), \quad \phi_1 = LK\rho^{(1-\beta)(1-\delta)},$$

$S_0 = t_0^-$, $S_1 = t_0 - \rho^4$, and $T = t_0 + \rho^4$, that for all $\lambda > 0$,

$$P\left\{\operatorname{osc}_{R_\rho}\left(N^{(1)}_{t_0,x_0,\rho}\right) > \lambda\right\} \leq C_0 \exp\left(-C_1\lambda^2\rho^{-2(1-\beta)(1-\delta)}(2\rho^4)^{-1/2}\right).$$

This proves the lemma. □

Oscillations of $N^{(2)}_{t_0,x_0,\rho}(t,x)$

Recall that

$$N^{(2)}_{t_0,x_0,\rho}(t,x) = \int_{t_0^-}^t \int_{[x_0-L_1,x_0+L_1]^c} G(t-s,x-y)\left[\sigma(u(s\wedge\tau_{K,1},y)) - \sigma(u(t_0^-\wedge\tau_{K,1},x_0))\right] W(dy,ds).$$

Our goal for this subsection is to establish the following lemma.

Lemma 4.2. *Fix $\kappa > 1$. There are constants $C_0, C_1 \in \mathbb{R}_+$ (which may depend on K and σ_1) such that, for all $(t_0, x_0) \in [1,2] \times [-1,1]$, $\rho \in]0, \frac{1}{2}]$ and $\lambda > 0$,*

$$P\left\{ \sup_{(t,x)\in R_\rho(t_0,x_0)} |N^{(2)}_{t_0,x_0,\rho}(t,x)| > \lambda \right\} \leq C_0 \exp\left(-C_1 \lambda^2 \rho^{-2\kappa}\right) \tag{4.9}$$

and

$$P\left\{ \operatorname{osc}_{R_\rho(t_0,x_0)} \left(N^{(2)}_{t_0,x_0,\rho} \right) > \lambda \right\} \leq C_0 \exp\left(-C_1 \lambda^2 \rho^{-2\kappa}\right), \tag{4.10}$$

with possibly different constants C_0 and C_1.

Proof. Since the oscillation of a function is bounded by twice its absolute maximum, we see that (4.9) implies (4.10). So we now prove (4.9).

First, we split up $N^{(2)}$ as follows:

$$N^{(2)}_{t_0,x_0,\rho}(t,x) = N^{(2a)}_{t_0,x_0,\rho}(t,x) + N^{(2b)}_{t_0,x_0,\rho}(t,x),$$

where

$$N^{(2a)}_{t_0,x_0,\rho}(t,x) = \int_{t_0^-}^{t} \int_{[x_0-L_1,x_0+L_1]^c} G(t-s, x-y)\sigma(u(s \wedge \tau_{K,1}, y))\, W(dy, ds),$$
$$N^{(2b)}_{t_0,x_0,\rho}(t,x) = \int_{t_0^-}^{t} \int_{[x_0-L_1,x_0+L_1]^c} G(t-s, x-y)\sigma(u(t_0^- \wedge \tau_{K,1}, 0))\, W(dy, ds).$$

It suffices to show (4.9) for $N^{(2)}$ replaced by $N^{(2a)}$ and $N^{(2b)}$. By Assumption 2.1(b), the factor $\sigma(\cdots)$ in both $N^{(2a)}$ and $N^{(2b)}$ is bounded by σ_1, and that is the only information about σ that we will use in our proof. So we will only deal with $N^{(2a)}$, since the proof for $N^{(2b)}$ is identical.

The intuition is the following.

Step 1: Since $R_\rho(t_0, x_0)$ is far away from $[t_0^-, t_0 + \rho^4] \times [x_0 - L_1, x_0 + L_1]^c$, we expect $G(t-s, x-y)$ to be small for $(t,x) \in R_\rho(t_0,x_0)$ and $(s,y) \in [t_0^-, t] \times [x_0 - L_1, x_0 + L_1]^c$. This will lead to a small value of $N^{(2a)}_{t_0,x_0,\rho}(t,x)$ when (t,x) is fixed, with high probability.

Step 2: To go further and show that the supremum of $|N^{(2a)}_{t_0,x_0,\rho}(t,x)|$ over $(t,x) \in R_\rho(t_0,x_0)$ is small, we divide $R_\rho(t_0,x_0)$ into even smaller subrectangles. Ignoring the helpful fact that the domain of integration is far away from each of these subrectangles, we simply take the size of each subrectangle to be so small that, by Corollary 6.7, the oscillation of $N^{(2a)}_{t_0,x_0,\rho}(t,x)$ over such a subrectangle is small with high probability.

We begin with Step 2, which is a bit easier. Divide $R_\rho(t_0,x_0)$ into a union of nonoverlapping subrectangles of dimensions $\rho^{4\kappa} \times \rho^{2\kappa}$, where $\kappa > 1$ is fixed in the statement of the lemma. Since $R_\rho(t_0,x_0)$ has dimensions $\rho^4 \times \rho^2$, the number of subrectangles required is bounded by $C\rho^{-6(\kappa-1)}$. On each of these subrectangles R', Corollary 6.7 with $\phi(r,z) = \sigma(u(r \wedge \tau_{K,1}, z))$ in (6.6), $\phi_1 = \sigma_1$, $S_0 = t_0^-$, $T - S_1 = 2\rho^{4\kappa}$, tells us that for all $\lambda > 0$,

$$P\left\{ \operatorname{osc}_{R'} \left(N^{(2a)}_{t_0,x_0,\rho} \right) > \lambda/2 \right\} \leq C_0 \exp\left(-C_1 \lambda^2 \rho^{-2\kappa}\right),$$

where the constant C_1 incorporates the constant σ_1.

Let $A_1(\rho,\kappa,\lambda)$ be the event that the oscillation of $N^{(2a)}_{t_0,x_0,\rho}$ over each of these rectangles is less than or equal to $\lambda/2$. Then for all $\lambda > 0$ and $\rho \in]0, \frac{1}{2}]$,

$$\begin{aligned} P\big(A_1(\rho,\kappa,\lambda)^c\big) &\leq C\rho^{6(1-\kappa)} \cdot C_0 \exp\left(-C_1\lambda^2\rho^{-2\kappa}\right) \\ &\leq C_0 \exp\left(-C_1\lambda^2\rho^{-2\kappa}\right), \end{aligned} \tag{4.11}$$

where C_0, C_1 may vary from line to line.

Now we turn to Step 1. Let R' be one of these subrectangles, and let $p' = (t', x') \in R'$ be a distinguished point. We wish to estimate $P\{|N^{(2a)}_{t_0,x_0,\rho}(p')| > \lambda/2\}$. For $r \in [t_0^-, t']$, let

$$M_r = \int_{t_0^-}^{r} \int_{[x_0-L_1,x_0+L_1]^c} G(t'-s, x'-y)\sigma(u(s\wedge\tau_{K,1}, y))\, W(dy, ds)$$

and note that (M_r) is an $(\mathcal{F}_r)$-martingale with quadratic variation

$$\begin{aligned} \langle M\rangle_r &= \int_{t_0^-}^{r} \int_{[x_0-L_1,x_0+L_1]^c} G^2(t'-s, x'-y)\sigma^2(u(s\wedge\tau_{K,1}, y))dyds \\ &\leq \sigma_1^2 \int_{t_0^-}^{t'} \int_{[x_0-L_1,x_0+L_1]^c} G^2(t'-s, x'-y)dyds \\ &= \sigma_1^2 \int_0^{t'-t_0^-} \int_{[x_0-L_1,x_0+L_1]^c} \frac{1}{2\pi r} \exp\left[-\frac{(x'-y)^2}{r}\right]. \end{aligned} \tag{4.12}$$

Note that for $a > 0$, $r \mapsto \frac{1}{r}\exp(-a^2/r)$ is increasing on the interval $]0, a^2]$, then decreasing. Let α and β be defined as in (4.1). For $r \in [0, t'-t_0^-]$ and $y \in [x_0 - L_1, x_0 + L_1]^c$,

$$r \leq t_0 + \rho^4 - t_0^- \leq C\rho^{4(1-\alpha)} \qquad \text{and} \qquad |x'-y| \geq \rho^{2(1-\beta)}.$$

Since $\beta > \alpha$, we replace r by $t' - t_0^-$ in (4.12) to see that

$$\langle M\rangle_r \leq \frac{\sigma_1^2}{2\pi} \int_{[x_0-L_1,x_0+L_1]^c} dy \, \exp\left[-\frac{(x'-y)^2}{t'-t_0^-}\right] \leq \frac{\sigma_1^2}{2\pi} \int_{[x_0-L_1,x_0+L_1]^c} dy \, \exp\left[-\frac{(x'-y)^2}{C\rho^{4(1-\alpha)}}\right].$$

This integral is a sum of two integrals, over $]-\infty, x_0 - L_1]$ and $[x_0 + L_1, +\infty[$. Both are bounded above by

$$\begin{aligned} \int_{\rho^{2(1-\beta)}} dz \, \exp\left[-\frac{z^2}{C\rho^{4(1-\alpha)}}\right] &= \int_{\rho^{2(\alpha-\beta)}} du \, \rho^{2(1-\alpha)} \exp\left[-\frac{u^2}{C}\right] \leq C\rho^{2(1-\alpha)} \exp\left[-\rho^{-4(\alpha-\beta)}\right] \\ &\leq C \exp\left[-\rho^{-4(\alpha-\beta)}\right]. \end{aligned}$$

Finally, we obtain

$$\langle M\rangle_r \leq C_0 \exp\left(-C_1\rho^{-4(\beta-\alpha)}\right), \qquad r \in [t_0^-, t'].$$

Since $\beta > \alpha$, this exponential is small for small ρ.

Thus, M_r is a time-changed Brownian motion with time scale bounded by our bound on $\langle M\rangle_r$, and by the reflection principle for Brownian motion,

$$\begin{aligned}
P\{|N^{(2a)}_{t_0,x_0,\rho}(t',x')| > \lambda/2\} &\leq P\left\{\sup_{t_0^-\leq r\leq t'} |M_r| > \lambda/2\right\} \\
&\leq 2P\left\{\sup_{0\leq t\leq \langle M\rangle_{t'}} B_t > \lambda/2\right\} \\
&\leq C_0 \exp\Big(-C_2\lambda^2 \exp\big(C_1\rho^{-4(\beta-\alpha)}\big)\Big). \qquad (4.13)
\end{aligned}$$

Let $A_2(\rho,\kappa,\lambda)$ be the event that for each subrectangle R' and for each distinguished point $p'\in R'$, we have $|N^{(2a)}_{t_0,x_0,\rho}(p')| < \lambda/2$. By (4.13),

$$P(A_2(\rho,\kappa,\lambda)^c) \leq C_0\rho^{-6(\kappa-1)}\exp\Big(-C_2\lambda^2\exp\big(C_1\rho^{-4(\beta-\alpha)}\big)\Big).$$

Therefore, using (4.11),

$$\begin{aligned}
P\left\{\sup_{(t,x)\in R_\rho} |N^{(2a)}_{t_0,x_0,\rho}(t,x)| > \lambda\right\} &\leq P(A_1(\rho,\kappa,\lambda)^c) + P(A_2(\rho,\kappa,\lambda)^c) \qquad (4.14)\\
&\leq C\rho^{-6(\kappa-1)}\cdot C_0\exp\left(-C_1\lambda^2\rho^{-2\kappa}\right) \\
&\quad + C_0\rho^{-6(\kappa-1)}\exp\Big(-C_2\lambda^2\exp\big(C_1\rho^{-4(\beta-\alpha)}\big)\Big) \\
&\leq C_0\exp\left(-C_2\lambda^2\rho^{-2\kappa}\right);
\end{aligned}$$

here, we have allowed the constants C_0, C_2 to vary from line to line. □

4.3 Oscillations of $\hat{u}_{t_0,x_0,\rho}(t,x)$ and $\hat{v}^{(1)}_{t_0,x_0,\rho}(t,x)$

Let $\hat{u}_{t_0,x_0,\rho}(t,x)$ and $\hat{v}^{(1)}_{t_0,x_0,\rho}(t,x)$ be as defined in (4.4).

Lemma 4.3. *Let α be as defined in* (4.1)*. There exists a constant C (which may depend on K and L) such that, for all $(t_0,x_0)\in[1,2]\times[-1,1]$, a.s., for all $\rho\in\,]0,\frac{1}{2}]$,*

$$\mathrm{osc}_{R_\rho(t_0,x_0)}\big(\hat{u}_{t_0,x_0,\rho}\big) \leq C\rho^{2\alpha} \qquad \textit{and} \qquad \mathrm{osc}_{R_\rho(t_0,x_0)}\big(\hat{v}^{(1)}_{t_0,x_0,\rho}\big) \leq C\rho^{2\alpha}.$$

Proof. The only difference between $\hat{u}_{t_0,x_0,\rho}(t,x)$ and $\hat{v}^{(1)}_{t_0,x_0,\rho}(t,x)$ is the initial data at $t=t_0^-$, which, in both cases, by the definitions of $\tau_{K,2}$ and $\tau_{K,3}$, have growth bounded by $K(1+|x|)$, and this bound is all that we use in our proof. Therefore we will only deal with $\hat{u}_{t_0,x_0,\rho}(t,x)$, and leave $\hat{v}^{(1)}_{t_0,x_0,\rho}(t,x)$ to the reader.

As mentioned,

$$|\hat{u}_{t_0,x_0,\rho}(t_0^-,x)| \leq K(1+|x|).$$

Recall from (4.4) that for $(t,x)\in R_\rho(t_0,x_0)$,

$$\hat{u}_{t_0,x_0,\rho}(t,x) = \int_{-\infty}^{\infty} G(t-t_0^-,x-z)\hat{u}_{t_0,x_0,\rho}(t_0^-,z)dz,$$

and so, using the inequalities $|z|+1 \le |x-z|+|x|+1 \le |x-z|+3$ since $|x|\le 2$, and the standard inequality

$$\left|\frac{\partial G(t,x)}{\partial x}\right| \le \frac{c}{\sqrt{t}} G(2t,x),$$

we obtain

$$\begin{aligned}
\left|\frac{\partial \hat{u}_{t_0,x_0,\rho}(t,x)}{\partial x}\right| &\le C \int_{-\infty}^{\infty} (t-t_0^-)^{-1/2}\, G(2(t-t_0^-),x-z)\,(|z|+1)dz \\
&\le C \int_{-\infty}^{\infty} (t-t_0^-)^{-1/2}\, G(2(t-t_0^-),x-z)\,(|x-z|+3)dz \\
&:= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= C \int_{-\infty}^{\infty} |x-z|\,(t-t_0^-)^{-1/2}\, G(2(t-t_0^-),x-z)\,dz, \\
I_2 &= 3C\,(t-t_0^-)^{-1/2} \int_{-\infty}^{\infty} G(2(t-t_0^-),x-z)\,dz.
\end{aligned}$$

Clearly, $I_2 = 3C(t-t_0^-)^{-1/2}$ and by change of variables, $I_1 \le C'$, therefore, since $t-t_0^- \le 1$,

$$\left|\frac{\partial \hat{u}_{t_0,x_0,\rho}(t,x)}{\partial x}\right| \le C(t-t_0^-)^{-1/2}. \tag{4.15}$$

Next, again using $|z|+1 \le |x-z|+3$ since $|x|\le 2$, and the standard inequality

$$\left|\frac{\partial G(t,x)}{\partial t}\right| \le \frac{c}{t} G(2t,x),$$

we find that

$$\begin{aligned}
\left|\frac{\partial \hat{u}_{t_0,x_0,\rho}(t,x)}{\partial t}\right| &\le C \int_{-\infty}^{\infty} \frac{1}{t-t_0^-}\, G(2(t-t_0^-),x-z)(|x-z|+3)dz \\
&:= I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_3 &= C \int_{-\infty}^{\infty} \frac{|x-z|}{t-t_0^-}\, G(2(t-t_0^-),x-z)\,dz, \\
I_4 &= 3C\, \frac{1}{t-t_0^-} \int_{-\infty}^{\infty} G(2(t-t_0^-),x-z)\,dz.
\end{aligned}$$

Clearly, $I_4 = 3C(t-t_0^-)^{-1}$, and again, a change of variables gives us $I_3 \le C(t-t_0^-)^{-1/2}$, so we conclude that

$$\left|\frac{\partial \hat{u}_{t_0,x_0,\rho}(t,x)}{\partial t}\right| \le C(t-t_0^-)^{-1}. \tag{4.16}$$

Now, since $t-t_0^- \ge \rho^{4(1-\alpha)}$, we can bound the oscillation of $\hat{u}_{t_0,x_0,\rho}$ over $R_\rho(t_0,x_0)$ as follows. First, consider oscillation in the x-direction. Let I be a line segment contained in

$R_\rho(t_0, x_0)$, consisting of points (t, x) with t fixed. The rectangle $R_\rho(t_0, x_0)$ has width ρ^2, so using (4.15), we get

$$\text{osc}_I\big(\hat{u}_{t_0,x_0,\rho}\big) \leq C\rho^2(t - t_0^-)^{-1/2} \leq C\rho^2\left(\rho^{4(1-\alpha)}\right)^{-1/2} = C\rho^{2\alpha}.$$

Second, consider oscillation in the t-direction. Let J be a line segment contained in $R_\rho(t_0, x_0)$, consisting of points (t, x) with x fixed. The rectangle $R_\rho(t_0, x_0)$ has height ρ^4, so using (4.16), we get

$$\text{osc}_J\big(\hat{u}_{t_0,x_0,\rho}\big) \leq C\rho^4(t - t_0^-)^{-1} \leq C\rho^4\left(\rho^{4(1-\alpha)}\right)^{-1} \leq C\rho^{4\alpha}.$$

Putting together these estimates establishes the conclusion of the lemma for $\hat{u}_{t_0,x_0,\rho}$. □

5 Existence of rectangles with small oscillations

For integers $q \geq 1$ and $\ell \geq 0$, set

$$r_{q,\ell} = 2^{-q}q^{-\ell} \qquad \text{and} \qquad \ell_q = \left\lfloor \frac{q}{\log_2 q} \right\rfloor$$

(where $\log_2$ is the base 2 logarithm), so that $r_{q,0} = 2^{-q}$ and $r_{q,\ell_q} \geq 2^{-2q}$ (and is of the same order as 2^{-2q}). Define

$$f(r) = r\left(\log_2\log_2\tfrac{1}{r}\right)^{-1/6}. \tag{5.1}$$

In [9, Prop.2.3], we established a result for Gaussian random fields satisfying [9, Assump 2.1]. In [9, Sect. 7], we showed that this assumption was satisfied for systems of linear stochastic heat equations with i.i.d. coefficients and vanishing initial condition (these last two assumptions are removed in [4]). Since $N^{(0)}$ is the solution of such a system of linear stochastic heat equations, we restate here the result of [9, Prop.2.3] for $N^{(0)}$, in the form that we will need.

Proposition 5.1. *There exist constants $\tilde{K}$ and q_0 with the following property: for all $q \geq q_0$ and for all $(t_0, x_0) \in [1, 2] \times [-1, 1]$,*

$$P\left\{\exists \ell \in \{0, \dots, \ell_q\} : \text{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(N^{(0)}) \leq \tilde{K}f(r_{q,\ell})\right\} \geq 1 - \exp\left(-\sqrt{q}\right).$$

Proof. This proposition is essentially equivalent to [9, Prop.2.3] (applied to $N^{(0)}$), but since the notation is different, we explain how Proposition 5.1 is obtained from the proof of [9, Prop 2.3].

In the proof of [9, Prop 2.3], we considered a sequence $r_\ell = r_0U^{-2\ell}$, we set $\ell_0 = \lfloor \frac{\log_2(1/r_0)}{2\log_2 U} \rfloor$ and we showed that

$$P\left\{\exists 1 \leq \ell \leq \ell_0 : \text{osc}_{R_{r_\ell}(t_0,x_0)}(N^{(0)}) \leq K_2 f(r_\ell)\right\} \geq 1 - \exp\left[-\left(\log_2\frac{1}{r_0}\right)^{1/2}\right].$$

Towards the end of the proof, U was chosen by setting

$$U = \left(\log_2 \frac{1}{r_0}\right)^{1/(2\beta)},$$

where this β, defined in the proof using the Hölder exponents of the Gaussian random field, takes the value $\beta = 1$ in the case of $N^{(0)}$.

Here, we take r_0 of the form $r_0 = 2^{-q}$, so $U = q^{1/2}$, and the r_ℓ, which now depend on q and which we denote $r_{q,\ell}$, take the value $r_{q,\ell} = 2^{-q} q^{-\ell}$, and $\ell_0 = \lfloor \frac{q}{\log_2 q} \rfloor$, which we now denote ℓ_q in the statement of Proposition 5.1. $\square$

In this section, we let $(t_0, x_0) \in R_0$, where

$$R_0 := [1, 2] \times [0, 1].$$

We will establish the following theorem.

Theorem 5.2. *Consider the process $w_{t_0,x_0,r}$ defined in* (4.5) *(with ρ there replaced by r). Let $\tilde{K}$ be the constant in Proposition 5.1 and f be the function defined in* (5.1)*. There is $\rho_0 \in]0, 1]$ with the following property. Given $0 < r_0 < \rho_0$, for all $(t_0, x_0) \in R_0$, we have*

$$P\left\{\exists r \in [r_0^2, r_0] : \operatorname{osc}_{R_r(t_0,x_0)}(w_{t_0,x_0,r}) \leq 2\sigma_1 \tilde{K} f(r)\right\} \geq 1 - 2\exp\left[-\left(\log_2 \frac{1}{r_0}\right)^{\frac{1}{2}}\right] \qquad (5.2)$$

(we will only use this for r_0 of the form 2^{-q}).

Remark 5.3. *The statement in this theorem should be compared with the statement for Gaussian processes in Proposition 5.1: notice the factor $2\sigma_1$ in front of $\tilde{K}$ and the factor 2 in front of the exponential on the right-hand side.*

Lemma 5.4. *Let $E_{t_0,x_0,\rho}$ be as defined in* (4.6)*. There are constants $a > 0$, $c_0 > 0$ and $c_1 > 0$ such that, for all $(t_0, x_0) \in R$ and all sufficiently large q,*

$$P\left\{\exists \ell \in \{0, \dots, \ell_q\} : \operatorname{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(E_{t_0,x_0,r_{q,\ell}}) \geq \sigma_1 \tilde{K} f(r_{q,\ell})\right\} \leq c_0 \exp\left(-c_1 2^{aq}\right).$$

Proof. Define the event

$$B_q = \left\{\exists \ell \in \{0, \dots, \ell_q\} : \operatorname{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(E_{t_0,x_0,r_{q,\ell}}) \geq \sigma_1 \tilde{K} f(r_{q,\ell})\right\}. \qquad (5.3)$$

Then

$$B_q \subset \bigcup_{\ell=0}^{\ell_q} \bigcup_{i=1}^{4} B_{q,\ell}^{(i)}, \qquad (5.4)$$

where

$$B_{q,\ell}^{(1)} = \left\{ \mathrm{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(N_{t_0,x_0,r_{q,\ell}}^{(1)}) \geq \tfrac{\sigma_1 \tilde{K}}{4} f(r_{q,\ell}) \right\},$$
$$B_{q,\ell}^{(2)} = \left\{ \mathrm{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(N_{t_0,x_0,r_{q,\ell}}^{(2)}) \geq \tfrac{\sigma_1 \tilde{K}}{4} f(r_{q,\ell}) \right\},$$
$$B_{q,\ell}^{(3)} = \left\{ \mathrm{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(\hat{v}_{t_0,x_0,r_{q,\ell}}^{(1)}) \geq \tfrac{\tilde{K}}{4} f(r_{q,\ell}) \right\},$$
$$B_{q,\ell}^{(4)} = \left\{ \mathrm{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(\hat{u}_{t_0,x_0,r_{q,\ell}}) \geq \tfrac{\sigma_1 \tilde{K}}{4} f(r_{q,\ell}) \right\}.$$

By the definition of $\tau_{K,1}$ and Lemma 4.1,

$$\begin{aligned} P(B_{q,\ell}^{(1)}) &\leq C_0 \exp\left[-C_1 (\sigma_1 \tilde{K} f(r_{q,\ell}))^2 \left(K r_{q,\ell}^{-(1-\beta)(1-\delta)-1} \right)^2 \right] \\ &= C_0 \exp\left[-C_1 \sigma_1^2 \tilde{K}^2 \frac{1}{(\log_2 \log_2 \frac{1}{r_{q,\ell}})^{2/6}} \tilde{K}^{-2} (r_{q,\ell})^{-2(1-\beta)(1-\delta)} \right]. \end{aligned}$$

Therefore, for $a = 2(1-\beta)(1-\delta)$, we have

$$\sum_{\ell=0}^{\ell_q} P(B_{q,\ell}^{(1)}) \leq c_0 \exp(-c_1 2^{aq}). \tag{5.5}$$

We develop similar estimates for $B_{q,\ell}^{(2)}$, $B_{q,\ell}^{(3)}$ and $B_{q,\ell}^{(4)}$. According to (4.10) in Lemma 4.2,

$$P(B_{q,\ell}^{(2)}) \leq C_0 \exp\left[-C_1 \left(\tfrac{\sigma_1 \tilde{K}}{4} f(r_{q,\ell}) \right)^2 (r_{q,\ell})^{-2\kappa} \right].$$

We take $\kappa = 2$, so that

$$\sum_{\ell=0}^{\ell_q} P(B_{q,\ell}^{(2)}) \leq C_0 \exp(-c_1 2^{2q}). \tag{5.6}$$

For large q and $\ell \in \{0, \ldots, \ell_q\}$, by Lemma 4.3, since $\alpha > \frac{1}{2}$,

$$P(B_{q,\ell}^{(3)}) = 0. \tag{5.7}$$

Finally, for $B_{q,\ell}^{(4)}$, also by Lemma 4.3, for large q and $\ell \in \{0, \ldots, \ell_q\}$,

$$P(B_{q,\ell}^{(4)}) = 0. \tag{5.8}$$

Putting together (5.4)–(5.8) establishes Lemma 5.4 with $a = 2(1-\beta)(1-\delta)$. □

Proof of Theorem 5.2. Define

$$A_{q,\ell} = \left\{ \mathrm{osc}_{R_{r_{q,\ell}}(t_0,x_0)}(N_{t_0,x_0,r_{q,\ell}}^{(0)}) \leq \tilde{K} f(r_{q,\ell}) \right\}.$$

Consider the event B_q defined in (5.3). If we set $r_0 = 2^{-q}$, then the event in (5.2) contains the event

$$\left(\bigcup_{\ell=0}^{\ell_q} A_{q,\ell} \right) \setminus B_q,$$

whose probability, using Proposition 5.1 and Lemma 5.4, is bounded below by

$$1 - \exp(-\sqrt{q}) - P(B_q). \tag{5.9}$$

From Lemma 5.4, we see that for q large enough, this is bounded below by $1 - 2\exp(-\sqrt{q})$, which proves the theorem. □

6 Probability estimates via chaining

Now we describe the chaining framework, which has been used in previous papers, such as [17].

Let R_1 be a rectangle with side lengths no greater than 1. By translating our coordinate system if necessary, we may assume that the lower left hand corner of R lies at the origin, so

$$R_1 = [0, a] \times [0, b], \qquad a, b \in [0, 1].$$

We consider the grid

$$\mathcal{G}_n = \{(k2^{-4n}, \ell 2^{-2n}) : \ k, \ell \in \mathbb{N}\}.$$

Note that the choice of exponents $-4n$, $-2n$ corresponds to the parabolic scaling: nearest neighbors in $\mathcal{G}_n$ have Δ-distance 2^{-n}. Finally, let

$$\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n.$$

Also, we say that a closed rectangle R is of type n if each of the four edges of R is an interval whose endpoints are nearest neighbors in $\mathcal{G}_n$. Two elements in a given rectangle of type n are at most at Δ-distance 2^{-n} of each other. We also say that a line segment (a step) is of type n if its endpoints are nearest neighbors of $\mathcal{G}_n$. Finally, a path of type n is a path whose steps are line segments of type n.

Lemma 6.1. *Let $(\delta_n, n \in \mathbb{N})$ be a sequence of nonnegative numbers. Let $g : R_1 \to \mathbb{R}$ and suppose that for all nearest neighbor pairs $p_n^{(1)}, p_n^{(2)} \in \mathcal{G}_n \cap R_1$, we have*

$$\left| g(p_n^{(1)}) - g(p_n^{(2)}) \right| \leq \delta_n, \qquad \text{for all } n \geq 0.$$

If $p^{(1)}, p^{(2)} \in \mathcal{G} \cap R_1$, then

$$\left| g(p^{(1)}) - g(p^{(2)}) \right| \leq 40 \sum_{n=n_0}^{\infty} \delta_n,$$

where $n_0 = n_0(p^{(1)}, p^{(2)})$ is the integer part of $\log_2(1/\Delta(p^{(1)}, p^{(2)}))$ (so $n_0 \geq 0$).

Lemma 6.1 follows from the triangle inequality and from the following lemma.

Lemma 6.2. *Let $p^{(1)}, p^{(2)} \in \mathcal{G} \cap R_1$, and let n_0 be the integer part of $\log_2(1/\Delta(p^{(1)}, p^{(2)}))$. Then we can connect $p^{(1)}$ and $p^{(2)}$ by a path consisting of line segments (steps) satisfying the following two conditions:*

(i) each line segment is of type n for some $n \geq n_0$.

(ii) for every $n \geq n_0$, there are at most 40 steps of type n.

Proof. Item (i) is a requirement which enters into the proof of (ii).

Item (ii): For $i = 1, 2$, let $R_i^{(n_0)}$ be the rectangle of type n_0 which contains $p^{(i)}$. First we claim that the rectangles $R_i^{(n_0)}$, $i = 1, 2$, are either the same or they share a corner $q_{n_0}^{(1)} = q_{n_0}^{(2)}$ that also belongs to R_1. We leave it to the reader to verify this assertion, which is similar to the statement that if two real numbers x and y are such that $|x - y| \leq 1$, then either x and y lie in the same interval with integer endpoints, or they belong to two adjacent such intervals.

In view of the above statement about rectangles sharing a corner, we see that it is enough to show that we can connect any one of the corners of $R_1^{(n_0)}$ to $p^{(1)}$ using a path with all corners in $\mathcal{G} \cap R_1$ and with at most 20 steps of type n for each $n \geq n_0$. Indeed, the same statement would hold for $i = 2$, giving a path of $20 + 20 = 40$ steps of type n altogether, using the shared corner as common starting point.

We can make a further reduction as follows. $i = 1$, we write p in place of $p^{(i)}$ in this paragraph. If $p^{(1)} \in \mathcal{G}_{n_0}$, then it is one of the corners of $R_1^{(n_0)}$ and only two steps of type n_0 are needed to connect $p^{(1)}$ to q_{n_0}. Assume that $p^{(1)} \notin \mathcal{G}_{n_0}$ and that n_1 is the smallest integer $> n_0$ such that $p \in \mathcal{G}_{n_1}$. For $n_0 < n < n_1$, let $R_1^{(n)} \subset R_1^{(n_0)}$ be a rectangle of type n which contains $p^{(1)}$. Since this rectangle must intersect R_1, we can choose one of its corners, denoted $q_n^{(1)}$, in $\mathcal{G}_n \cap R_1$. We also require that $q_{n_0}^{(1)}$ is the shared corner mentioned above. For $n = n_1$, we let $q_{n_1}^{(1)} = p^{(1)}$. It suffices to show that for $n_1 \geq n > n_0$, we can find a path of type n between $q_{n-1}^{(1)}$ and $q_n^{(1)}$, with at most 20 steps. However, $R_1^{(n-1)} \cap \mathcal{G}_n$ consists of a $2^4 \times 2^2$ grid of points. Given one of these points, and one of the corners of $R_1^{(n-1)}$, we can connect them by a path of type n by taking at most $2^4 = 16$ steps of type n in the t-direction and at most $2^2 = 4$ steps of type n in the x-direction. Altogether, this gives at most 20 steps of type n, as we claimed.

Thus we have a path from $q_{n_0}^{(1)}$ to $q_{n_1}^{(1)}$ of the required type. To get the full path from $p^{(1)}$ to $p^{(2)}$, we put together the two paths from $q_{n_0}^{(i)}$ to $p^{(i)}$, $i = 1, 2$, and we recall that $q_{n_0}^{(1)} = q_{n_0}^{(2)}$. □

Probability estimate for chaining

We use the notation of Lemmas 6.1 and 6.2 .

Lemma 6.3. *Let n_{R_1} be the largest value of n such that $\mathcal{G}_n \cap R_1$ is contained in a single rectangle of type n. For $n \geq n_{R_1}$, let $N(n)$ be the number of nearest neighbor pairs in $\mathcal{G}_n \cap R_1$, so that for all $n \geq n_{R_1}$,*

$$N(n) \leq 2^{6n+1} + 2^{4n} + 2^{2n} \leq 2^{6n+2}.$$

Let $(Y(t,x),\ (t,x)\in R_1)$ *be a stochastic process. Let* $(\delta_n,\ n\in\mathbb{N})\subset\mathbb{R}_+$ *and* $(\varepsilon_n,\ n\in\mathbb{N})\subset\mathbb{R}_+$ *be two sequences of nonnegative numbers. Suppose that for all* $n\geq n_{R_1}$ *and for all nearest neighbor pairs* $p_n^{(1)},p_n^{(2)}\in\mathcal{G}_n\cap R_1$*, we have*

$$P\left\{\left|Y(p_n^{(1)})-Y(p_n^{(2)})\right|>\delta_n\right\}\leq\varepsilon_n. \tag{6.1}$$

Let

$$\varepsilon=4\sum_{n=n_{R_1}}^{\infty}2^{6n}\varepsilon_n. \tag{6.2}$$

Let A *be the event that for all* $p^{(1)},p^{(2)}\in\mathcal{G}\cap R_1$*, we have*

$$\left|Y(p^{(2)})-Y(p^{(1)})\right|\leq 40\sum_{n=n_0(p^{(1)},p^{(2)})}^{\infty}\delta_n,$$

where $n_0(p^{(1)},p^{(2)})$ $(\geq n_{R_1})$ *is as in Lemma 6.1. Then* $P(A^c)\leq\varepsilon$*.*

Proof. Let F_n be the event that for all nearest neighbor pairs $p_n^{(1)},p_n^{(2)}\in\mathcal{G}_n\cap R_1$, we have $|Y(p_n^{(1)})-Y(p_n^{(2)})|\leq\delta_n$, and let $F=\cap_{n=n_{R_1}}^{\infty}F_n$. By assumption (6.1), we have

$$P(F^c)\leq\sum_{n=n_{R_1}}^{\infty}N(n)\varepsilon_n\leq 4\sum_{n=n_{R_1}}^{\infty}2^{6n}\varepsilon_n=\varepsilon.$$

Next we claim that on the set F, for all points $p^{(1)},p^{(2)}\in\mathcal{G}\cap R_1$, we have

$$|Y(p^{(2)})-Y(p^{(1)})|\leq 40\sum_{n=n_0(p^{(1)},p^{(2)})}^{\infty}\delta_n. \tag{6.3}$$

Indeed, this follows from Lemma 6.1 with $g=Y$. Therefore $A^c\subset F^c$, and this finishes the proof of Lemma 6.3. □

The following estimates are standard [14]. Here, $T>0$ and $0\leq s<t\leq T$.

Lemma 6.4. *There exists a constant* $C>0$ *such that for all* $0<s<t$, $x,y\in\mathbb{R}$*,*

$$\int_0^t\int_{-\infty}^{\infty}\left[G(t-r,x-z)-G(t-r,y-z)\right]^2dzdr\leq C|x-y|,$$
$$\int_0^s\int_{-\infty}^{\infty}\left[G(t-r,x-z)-G(s-r,x-z)\right]^2dzdr\leq C|t-s|^{1/2},$$
$$\int_s^t\int_{-\infty}^{\infty}\left[G(t-r,x-z)\right]^2dzdr\leq C|t-s|^{1/2}.$$

We also give a probability estimate which we will use together with Lemma 6.4.

Lemma 6.5. *There exist constants $C_0, C_1 > 0$ such that the following holds. Suppose that $p^{(i)} = (t_i, x_i) \in [0,\infty) \times \mathbb{R}$ for $i = 1, 2$. Let $\phi(t,x)$ be jointly measurable and $(\mathcal{F}_t)$-adapted, and assume that there is $\phi_1 \in \mathbb{R}_+$ such that*

$$\sup_{t,x} |\phi(t,x)| \le \phi_1, \qquad a.s.$$

Define

$$X_i = \int_0^{t_i} \int_{-\infty}^{\infty} G(t_i - s, x_i - y)\phi(s,y)W(dy,ds).$$

Then for $\lambda > 0$,

$$P\{|X_1 - X_2| > \lambda\} \le C_0 \exp\left(-C_1 \lambda^2 \phi_1^{-2} \Delta(p^{(1)}, p^{(2)})^{-2}\right).$$

Proof. Note that by replacing ϕ by ϕ/ϕ_1 and λ by λ/ϕ_1, we can reduce to the case $\phi_1 = 1$, so we assume from now on that $\phi_1 = 1$ and so $|\phi(s,y)| \le 1$. By possibly changing indices, we may assume $t_1 \le t_2$. Then

$$\begin{aligned} X_2 - X_1 &= \int_0^{t_1} \int_{-\infty}^{\infty} \left[G(t_2 - r, x_2 - z) - G(t_1 - r, x_2 - z)\right] \phi(r,z) W(dz,dr) \\ &\quad + \int_0^{t_1} \int_{-\infty}^{\infty} \left[G(t_1 - r, x_2 - z) - G(t_1 - r, x_1 - z)\right] \phi(r,z) W(dz,dr) \\ &\quad + \int_{t_1}^{t_2} \int_{-\infty}^{\infty} G(t_2 - r, x_2 - z) \phi(r,z) W(dz,dr) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We analyze term I_1, leaving I_2 and I_3 to the reader using very similar arguments. For $0 \le t \le t_1$, let

$$M_t = \int_0^{t} \int_{-\infty}^{\infty} \left[G(t_2 - r, x_2 - z) - G(t_1 - r, x_2 - z)\right] \phi(r,z) W(dz,dr)$$

We note that M_t is an $(\mathcal{F}_t)$-martingale with quadratic variation

$$\begin{aligned} \langle M \rangle_t &= \int_0^{t} \int_{-\infty}^{\infty} \left[G(t_2 - r, x_2 - z) - G(t_1 - r, x_2 - z)\right]^2 \phi(r,z)^2 dz dr \\ &\le \int_0^{t} \int_{-\infty}^{\infty} \left[G(t_2 - r, x_2 - z) - G(t_1 - r, x_2 - z)\right]^2 dz dr \\ &\le C\Delta(p^{(1)}, p^{(2)})^2 \end{aligned}$$

by Lemma 6.4. Thus M_t is a time-changed Brownian motion with time scale $\tau(t)$ bounded by

$$T_{\max} = C\Delta(p^{(1)}, p^{(2)})^2$$

for $0 \le t \le t_1$. Noting that $I_1 = M_{t_1}$, we obtain from the reflection principle for Brownian motion and standard Gaussian estimates that

$$P\{|I_1| > \lambda/3\} \le P\left\{\sup_{0 \le s \le T_{\max}} |B_s| > \lambda/3\right\} \le CP\{|B_{T_{\max}}| > \lambda/3\} \le C_0 \exp\left(-C_1 \lambda^2 T_{\max}^{-1}\right)$$

for appropriate constants $C_0, C_1 > 0$.

Similar estimates hold for I_2 and I_3. Combining these estimates and using the definition of $T_{\max}$ finishes the proof of Lemma 6.5. □

Probability bounds for the modulus of continuity

In this section we combine Lemmas 6.3 and 6.5 to get the probability bound in Lemma 6.6 below. For this section, let

$$N^{(3)}(t,x) = N^{(3)}(t,x,\phi) = \int_0^t \int_{-\infty}^{\infty} G(t-r, x-z)\phi(r,z)W(dz,dr), \tag{6.4}$$

where $\phi(r,z)$ is a jointly measurable and $(\mathcal{F}_t)$-adapted process, and for some $\phi_1 \in \mathbb{R}_+$,

$$\sup_{r,z} |\phi(r,z)| \leq \phi_1, \qquad \text{a.s.} \tag{6.5}$$

We will be using the jointly continuous version of $N^{(3)}$ (which exists by Lemma 6.4 and [2, Propositions 4.3 & 4.4]).

Lemma 6.6. *Fix $\lambda_0 > 0$. There exist constants C_0 and C_1 such that the following holds. For $\rho \in]0,1]$ and $\lambda \geq \lambda_0$, for each rectangle $R \subset R_0 = [1,2] \times [0,1]$ of dimensions $\rho^4 \times \rho^2$, let $A_\lambda(R)$ be the event that for all $p^{(1)}, p^{(2)} \in R$,*

$$\left|N^{(3)}(p^{(1)}) - N^{(3)}(p^{(2)})\right| \leq \lambda \Delta(p^{(1)}, p^{(2)}) \log_+\left(1/\Delta(p^{(1)}, p^{(2)})\right),$$

where for $\gamma > 0$, $\log_+(\gamma) := \max(1, \log_2(\gamma))$. Then

$$P(A_\lambda(R)^c) \leq C_0 \exp\left(-C_1 \lambda^2 \phi_1^{-2} \log_+^2(1/\rho)\right).$$

Proof. As in the proof of Lemma 6.5, first note that by replacing ϕ by ϕ/ϕ_1 and λ by λ/ϕ_1, we can reduce to the case where $|\phi(r,z)| \leq 1$, so we assume that that $\phi_1 = 1$.

Now let $n_1 \in \mathbb{N}$ be such that $2^{-n_1-1} < \rho \leq 2^{-n_1}$, and for $n \in \mathbb{N}$, set

$$\delta_n = c\lambda(n+1)2^{-n}, \qquad \varepsilon_n = C_0 \exp\left(-C_1 c^2 \lambda^2 n^2\right),$$

with $c > 0$ to be defined later, and C_0 and C_1 are the constants from Lemma 6.5. We want to use Lemma 6.3, but this lemma was stated for rectangles with one corner at the origin, and since this is not the case for R, we are going to shift the grid $\mathcal{G}$. Suppose that $R = [s_0, s_0 + \rho^4] \times [y_0, y_0 + \rho^2]$, where $p_0 = (s_0, y_0) \in [1,2] \times [0,1]$. In the statement of Lemma 6.3, we replace R_1 by R, $\mathcal{G}_n$ by $p_0 + \mathcal{G}_n$ and $\mathcal{G}$ by $p_0 + \mathcal{G}$, without affecting the validity of the statement in Lemma 6.3.

Next, we use Lemma 6.5 in order to check (6.1), for $p_n^{(i)} \in p_0 + \mathcal{G}_n$ for $i = 1, 2$. Lemma 6.5 yields

$$P\left\{\left|N^{(3)}(p_n^{(1)}) - N^{(3)}(p_n^{(2)})\right| > \delta_n\right\} \leq C_0 \exp\left(-C_1 \delta_n^2 \Delta(p_n^{(1)}, p_n^{(2)})^{-2}\right).$$

For $p_n^{(i)} \in (p_0 + \mathcal{G}_n) \cap R$ nearest neighbor pairs in $p_0 + \mathcal{G}_n$, we have

$$\Delta(p_n^{(1)}, p_n^{(2)}) = 2^{-n},$$

and so, using the definition of δ_n,

$$P\left\{\left|N^{(3)}(p_n^{(1)}) - N^{(3)}(p_n^{(2)})\right| > \delta_n\right\} \le C_0 \exp\left(-C_1\delta_n^2 2^{2n}\right) = C_0 \exp\left(-C_1 c^2\lambda^2(n+1)^2\right).$$

Now we use Lemma 6.3 with the shifted grid to complete the proof. In that lemma, let $Y(p) = N^{(3)}(p)$ and note that we have verified condition (6.1) with $\varepsilon_n = C_0 \exp(-C_1c^2\lambda^2n^2)$.

Also, for $p^{(1)}, p^{(2)} \in (p_0 + \mathcal{G}) \cap R$ and $n_0 = n_0(p^{(1)}, p^{(2)})$ (defined in Lemma 6.1), we compute

$$40 \sum_{m=n_0}^{\infty} \delta_m \le C_5\, c\lambda\, (n_0+1)\, 2^{-n_0} \le \lambda\Delta(p^{(1)},\, p^{(2)}) \log_+\left(1/\Delta(p^{(1)},\, p^{(2)})\right)$$

for some constant C_5 and for c small enough.

Continuing with formula (6.2), we define (with $n_{R_1} = n_1$)

$$\begin{aligned}\varepsilon = 2\sum_{n=n_1}^{\infty} 2^{6n}\varepsilon_n &= 2\sum_{n=n_1}^{\infty} 2^{6n} C_0 \exp\left(-C_1c^2\lambda^2(n+1)^2\right) \\ &\le C_0 \exp\left(-C_1c^2\lambda^2(n_1+1)^2\right)\end{aligned}$$

for $\lambda \ge \lambda_0$ and as usual, C_0, C_1 changing from line to line.

Since $n_1 + 1 > \log_+(1/\rho)$, the conclusion of Lemma 6.3 directly implies the conclusion of Lemma 6.6. □

Probability bounds for oscillation over a rectangle

Now we prove an estimate similar to Lemma 6.6, but for the modulus of continuity over a rectangle.

Assume that

$$0 \le S_0 \le S_1 \le t \le T, \qquad x \in \mathbb{R},$$

and let

$$N^{(4)}(t,x) = N^{(4)}(t,x,\phi,S_0,S_1) = \int_{S_0}^{t}\int_{-\infty}^{\infty} G(t-r, x-z)\phi(r,z)W(dz,dr), \tag{6.6}$$

where ϕ is a jointly measurable and $(\mathcal{F}_t)$-adapted process, and for some $\phi_1 \in \mathbb{R}_+$,

$$\sup_{t,x} |\phi(t,x)| \le \phi_1 \qquad \text{a.s.} \tag{6.7}$$

By Lemma 6.4 and [2, Propositions 4.3 & 4.4], $N^{(4)}$ has a continuous version.

Corollary 6.7. *Let $N^{(4)}(t,x)$ be as in (6.6). There exist constants $C_0, C_1 > 0$ such that for $0 \le S_0 \le S_1 \le T$ and $y_0 \in \mathbb{R}$, letting*

$$R_1 = [S_1, T] \times \left[y_0, y_0 + (T-S_1)^{1/2}\right]$$

we have for all $\lambda > 0$,

$$P\left\{\mathrm{osc}_{R_1}(N^{(4)}) > \lambda\right\} \le C_0 \exp\left(-C_1\lambda^2\phi_1^{-2}(T-S_1)^{-1/2}\right). \tag{6.8}$$

Proof. First, note that by translation of the time and space variables, which preserves the space-time white noise and property (6.7), it suffices to consider the case $S_0 = 0$ and $y_0 = 0$, so we will assume $S_0 = 0$ and $y_0 = 0$ from now on.

Secondly, by considering $N^{(4)}(t, x, \phi/\phi_1, 0, S_1)$ and noting that

$$N^{(4)}(t, x, \phi/\phi_1, 0, S_1) = \phi_1^{-1} N^{(4)}(t, x, \phi, 0, S_1)$$

we can remove the dependence on ϕ_1 and assume that $\phi_1 = 1$. Thirdly, we will remove the dependence on $T - S_1$ by scaling: let

$$\begin{aligned}\tilde{N}(t, x) &= \phi_1^{-1}(T - S_1)^{-1/4} N^{(4)}\left(t(T - S_1), x(T - S_1)^{1/2}\right) \\ \tilde{\phi}(t, x) &= \phi_1^{-1}\phi(t(T - S_1), x(T - S_1)^{1/2})\end{aligned}$$

and note that $\tilde{N}(t, x)$ satisfies (6.6) with the following modifications,

$$\tilde{N}(t, x) = \int_0^t \int_{-\infty}^{\infty} G(t - r, x - z)\tilde{\phi}(r, z)\tilde{W}(dzdr)$$

where $\tilde{W}$ is another two-parameter white noise and $|\tilde{\phi}| \leq 1$. With these transformation, the rectangle R_1 is replaced by

$$R_2 = [a, a + 1] \times [0, 1],$$

with $a = S_1/(T - S_1)$. In particular,

$$\begin{aligned}P\left\{\text{osc}_{R_1}(N^{(4)}) > \lambda\right\} &= P\left\{\text{osc}_{R_2}(\phi_1(T - S_1)^{1/4}\tilde{N}) > \lambda\right\} \\ &= P\left\{\text{osc}_{R_2}(\tilde{N}) > \lambda\phi_1^{-1}(T - S_1)^{-1/4}\right\}.\end{aligned} \tag{6.9}$$

Thus it suffices to prove Corollary 6.7 for $\tilde{N}$ and $S_0 = 0$, $y_0 = 0$, $T = S_1 + 1$. With these changes, since $S_0 = 0$, we can replace $\tilde{N}$ by $N^{(3)}$, as defined in (6.4).

We now reduce Corollary 6.7 to Lemma 6.6. In Lemma 6.6, let $R = R_2$, which is a 1×1 rectangle, so $\rho = 1$. Let $\lambda_0 = 1$. Observe that $|x \log_+(1/x)| \leq 1$ for $x \in]0, 1]$, and for points $p^{(1)}, p^{(2)} \in R_2$, we have $\Delta(p^{(1)}, p^{(2)}) \leq 1$, so on the event $A_\lambda(R_2)$ in Lemma 6.6, we have

$$\left|N^{(3)}(p^{(1)}) - N^{(3)}(p^{(2)})\right| \leq \lambda$$

for all points $p^{(1)}, p^{(2)} \in R_2$ and thus $\text{osc}_{R_2}(N^{(3)}) \leq \lambda$. Therefore, if A' is the event that $\text{osc}_{R_2}(N^{(3)}) \leq \lambda$, then we have $A_\lambda(R_2) \subset A'$ and also $(A')^c \subset A_\lambda(R_2)^c$. Therefore, Corollary 6.7 will follow if we can show that for all $\lambda > 0$,

$$P(A_\lambda(R_2)^c) \leq C_0 \exp\left(-C_1\lambda^2\right), \tag{6.10}$$

where we recall that we are in the case $\phi_1 = 1$ and $\rho = T - S_1 = 1$. However, the conclusion of Lemma 6.6 gives (6.10) for $\lambda \geq \lambda_0 = 1$. To deal with $0 < \lambda < \lambda_0$, it suffices to increase C_0 if necessary, so that $C_0 \exp\left(-C_1\lambda_0^2\right) \geq 1$.

This establishes (6.10) and finishes the proof of Corollary 6.7. □

Growth of $N^{(3)}(t,x)$ as $|x| \to \infty$

Let $N^{(3)}(t,x)$ be the jointly continuous version of the process defined in (6.4).

Lemma 6.8. *Fix $T > 0$. There exists an almost surely finite random variable Z such that with probability one, for all $x \in \mathbb{R}$,*

$$\sup_{t \le T} \left| N^{(3)}(t,x) \right| \le Z(|x|+1).$$

Proof. We split up $[0,T] \times \mathbb{R}$ into unit blocks $B_n = [0,T] \times [n, n+1]$ and let

$$N_n = \sup_{(t,x) \in B_n} |N^{(3)}(t,x)|.$$

Since $N^{(3)}(0,x) \equiv 0$, and B_n is the union of at most $[T]+1$ squares, to each of which Corollary 6.7 applies, for each $n \in \mathbb{Z}$, we have

$$P\{N_n > \lambda(n+1)\} \le C_0 \exp\left(-C_1 \lambda^2 (n+1)^2\right).$$

Here we have incorporated ϕ_1 into C_1. By the Borel-Cantelli lemma, there exists an almost surely finite random variable Z such that with probability one, for all $x \in \mathbb{R}$,

$$\sup_{t \le T} \left| N^{(3)}(t,x) \right| \le Z(|x|+1).$$

□

7 Establishing polarity of almost all points for $d \ge 6$

The goal of this section is to prove Theorem 2.2. We want to study the range of $\tilde{u}$, which takes values in $\mathbb{R}^d$ ($d \ge 6$), as (t,x) varies in the time-space rectangle $R_0 = [1,2] \times [0,1]$.

Let $w_{s,y,r}$ be as defined in (4.5) with t_0, x_0, ρ there replaced respectively by s, y, r. Let $\tilde{K}$ be the constant in Proposition 5.1. For $q \ge 1$, consider the random set

$$G_q = \left\{ (s,y) \in R_0 : \exists r \in [2^{-2q}, 2^{-q}[\text{ with } \operatorname{osc}_{R_r(s,y)}(w_{s,y,r}) \le 2\sigma_1 \tilde{K} f(r) \right\},$$

where the function f is defined in (5.1), and the event

$$\Omega_{q,1} = \{\lambda_2(G_q) \ge \lambda_2(R_0)\,(1 - \exp(-\sqrt{q}/4))\}$$

(here, λ_2 denotes Lebesgue measure on $\mathbb{R}^2$, so $\lambda_2(R_0) = 1$). Then

$$\begin{aligned}(\Omega_{q,1})^c &= \{\lambda_2(G_q) < \lambda_2(R_0)\,(1 - \exp(-\sqrt{q}/4))\} \\ &= \{\lambda_2(R_0 \setminus G_q) > \lambda_2(R_0) \exp(-\sqrt{q}/4)\}.\end{aligned}$$

By Markov's inequality,

$$P((\Omega_{q,1})^c) \le \frac{E[\lambda_2(R_0 \setminus G_q)]}{\lambda_2(R_0)\exp(-\sqrt{q}/4)}. \tag{7.1}$$

The numerator is equal to

$$E\left[\int_{R_0} 1_{R_0\setminus G_q}(s,y)\,dsdy\right] = \int_{R_0} P\{(s,y)\in R_0\setminus G_q\}\,dsdy.$$

By definition of G_q and Theorem 5.2, for all $(s,y)\in R_0$,

$$P\{(s,y)\notin G_q\} \le 2\exp\left[-\left(\log_2 \frac{1}{2^{-q}}\right)^{\frac{1}{2}}\right] = 2\exp(-\sqrt{q}),$$

therefore, by (7.1),

$$P((\Omega_{q,1})^c) \le 2\exp\left[-\frac{3}{4}\sqrt{q}\right].$$

In particular,

$$\sum_{q=1}^{\infty} P((\Omega_{q,1})^c) < +\infty. \tag{7.2}$$

On $\Omega_{q,1}$, for each $(s,y)\in G_q$, there exists $r\in[2^{-2q},2^{-q}]$ such that

$$\mathrm{osc}_{R_r(s,y)}(w_{s,y,r}) \le 2\sigma_1 \tilde{K} f(r). \tag{7.3}$$

Define an "anisotropic dyadic rectangle" of order ℓ as a rectangle in $\mathbb{R}_+\times\mathbb{R}$ of the form

$$[m_1 2^{-4\ell},\,(m_1+1)2^{-4\ell}]\times[m_2 2^{-2\ell},\,(m_2+1)2^{-2\ell}],$$

where $m_1,m_2\in\mathbb{N}$. For $(s,y)\in R_0$, let $Q_\ell(s,y)$ denote the anisotropic dyadic rectangle of order ℓ that contains (s,y). This rectangle is called "good" if

$$\mathrm{osc}_{Q_\ell(s,y)}(\tilde{u}) \le d_\ell, \tag{7.4}$$

where

$$d_\ell = 8\sigma_1\tilde{K} f(2^{-\ell}).$$

By (7.3), when $\Omega_{q,1}\cap\{\tau_{K,2}\wedge\tau_{K,3}=T_0\}$ occurs (so that, by (4.7), all the $w_{s,y,r}$ are equal to $\tilde{u}$), we can find a family $\mathcal{H}_{q,1}$ of non-overlapping good dyadic rectangles, each of some order $\ell\in[q,2q]$, that covers G_q. This family is determined by the random field $\tilde{u}$.

Let $\mathcal{H}_{q,2}$ be the family of non-overlapping dyadic rectangles of order $2q$ that meet R_0 but are not contained in any of rectangle of $\mathcal{H}_{q,1}$. For q large enough, these rectangles are contained in $[1,2]\times[0,1]$. Therefore, when $\Omega_{q,1}\cap\{\tau_{K,2}\wedge\tau_{K,3}=T_0\}$ occurs, their number is at most N_q, where

$$N_q\, 2^{-2q\cdot 6} \le \lambda_2(R)\exp(-\sqrt{q}/4),$$

so

$$N_q \le C 2^{12q}\exp(-\sqrt{q}/4), \tag{7.5}$$

where C does not depend on q.

Let $\Omega_{q,2}$ be the event "for all dyadic rectangles R of order $2q$ that meet R_0, the inequality

$$\mathrm{osc}_R(\tilde{u}) \le K_2 2^{-2q} q \tag{7.6}$$

holds." The next statement is a consequence of Corollary 6.7.

Lemma 7.1. *There are constants c_1, c_2 such that, for K_2 large enough, for all $q \geq 1$, we have*

$$P(\Omega_{q,2}^c) \leq c_1 \exp[-c_2 K_2^2 q^2].$$

Proof. Let R be a dyadic rectangle of order $2q$ that meets R_0. Consider the event

$$H(R) = \left\{ \operatorname{osc}_R(\tilde{u}) \leq K_2 2^{-2q} q \right\}.$$

Recall from (3.5) that $\tilde{u}(t,x) = I(t,x) + N(t,x)$, where $I(t,x)$ is a deterministic integral and $N(t,x)$ is a stochastic integral of the same form as the process $N^{(4)}(t,x)$ in (6.6), with the bound $\phi_1 = \sigma_1$ given by Assumption 2.1(b). We note that on $R_0 = [1,2] \times [0,1]$, $(t,x) \mapsto I(t,x)$ is C^∞, hence Lipschitz continuous with some Lipschitz constant $\tilde{L}$. Choose $K_2 \geq 2\tilde{L}$. Then for $q \geq 1$, $\operatorname{osc}_R(I) \leq \tilde{L}\, 2^{-4q+1} \leq \tilde{L} 2^{-2q}$, therefore,

$$H(R)^c = \{\operatorname{osc}_R(\tilde{u}) > K_2 2^{-2q} q\} \subseteq \left\{ \operatorname{osc}_R(N) \geq \tfrac{K_2}{2} 2^{-2q} q \right\}.$$

By Corollary 6.7 applied to $N(t,x)$ with $T - S_1 = 2^{-8q}$, we see that

$$\begin{aligned} P(H(R)^c) &\leq P\left\{ \operatorname{osc}_R(N) \geq \tfrac{K_2}{2} 2^{-2q} q \right\} \\ &\leq C_0 \exp\left[-C_1 \left(\tfrac{K_2}{2} 2^{-2q} q \right)^2 \sigma_1^{-2} (2^{-8q})^{-1/2} \right] \\ &= C_0 \exp\left[-\tilde{C}_1 K_2^2 q^2 \right]. \end{aligned}$$

It follows that

$$P(\Omega_{q,2}^c) \leq 2^{12q} C_0 \exp\left[-\tilde{C}_1 K_2^2 q^2 \right] \leq c_1 \exp[-c_2 K_2^2 q^2]$$

for K_2 large enough. This proves Lemma 7.1. □

We continue working towards the proof of Theorem 2.2: We choose K_2 large enough so that

$$\sum_{q=1}^{\infty} P((\Omega_{q,2})^c) < +\infty : \tag{7.7}$$

this is possible by Lemma 7.1.

Set $\mathcal{H}_q = \mathcal{H}_{q,1} \cap \mathcal{H}_{q,2}$. This is a non-overlapping cover of R_0 (because of how dyadic rectangles fit together). Set

$$\begin{aligned} r_A &= d_\ell & &\text{if } A \in \mathcal{H}_{q,1} \text{ and } A \text{ is of order } \ell \in [q, 2q], \\ r_A &= K_2\, 2^{-2q} q & &\text{if } A \in \mathcal{H}_{q,2}. \end{aligned}$$

Define

$$\Omega_q = \Omega_{q,1} \cap \Omega_{q,2}.$$

Lemma 7.2. *For $x > 0$, let*

$$\zeta(x) = x^6 \log_2 \log_2 \frac{1}{x}. \tag{7.8}$$

For q large enough, if $\Omega_{q,1} \cap \{\tau_{K,2} \wedge \tau_{K,3} = T_0\}$ occurs, then

$$\sum_{A \in \mathcal{H}_q} \zeta(r_A) \leq K \lambda_2(R_0).$$

Proof. For $A \in \mathcal{H}_{q,1}$, if A is of order $\ell \in [q, 2q]$, then

$$\zeta(r_A) \leq K \left(\frac{2^{-\ell}}{(\log_2 \ell)^{1/6}} \right)^6 \log_2 \log_2 2^\ell = K 2^{-6\ell},$$

and the right-hand side is the volume of an anisotropic rectangle of order ℓ.

For q large enough and for $A \in \mathcal{H}_{q,2}$,

$$\zeta(r_A) \leq K(2^{-2q} q)^6 \log_2(2q),$$

hence by (7.5), on $\Omega_{q,1} \cap \{\tau_{K,2} \wedge \tau_{K,3} = T_0\}$,

$$\begin{aligned}\sum_{A \in \mathcal{H}_{q,2}} \zeta(r_A) &\leq K(2^{-2q} q)^6 \log_2(2q) \cdot C 2^{12q} \exp(-\sqrt{q}/4) \\ &= CKq^6 \log_2(2q) \exp(-\sqrt{q}/4).\end{aligned}$$

Therefore, since rectangles in $\mathcal{H}_{q,1}$ are non-overlapping and contained in $R_0 = [1, 2] \times [0, 1]$,

$$\sum_{A \in \mathcal{H}_q} \zeta(r_A) \leq K\lambda_2(R_0) + CKq^6 \log_2(2q) \exp(-\sqrt{q}/4).$$

The first term on the right-hand side does not depend on q, while the second has limit 0 as $q \to \infty$, so Lemma 7.2 is proved. □

For each $A \in \mathcal{H}_q$, we pick a distinguished point $(s_A, y_A) \in A$ (say, the lower left corner). Let B_A be the Euclidean ball in $\mathbb{R}^d$ centered at $\tilde{u}(s_A, y_A)$ with radius r_A.

Lemma 7.3. *Let $\mathcal{F}_q$ be the family of balls $(B_A,\ A \in \mathcal{H}_q)$. For q large enough, on $\Omega_q \cap \{\tau_{K,2} \wedge \tau_{K,3} = T_0\}$, $\mathcal{F}_q$ covers the random set*

$$\tilde{M} = \{\tilde{u}(s, y) : (s, y) \in R_0\}.$$

Notice that $\tilde{M} \subset \mathbb{R}^d$ is the range of $\tilde{u}$ as (s, y) varies in $R_0 = [1, 2] \times [0, 1]$.

Proof. Fix $z \in \tilde{M}$. By definition, there is $(s, y) \in R_0$ such that $z = \tilde{u}(s, y)$. Since $\mathcal{H}_q$ is a cover of R_0, the point (s, y) belongs to some rectangle A of $\mathcal{H}_q$.

Consider first the case where $A \in \mathcal{H}_{q,1}$. Suppose that A is of order $\ell \in [q, 2q]$. By (7.4),

$$|\tilde{u}(s, y) - \tilde{u}(s_A, y_A)| \leq d_\ell, \qquad \text{that is,} \qquad |z - \tilde{u}(s_A, y_A)| \leq r_A.$$

This means that $z \in B_A$.

Now consider that case where $A \in \mathcal{H}_{q,2}$. Then on $\Omega_{q,2}$, by (7.6),

$$|z - \tilde{u}(s_A, y_A)| = |\tilde{u}(s, y) - \tilde{u}(s_A, y_A)| \leq K_2\, 2^{-2q} q = r_A,$$

so $z \in B_A$. The lemma is proved. □

Proposition 7.4. *Let λ_6 denote 6-dimensional Hausdorff measure. Then $\lambda_6(\tilde{M}) = 0$ a.s.*

Proof. For q large enough so that Ω_q occurs, on $\{\tau_{K,2} \wedge \tau_{K,3} = T_0\}$, by the definition of ζ in (7.8) and Lemma 7.2,

$$\sum_{A \in \mathcal{H}_q} r_A^6 \leq \frac{1}{\log_2 q} \sum_{A \in \mathcal{H}_q} \zeta(r_A) \leq \frac{K\lambda_2(R_0)}{\log_2 q} \to 0$$

as $q \to \infty$. Since the family of balls $(B_A,\, A \in \mathcal{H}_q)$ covers $\tilde{M}$ by Lemma 7.3, we conclude that $\lambda_6(\tilde{M}) = 0$ on $\Omega_q \cap \{\tau_{K,2} \wedge \tau_{K,3} = T_0\}$. Since $\lim_{K \uparrow \infty} P\{\tau_{K,2} \wedge \tau_{K,3} = T_0\} = 1$ and Ω_q occurs for large enough q (by (7.2) and (7.7)), the proposition is proved. □

Proof of Theorem 2.2. We first prove that $\lambda_6(M) = 0$, where

$$M = \{u(s, y) : (s, y) \in R_0\}.$$

On the event $\{\tau_{K,1} = T_0\}$, u and $\tilde{u}$ coincide on $[0, T_0] \times \mathbb{R}$, so $M = \tilde{M}$, where $\tilde{M}$ is defined in Lemma 7.3, and therefore, by Proposition 7.4,

$$\lambda_6(M) = 0 \qquad \text{a.s. on } \{\tau_{K,1} = T_0\}.$$

Since $\lim_{K \uparrow \infty} P\{\tau_{K,1} = T_0\} = 1$, we conclude that $\lambda_6(M) = 0$ a.s.

Let $u(]0, \infty[\, \times \mathbb{R})$ denote the random set $\{u(s, y) : (s, y) \in]0, \infty[\, \times \mathbb{R}\}$. Since in the entire paper, the rectangle R_0 could have been replaced by any other rectangle in $]0, \infty[\times\mathbb{R}$, we deduce that $\lambda_6(u(]0, \infty[\, \times \mathbb{R})) = 0$. Therefore, for $d \geq 6$, $\lambda_d(u(]0, \infty[\, \times \mathbb{R})) = 0$, where λ_d denotes Lebesgue-measure on $\mathbb{R}^d$. By Fubini's theorem,

$$0 = E\left[\int_{\mathbb{R}^d} 1_{u(]0,\infty[\, \times \mathbb{R})}(z)\, \lambda_d(dz)\right] = \int_{\mathbb{R}^d} P\{z \in u(]0, \infty[\, \times \mathbb{R})\}\, \lambda_d(dz),$$

that is, for Lebesgue-almost all $z \in \mathbb{R}^d$, $P\{z \in u(]0, \infty[\, \times \mathbb{R})\} = 0$. This proves Theorem 2.2. □

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