Strong Uniqueness for an SPDE via backward doubly stochastic differential equations

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Abstract

We prove strong uniqueness for a parabolic SPDE involving both the solution $v(t,x)$ and its derivative $\partial_x v(t,x)$. The familiar Yamada-Watanabe method for proving strong uniqueness might encounter some difficulties here. In fact, the Yamada-Watanabe method is essentially one dimensional, and in our case there are two unknown functions, v and $\partial_x v$. However, Pardoux and Peng's method of backward doubly stochastic differential equations, when used with the Yamada-Watanabe method, gives a short proof of strong uniqueness.

1 Introduction

In this paper we prove strong uniqueness for the following stochastic partial differential equation (SPDE)

$$\begin{cases} dv(t,x) = \dfrac{1}{2}\partial_{xx}v(t,x)dt + g(v(t,x), \partial_x v(t,x))dF(t,x)\,, \\ v(0,x) = v_0(x)\,. \end{cases} \tag{1}$$

Our motivation was to generalize the equation $dv(t,x) = \frac{1}{2}\partial_{xx}vdt + L\partial_x vdB$. We leave the question of existence open. Here $x \in \mathbb{R}$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete filtered probability space. We seek a unique non-anticipating solution $v(t,x) = v(\omega,t,x) : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that with probability 1, $v(t,x), \partial_x v(t,x)$ are jointly continuous. For ease of notation, we often simply write v instead of $v(t,x)$. Here we assume that $F(t,x) = F(\omega,t,x)$ is a mean-zero Gaussian random field with the covariance function

$$\mathbb{E}\,[F(t,x_1)F(s,x_2)] = \min(t,s)\sigma(|x_1 - x_2|) \tag{2}$$

such that F has nuclear covariance with respect to x. So we can write

$$F(t,x) = \sum_{n=1}^{\infty} e_n(x)B^{(n)}(t) \tag{3}$$

where $B^{(n)}(t)$ are independent Browninan motions. We understand (1) as an SDE in $\mathbf{L}^2$, which fits into the framework described in Da Prato and Zabczyk [1]. The readers can consult that reference for more details about this formulation.

Next we give some history and motivation for this problem. Itô had already proved strong uniqueness for stochastic differential equations (SDE) equipped with Lipschitz coefficients. In [2], Yamada and Watanabe dealt with such equations as $dX = a(X)dB$ under the assumption that $a(\cdot)$ is Hölder of order $1/2$. Their method was designed for problems in which X takes values in $\mathbb{R}^1$, or has rotational symmetry which allows us to reduce to one-dimensional case. Even with these limitations, their method is still one of the few tools available for proving strong uniqueness for stochastic equations with coefficients which are non-Lipschitz; the reader can consult Bass, Burdzy, and Chen [3] for a list of known techniques.

In the case of SPDEs, Viot [4] extended Yamada and Watanabe's method to deal with equations such as

$$dv = \partial_{xx} v dt + g(v) dB \tag{4}$$

where g is Hölder of order $1/2$, and B is a Brownian motion which does not depend on x. In [5], Mytnik, Perkins and Sturm used Viot's idea to analyze the same equation with B being Gaussian noise that is white in time and colored in space. In a later and more difficult paper, Mytnik and Perkins [6] solved the long-standing problem of proving strong uniqueness for

$$\partial_t v = \partial_{xx} v + g(v)\dot{W} \tag{5}$$

with one-dimensional spatial variable, $\dot{W} = \dot{W}(t, x)$ is the space-time white noise, and g is Hölder continuous of order greater than $3/4$.

For the case when $g(v) = v^{1/2}$, the problem is even more challenging and interesting. In this setting, the equation (5) describes the superprocess (or sometime called Dawson-Watanabe process), which has been an object of intense interests by many probabilists (see Dawson [7] and Perkins [8]). If we allow signed solutions, Mueller, Mytnik and Perkins [9] showed that nonuniqueness can hold if $g(v) = v^\alpha$ and $0 < \alpha < 3/4$. But strong uniqueness among nonnegative solutions remains open, and is considered as one of the main unsolved problems on superprocesses. Recently Dawson and Li [10] gave an alternative framework for studying the superprocesses in the strong sense. Xiong [11] followed their framework, and built up a connection between the SPDE and the backward doubly stochastic differential equations (BDSDE) based on Pardoux and Peng's method [12]. By analyzing the corresponding BDSDE instead, Xiong was able to verify strong uniqueness for nonnegative solutions of (5) when $g(v) = v^{1/2}$.

Our purpose is to illustrate the usefulness of BDSDE in proving strong uniqueness for SPDE by studying a case in which Yamada and Watanabe's method alone could face difficulties. As mentioned above, (1) involves both v and $\partial_x v$ as unknown functions, causing difficulties for the one-dimensional nature of the Yamada-Watanabe argument. Mytnik, Perkins, and Sturm faced similar difficulties in [5] and [6]. They overcame them by an intricate argument based on studying the region where $v_1 - v_2$ is small, where v_1, v_2 are two solutions. If $\partial_x v_1 - \partial_x v_2$ is large, then this region must be small. See [6] for details. We could conceivably apply this reasoning, but the argument using BDSDE seems much simpler. Lastly, proving uniqueness using BDSDE is not well known, so this paper may illustrate the method without involving a large number of technical difficulties.

2 BDSDE and uniqueness

First we state our assumptions on g.

Assumption 2.1 *We assume that there exist constants $K,\ L > 0$ such that*

$$|g(y_1, z_1) - g(y_2, z_2)| \leq K\rho(|y_1 - y_2|)(1 + |z_1| + |z_2|) + L|z_1 - z_2|. \tag{6}$$

where the function ρ satisfies

$$\int_0^1 \frac{1}{\rho^2(u)} du = \infty. \tag{7}$$

For existence, we would need to assume, in addition to the condition (2) on F, that $|g^2(\cdot,\cdot)\sigma(\cdot)| \leq \varepsilon$ for some constant $\varepsilon > 0$. It is well known that the SPDE

$$dv(t, x) = \frac{1}{2}\partial_{xx} v(t, x)dt + L\partial_x v(t, x)dB(t)$$

is not well posed unless the constant L is sufficiently small, where $B(t)$ is a one-dimensional Brownian motion. This observation is important for existence, but we do not use it to study uniqueness.

We also assume that F has covariance

$$\mathbb{E}\left[F(t, x)F(s, y)\right] = \delta(t - s)R(x - y)$$

and that $|R(z)| \leq K$ for some constant K.

To obtain strong uniqueness for (1), we first consider the corresponding BDSDE. As in Xiong [11], we define

$$u_t(x) = v_{T-t}(x), \text{ and } H(x, t) = F(x, T - t).$$

Then H is a Gaussian noise in $[0, T] \times \mathbb{R}$ which is white in time and colored in space. From (2) we have:

$$\mathbb{E}[H(x, dt)H(y, dt)] = \sigma(x - y)dt.$$

The backward version of (1) is then

$$du = \frac{1}{2}\Delta u + g(u, \nabla u)H(\cdot, \hat{d}t)/dt, \qquad u_T = v_0 \tag{8}$$

where $\hat{d}$ denotes the backward derivative (cf. Xiong [13]), that is, in the Riemann sum approximating the stochastic integral, we take the right end-points instead of the left ones.

With the backward version of the equation, we then follow the ideas in [12] and [11], and construct the following BDSDE of $(Y_s^{t,x}, Z_s^{t,x})$:

$$Y_s^{t,x} = v_0(X_T^{t,x}) + \int_s^T g(Y_r^{t,x}, Z_r^{t,x}) H(X_r^{t,x}, \hat{d}r) - \int_s^T Z_r^{t,x} dB_r \,, \tag{9}$$

where $X^{t,x}$ is given by

$$X_s^{t,x} = x + B_t - B_s, \qquad s \geq t \,,$$

with t, s fixed and B_t the standard Brownian motion. The correspondence between (8) and (9) is given by the following theorem.

Theorem 2.1 *If u is a solution to (8), then*

$$Y_s^{t,x} = u_s(X_s^{t,x}) \text{ and } Z_s^{t,x} = \partial u_s(X_s^{t,x})$$

is a pair of solutions to the BDSDE (9), where ∂ denotes the partial derivative with respect to the space variable. As a consequence, we have $u_t(x) = Y_t^{t,x}$.

Proof: We want to show:

$$u_s(X_s^{t,x}) = v_0(X_T^{t,x}) + \int_s^t g(Y_r^{t,x}, Z_r^{t,x}) H(\hat{d}r, X_r^{t,x}) - \int_s^t \partial u_r(X_r^{t,x}) dB_r, \tag{10}$$

We will construct an approximation to u that is smooth enough, we will work with this approximation to show that it satisfies a similar equation to (10) and finally we show that each term converges to the corresponding term. For any $\delta > 0$, let

$$u_t^\delta(x) = T_\delta u_t(x), \qquad \forall\, x \in \mathbb{R},$$

where T_δ is the Brownian semigroup. Namely,

$$T_\delta f(x) = \int_{\mathbb{R}} p_\delta(x-\xi) f(\xi) d\xi \text{ and } p_\delta(x) = \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{x^2}{2\delta}\right).$$

It is well-known that for any $t \geq 0$ and $\delta > 0$, $u_t^\delta(\cdot) \in C^\infty$. Applying T_δ to both sides of (8), we get

$$u_t^\delta(x) = T_\delta v_0(x) + \int_{\mathbb{R}} \int_t^T p_\delta(x-\xi) g(u_r(\xi), \partial u_r(\xi)) H(\hat{d}r, \xi) d\xi + \int_t^T \frac{1}{2} \partial_{\xi\xi} u_r^\delta(\xi) d\xi. \tag{11}$$

Let $s = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[s, T]$. Then using a telescoping sum and Itô's formula on the y variable for $u_{t_i}^\delta(y)$ (note that $u_{t_i}^\delta$ is independent of $X_r^{t,x}$ and B_r), and the SPDE (11) with $x = X_{t_{i+1}}^{t,x}$, we have:

$$
\begin{aligned}
& u_s^\delta(X_s^{t,x}) - T_\delta v_0(X_T^{t,x}) \\
= \;& u_s^\delta(X_s^{t,x}) - u_T^\delta(X_T^{t,x}) \\
= \;& \sum_{i=0}^{n-1} \left(u_{t_i}^\delta(X_{t_i}^{t,x}) - u_{t_i}^\delta(X_{t_{i+1}}^{t,x}) \right) + \sum_{i=0}^{n-1} \left(u_{t_i}^\delta(X_{t_{i+1}}^{t,x}) - u_{t_{i+1}}^\delta(X_{t_{i+1}}^{t,x}) \right) \\
= \;& -\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2}\partial^2 u_{t_i}^\delta(X_r^{t,x}) dr - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \partial u_{t_i}^\delta(X_r^{t,x}) dB_r \\
& + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2}\partial^2 u_r^\delta(X_{t_{i+1}}^{t,x}) dr \\
& + \sum_{i=0}^{n-1} \int_{\mathbb{R}} \int_{t_i}^{t_{i+1}} p_\delta(X_{t_{i+1}}^{t,x} - \xi) g(u_r(\xi), \partial u_r(\xi)) H(\hat{d}r, \xi) d\xi,
\end{aligned}
$$

Note that if the mesh size of the partition goes to 0 we have:

$$
\begin{aligned}
i) \quad & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2}\partial^2 u_{t_i}^\delta(X_r^{t,x}) dr \to \int_s^T \frac{1}{2}\partial^2 u_r^\delta(X_r^{t,x}) dr \\
ii) \quad & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \partial u_{t_i}^\delta(X_r^{t,x}) dB_r \to \int_s^T \partial u_r^\delta(X_r^{t,x}) dB_r \\
iii) \quad & \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{2}\partial^2 u_r^\delta(X_{t_{i+1}}^{t,x}) dr \to \int_s^T \frac{1}{2}\partial^2 u_r^\delta(X_r^{t,x}) dr \\
iv) \quad & \sum_{i=0}^{n-1} \int_{\mathbb{R}} \int_{t_i}^{t_{i+1}} p_\delta(X_{t_{i+1}}^{t,x} - \xi) g(u_r(\xi), \partial u_r(\xi)) H(\hat{d}r, \xi) d\xi \\
& \qquad \to \int_{\mathbb{R}} \int_s^T p_\delta(X_r^{t,x} - \xi) g(u_r(\xi), u_r(\xi)) H(\hat{d}r, \xi) d\xi,
\end{aligned}
$$

$i)$ and $ii)$ follow because $\partial^2 u_\cdot^\delta(x)$ and $\partial u_\cdot^\delta(x)$ are continuous, $iii)$ because $u_t^\delta(\cdot) \in C^\infty$ and $X_\cdot^{t,x}$ is continuous, and $iv)$ follow because the Browninan semigroup has a strong decay in the space variable and $X_\cdot^{t,x}$ is continuous. Therefore the terms $i)$ and $iii)$ cancel each other and we obtain:

$$
\begin{aligned}
& u_s^\delta(X_s^{t,x}) - T_\delta v_0(X_T^{t,x}) \qquad (12) \\
= \;& -\int_s^T \partial u_r^\delta(X_r^{t,x}) dB_r + \int_{\mathbb{R}} \int_s^T p_\delta(X_r^{t,x} - \xi) g(u_r(\xi), \partial u_r(\xi)) H(\hat{d}r, \xi) d\xi.
\end{aligned}
$$

Finally we show that each term converges when we take $\delta \to 0$ to the corresponding terms in (10). We show the calculations for the last term, noting that for $s > t$,

$$
\begin{aligned}
&\mathbb{E}\Big|\int_{\mathbb{R}}\int_s^T p_\delta(X_r^{t,x}-\xi)g(u_r(\xi),\partial u_r(\xi))H(\hat{d}r,\xi)d\xi \\
&\qquad -\int_s^T g(u_r(X_r^{t,x}),\partial u_r(X_r^{t,x}))H(\hat{d}r,X_r^{t,x})\Big|^2 \\
= \;& \mathbb{E}\int_s^T \int_{\mathbb{R}}\int_{\mathbb{R}} p_\delta(X_r^{t,x}-\xi)p_\delta(X_r^{t,x}-\zeta)g(u_r(\xi),\partial u_r(\xi)) \\
&\qquad \times g(u_r(\zeta),\partial u_r(\zeta))\sigma(|\xi-\zeta|)d\xi d\zeta dr \\
&-2\mathbb{E}\int_s^T \int_{\mathbb{R}} p_\delta(X_r^{t,x}-\xi)g(u_r(\xi),\partial u_r(\xi)) \\
&\qquad \times g(u_r(X_r^{t,x}),\partial u_r(X_r^{t,x}))\sigma(|X_r^{t,x}-\xi|)d\xi dr \\
&+\mathbb{E}\int_s^T (g(u_r(X_r^{t,x}),\partial u_r(X_r^{t,x}))^2\sigma(0)dr \\
\to \;& 0.
\end{aligned}
$$

Since $u_s^\delta \to u_s$, $T_\delta v_0 \to v_0$ and $\nabla u_r^\delta \to \nabla u_r$ as $\delta \to 0$, the convergence of the other terms of (12) can be proved as in [11]. (9) follows from (12) by taking $\delta \to 0$. ∎

Next, under Assumption (7) we prove the uniqueness for the solution to the BDSDE (9). Suppose that $\{a_k\}$ is a decreasing sequence such that we have

$$\int_{a_k}^{a_{k-1}} \frac{1}{\rho^2(u)}\,du = k \tag{13}$$

which is possible by Condition (7). We construct a function ϕ_k such that the following conditions hold

$$i)\quad \phi_k'(u) = \begin{cases} 0 & 0 \le u < a_k \\ \text{between 0 and 1} & a_k < u < a_{k-1} \\ 1 & a_{k-1} \le u \end{cases} \tag{14}$$

$$ii)\quad \phi_k''(u) = \begin{cases} 0 & 0 \le u < a_k \\ \text{between 0 and } \dfrac{2}{k}\rho^{-2}(u) & a_k < u < a_{k-1} \\ 0 & a_{k-1} \le u \end{cases} \tag{15}$$

Then we have that $\phi_k(\cdot)$ converges to $|\cdot|$, but for every k it is two times differentiable and therefore we can apply Itô's formula.

Theorem 2.2 *If Assumption (7) holds with $L < \sigma(0)^{-1}$, then (9) has at most one solution.*

Proof: Suppose that (9) has two solutions (Y_s^i, Z_s^i), $i = 1, 2$. Then,

$$Y_s^1 - Y_s^2 = \int_s^T \left(g(Y_r^1, Z_r^1) - g(Y_r^2, Z_r^2)\right) H(\hat{d}r, X_r) - \int_s^T \left(Z_r^1 - Z_r^2\right) dB_r.$$

By the extended Itô's formula, we have

$$\begin{aligned}\phi_k(Y_s^1-Y_s^2) &= \int_s^T \phi_k'(Y_r^1-Y_r^2)\left(g(Y_r^1,Z_r^1)-g(Y_r^2,Z_r^2)\right)H(\hat{d}r,X_r)\\ &\quad -\int_s^T \phi_k'(Y_r^1-Y_r^2)\left(Z_r^1-Z_r^2\right)dB_r\\ &\quad +\int_s^T \frac{1}{2}\phi_k''(Y_r^1-Y_r^2)\left(g(Y_r^1,Z_r^1)-g(Y_r^2,Z_r^2)\right)^2\sigma(0)dr\\ &\quad -\int_s^T \frac{1}{2}\phi_k''(Y_r^1-Y_r^2)\left(Z_r^1-Z_r^2\right)^2 dr.\end{aligned}$$

Taking expectation, it follows from (6) that

$$\begin{aligned}\mathbb{E}\phi_k(Y_s^1-Y_s^2) &= \mathbb{E}\int_s^T \frac{1}{2}\phi_k''(Y_r^1-Y_r^2)\left(g(Y_r^1,Z_r^1)-g(Y_r^2,Z_r^2)\right)^2\sigma(0)dr\\ &\quad -\mathbb{E}\int_s^T \frac{1}{2}\phi_k''(Y_r^1-Y_r^2)\left(Z_r^1-Z_r^2\right)^2 dr\\ &\le \mathbb{E}\int_s^T \frac{K\sigma(0)}{2}\phi_k''(Y_r^1-Y_r^2)\rho(|Y_r^1-Y_r^2|)^2\left(1+|Z_r^1|^2+|Z_r^2|^2\right)dr\\ &\quad -\mathbb{E}\int_s^T \frac{1}{2}\phi_k''(Y_r^1-Y_r^2)(1-L'\sigma(0))\left(Z_r^1-Z_r^2\right)^2 dr\\ &\le \frac{K\sigma(0)}{2}k^{-1}\int_s^T \mathbb{E}\left(|Z_r^1|^2+|Z_r^2|^2\right)dr,\end{aligned}$$

where $L'>L$ is a constant such that $L'\sigma(0)<1$. Taking $k\to\infty$, we then get

$$\mathbb{E}|Y_s^1-Y_s^2|=0.$$

This implies the strong uniqueness of the BDSDE (9), and therefore the SPDE (8). ∎

Remark 2.3 *The condition (7) is satisfied by the function*

$$g(y,z)=(|y|^\alpha\wedge 1)z$$

for $\alpha\in[1/2,1]$ *if* $\sigma(0)<1$.

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