Tracy-Widom fluctuations for perturbations of the log-gamma polymer in intermediate disorder

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Abstract

The free-energy fluctuations of the directed polymer in $1+1$ dimensions is conjecturally in the Tracy-Widom universality class at all finite temperatures and in the intermediate disorder regime. Seppäläinen's log-gamma polymer was proven to have GUE Tracy-Widom fluctuations in a restricted temperature range by Borodin et al. [11]. We remove this restriction, and extend this result into the intermediate disorder regime. This result also identifies the scale of fluctuations of the log-gamma polymer in the intermediate disorder regime, and thus verifies a conjecture of Alberts et al. [3]. Using a perturbation argument, we show that any polymer that matches a certain number of moments with the log-gamma polymer also has Tracy-Widom fluctuations in intermediate disorder.

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1 Introduction

In 1999-2000 Baik et al. [8] and Johansson [20] proved that the asymptotic fluctuations of the energy in certain point-to-point last passage problems were governed by the same Tracy-Widom law which arises in the large N limit of the top eigenvalue of an $N \times N$ matrix from the Gaussian Unitary Ensemble (GUE). It was then conjectured that this holds for very general distributions, and furthermore that it extends to the *positive temperature* case of directed polymers in a random environment in $1+1$ dimensions. Here, the free energy takes the form

$$F(\beta, N) = \log \sum_{x(\cdot)} \exp\left(\beta \sum_{i=1}^{N} \xi_{i,x(i)}\right) \tag{1}$$

where the paths $x(\cdot)$ have fixed starting and end point, and the $\xi_{i,j}$ are independent identically distributed (iid) random variables, collectively referred to as the disorder.

To date, the only progress that has been made on the positive temperature conjecture is: 1) It has been verified for the special exactly solvable log-gamma case [24] in a certain (low temperature) range of parameter values [11]; 2) It has been shown to hold under certain double scaling regimes a) for long thin rectangles [7], and b) in a special case of the intermediate disorder limit [3].

In $1+1$ dimensions, the directed random polymers are in the strong disorder regime for all values of inverse temperature $\beta > 0$. The intermediate disorder regime means to take $\beta \to 0$ with the length of the polymer to probe the transition: The more slowly β is taken to 0, the closer one is to the Tracy-Widom GUE asymptotics at fixed, non-zero beta. The special case where $\beta_N = \mathcal{O}(N^{-1/4})$ was studied in detail in [3]. It probes the regime governed by the Kardar-Parisi-Zhang (KPZ) equation, crossing over between the Gaussian (Edwards-Wilkinson) regime $\beta_N \ll \mathcal{O}(N^{-1/4})$, and the Tracy-Widom GUE regime $\beta_N \gg \mathcal{O}(N^{-1/4})$.

In this article we use a standard perturbation argument (Theorem 2.4) which shows the universality of the Tracy-Widom GUE distribution in the last regime $1 \gg \beta_N \gg \mathcal{O}(N^{-1/4})$. If one has two disorder distributions whose moments up to a certain order are sufficiently close, then there is a β_N in that regime such that the appropriately rescaled free energy fluctuations are the same asymptotically.

In principle, one would like to use this to prove some universality of the Tracy-Widom law in intermediate disorder for directed polymers free energies of the form (1). However, the only case in which the Tracy-Widom law is known, the log-gamma polymer, is not even really of the form (1). A log-gamma random variable is the log of Gamma distributed random variable; i.e., it has the exp-gamma distribution. The exp-gammadistributions form a two parameter family corresponding to the scale and shape parameter of the Gamma distribution. The scale is a trivial parameter in the directed polymer (1) since it corresponds to centering the weights. The shape (θ) affects the properties of the exp-gamma distribution nonlinearly. However, at least at high temperature ($\theta \to \infty$), the shape parameter approximately controls the variance just like the inverse temperature β does in the standard polymer (see (7)). Since the shape/temperature parameter of the log-gamma random variable does not appear multiplicatively, the statement (Corollary 2.1) is not as simple as it would be if there were a solvable model of the form (1). Nevertheless, the result shows that the log-gamma polymer can be significantly perturbed in the intermediate disorder regime without changing the GUE Tracy-Widom fluctuations (see Example 1).

Finally, the intermediate disorder regime of the log-gamma polymer ($\beta_N \to 0$) turns out to be outside of the range of the best available result [11], which requires $\beta \geq \beta^* > 0$. Most of the present article is devoted to removing this restriction, caused by the form of the contours employed in the exact formula for the log-gamma polymer in [11]. We start with a different exact formula from Borodin et al. [12] that has more convenient contours, and we thank I. Corwin for pointing us towards this paper. We also thank an anonymous reviewer and I. Corwin for comments about a small error in the Theorem from [12]. Since we rely on this theorem, we sketch a way to fix their oversight in Remark 5.

In this way, we obtain the Tracy-Widom GUE law for the point-to-point log-gamma polymer for all temperature parameter values and appropriate "nearby" distributions.

2 Perturbation Theorem

We now precisely describe the discrete random polymer model. The disorder is a random field given by variables $\xi_{i,j}(\beta)$, $i, j \in \{1, 2, \dots\}$ which are independent for each $\beta > 0$. The polymer is represented as an up-right directed lattice path $\mathbf{x}$ from $(1, 1)$ to (N, N). The energy of such a path is given by

$$H_\beta(\mathbf{x}) = -\sum_{(i,j)\in\mathbf{x}} \xi_{i,j}(\beta).$$

The partition function is given by

$$Z_\beta^N = \sum_{\mathbf{x}} e^{-H_\beta(\mathbf{x})}. \tag{2}$$

Typically one would have $\xi_{i,j}(\beta) = \beta\xi_{i,j}$, but since we want to also consider the the log-gamma polymer, we allow for a more complicated dependence on β. The limiting free energy is given by

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N} \log Z_\beta^N.$$

The limit was proved to exist when the $\xi_{i,j}$ are iid Gaussians by Carmona and Hu [13, Proposition 1.4]. Comets et al. [16, Proposition 2.5] extended their result to general iid weights with exponential moments.

The scaled and centered free energy fluctuations are given by

$$h_N := \frac{\log Z_\beta^N - NF(\beta)}{\sigma(\beta)N^{1/3}}. \tag{3}$$

In general, one expects the right scaling to be

$$\sigma(\beta) \approx C\beta^{4/3} \qquad \text{as} \qquad \beta \searrow 0, \tag{4}$$

with a constant C depending only on the distribution of the weights ξ (see (11)). This scaling was conjectured in [2], and we prove it in this paper for the log-gamma and nearby polymers in this paper.

We will be primarily interested in the intermediate disorder regime, in which β goes to zero as $N \to \infty$, but $\lim_{N\to\infty} \sigma(\beta)N^{1/3} > 0$. In particular, if

$$\lim_{N\to\infty} \sigma(\beta)N^{1/3} = \infty, \tag{5}$$

we expect the fluctuations to be Tracy-Widom GUE. If $\lim_{N\to\infty} \sigma(\beta)N^{1/3} \in (0,\infty)$, we get a KPZ crossover distribution. If $\lim_{N\to\infty} \sigma(\beta)N^{1/3} = 0$ the fluctuations are Gaussian, as can be seen by doing a chaos expansion in β and checking that only the leading term, linear in the noise, survives. The case $\beta = CN^{-1/4}$ was studied in [3].

In the case (5) the limiting fluctuations are supposed to have the GUE Tracy-Widom law in wide generality, but the only case where there are any results is the special log-gamma polymer. Here $e^{-\xi(\beta)}$ have the Gamma distribution, or $e^{\xi(\beta)}$ have the inverse Gamma distribution which is supported on $x > 0$, with density

$$P(e^{\xi(\beta)} \in dx) = \frac{1}{\Gamma(\theta)} x^{-\theta-1} e^{-1/x} dx \tag{6}$$

where

$$\theta = \beta^{-2}. \tag{7}$$

The cumulant generating function of the exp-gamma distribution is given by

$$\log \mathbb{E}\left[\exp(t\log(X))\right] = \log\left(\frac{\Gamma(t+\theta)}{\Gamma(\theta)}\right). \tag{8}$$

Differentiating this k times, we see that the k^{th} cumulant of the log-gamma distribution is given by $\Psi^{(k-1)}(\theta)$, where

$$\Psi(\theta) = \frac{\Gamma'(\theta)}{\Gamma(\theta)} \tag{9}$$

is the digamma function. In particular, the variance is $\mathrm{Var}(\xi) = \Psi'(\theta)$. Using a series expansion for the digamma function (47) as $\theta \to \infty$, we can estimate the variance of ξ as

$$\mathrm{Var}(\xi) \approx \beta^2, \qquad \beta \searrow 0 \tag{10}$$

mimicking the way in which the inverse temperature β would enter the standard polymer $\xi(\beta) = \beta\xi$. In other words, choosing $\beta = \theta^{-1/2}$ *in* (7) *ensures that at high-temperature, β plays the role of inverse temperature in the log-gamma model.*

For the log-gamma polymer,

$$F(\beta) = -2\Psi(\theta/2), \qquad \sigma(\beta) = (-\Psi''(\theta/2))^{1/3}. \tag{11}$$

The limiting free-energy was identified in [24] and the variance in [11].

Our first theorem concerns the fluctuations of the log-gamma model

Theorem 2.1. *Let $-\xi_{i,j}(\beta)$ have the log-gamma distribution* (6) *and $\beta_N \to \beta \in [0,\infty)$ with $\sigma(\beta)N^{1/3} \to \infty$. Then*

$$\lim_{N\nearrow\infty} P(h_N < r) = F_{\mathrm{GUE}}(r)$$

where h_N is the scaled-centered log partition function in (3)*, and F_{GUE} is the GUE Tracy-Widom distribution.*

This was proved for $\beta \geq \beta^* > 0$ in Borodin et al. [11], where β^* is some unidentified but finite number. Our result removes this restriction, and further extends it into the intermediate disorder regime.

Our next result extends the $\beta_N \searrow 0$ part of this result to "nearby" distributions.

Definition 2.2 (Moment matching condition)**.** Two parametrized families of weights $\xi = \xi(\beta)$ and $\tilde{\xi} = \tilde{\xi}(\beta)$ are said to *match moments up to order* k if for some $C < \infty$ and for all sufficiently small β,

$$|\mathbb{E}\,[\xi^n] - \mathbb{E}\,[\tilde{\xi}^n]| \leq C\beta^k \qquad n = 1, \ldots, k-1,$$

and

$$|\mathbb{E}\,[\xi^k]|, |\mathbb{E}\,[\tilde{\xi}^k]| \leq C\beta^k. \tag{12}$$

Denote by $C^k(\mathbb{R})$ the space of functions on $\mathbb{R}$ whose derivatives up to order k are all uniformly bounded on all of $\mathbb{R}$.

Lemma 2.3. *Suppose two families of weights match moments up to order k (as in Definition 2.2) and let $\varphi \in C^k(\mathbb{R})$. Then there is a $C < \infty$ such that,*

$$|\mathbb{E}\,[\varphi(h_N)] - \mathbb{E}\,[\varphi(\tilde{h}_N)]| \leq C\frac{N^{2-\frac{1}{3}}\beta^k}{\sigma(\beta)}. \tag{13}$$

Lemma 2.3 is proved in Section 3, and the following theorem is a consequence of Lemma 2.3 and the fact that weak convergence is equivalent to convergence of expectations of all $C^k(\mathbb{R})$ functions (see, for example, [9, Theorem 2.1]).

Theorem 2.4 (Perturbation theorem)**.** *Suppose $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order k (as in Definition 2.2) and $\beta_N \searrow 0$ with*

$$\lim_{N \nearrow \infty} \frac{N^{2-\frac{1}{3}}\beta_N^k}{\sigma(\beta_N)} = 0. \tag{14}$$

Then

$$\lim_{N\to\infty} \mathbb{P}(h_N \leq r) = \lim_{N\to\infty} \mathbb{P}(\tilde{h}_N \leq r).$$

We do not expect this perturbation technique to extend to positive temperature. The reason it works here is because the k^{th} term of the Taylor expansion of the log-partition function is of order β_N^k (see (18)), and $\beta_N \to 0$ in intermediate disorder.

Corollary 2.5. *Suppose $\xi(\beta)$ and $\tilde{\xi}(\beta)$ match moments up to order k with $-\tilde{\xi}(\beta)$ having exp-gamma distribution as above and (14) holds. Then*

$$\lim_{N\to\infty} \mathbb{P}(h_N \leq r) = F_{\text{GUE}}(r).$$

For example, if $\beta_N = N^{-\alpha}$, $\alpha \in (0, 1/4]$, we need to match

$$k > \frac{5}{3\alpha} + \frac{4}{3}$$

moments with an exp-gamma to prove the GUE Tracy-Widom law. For the exp-gamma distribution, Corollary 2.5 shows that $c = 1/2$.

Remark 1. In the standard polymer, it's known when $\alpha = 1/4$ that 6 moments suffice to get the crossover law [17], so the result is slightly suboptimal. Note that it gets worse as α decreases, whereas the truth is supposed to be that 5 moments suffice when $\alpha = 0$ [10].

Remark 2. The perturbation theorem can clearly be stated in higher dimensions; we only need to modify (2.2). However, little is known about the exact scale of the fluctuations in higher dimensions, or the intermediate disorder scaling regime. Since the critical temperature is positive at higher temperature, our perturbation argument would have to be modified as well. Therefore we do not explore this here.

Example 1. Let $\{X_{i,j}\}$ be a family of independent random variable with k bounded moments. Let $\{\xi_{i,j}\}$ be another independent iid family of centered exp-gamma random variables with parameter $\theta = \beta^{-2}$. Then the polymer with weights

$$\tilde{\xi}_{i,j} = \xi_{i,j}(1 + X_{i,j}\beta^k) \tag{15}$$

satisfies the moment matching condition, and hence its log-partition function has GUE Tracy-Widom fluctuations in the limit $N \to \infty$.

3 Proof of the perturbation theorem

The Lindeberg proof of the central limit theorem is a now standard argument for proving universality [21, 22]. Let f be a function on $\mathbb{R}^n$ and consider two sets of iid random variables $\xi = (\xi_1, \ldots, \xi_n)$ and $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ that match some number of moments. For any bounded smooth function φ, the Lindeberg strategy allows one to show $|\mathbb{E}[\varphi \circ f(\xi)] - \mathbb{E}[\varphi \circ f(\tilde{\xi})]| = o(n)$ for all smooth and bounded functions φ. This is done by replacing the ξ variables one by one with $\tilde{\xi}$ variables, and using Taylor expansion to control the error. This estimate controls the weak-$*$ distance between $f(\tilde{\xi})$ and $f(\xi)$ and thus shows that they converge to the same distributional limit if it exists for either one of them. The technique has been applied to show, for example, that the limiting free energy of the Sherrington-Kirkpatrick (SK) spin glass, and the semi-circle distribution in Wigner random matrices are *universal*; i.e., that they're independent of the distributions of the variables involved [14].

There is another related technique in spin glass theory called Guerra's interpolation method that also relies on Taylor expansion. It uses the Ornstein-Uhlenbeck process to interpolate between a vector of iid random variables $\tilde{\xi}$ and an independent iid *Gaussian* vector ξ. In the SK model, the partition function is of the form $Z = \sum_{\sigma \in \{-1,1\}^N} e^{\beta_N H(\sigma)}$ where $H(\sigma) = \sum_{ij} \xi_{ij}\sigma_i\sigma_j$ and $\beta_N = N^{-1/2}$. Again, $N^{-1}\log Z$ has a deterministic limit called the free-energy as $N \to \infty$. For iid Gaussian weights, the limit was shown to be given by the celebrated Parisi formula by Talagrand [26]. The limit was shown to be the same for symmetric random variables with four moments by Guerra and Toninelli [19]. They used the aforementioned interpolation technique and the so-called approximate integration by parts for weights that match the moments of a Gaussian up to some order . Their ideas were extended by Carmona and Hu [13] to include distributions that match two moments with the Gaussian and have finite third moment. Chatterjee's approach using truncation and the Lindeberg technique removed the finite third moment requirement [14]. Other results like [5] and [18] extend the interpolation technique using higher order Taylor expansions.

In particular, Auffinger and Chen [5] showed that the *Gibbs measure* – the random measure on configurations given by $P_\xi(\sigma) = Z^{-1}\exp(\beta_N H(\sigma))$ – also converged to a universal limit as long as the weights matched a certain number of moments with the Gaussian. Since their results are for general spin-systems, and not just for the SK model, they also apply to polymer models in intermediate disorder. In a personal communication, [4] applied the theorems in [5] to show that the limiting Gibbs measure associated with the polymer path is universal: Let $\beta_N = \beta N^{-\alpha}$, and let $\gamma = (\gamma_i)_{i=1}^{Nd}$ be a directed path from the origin to $N(1,\ldots,1) \in \mathbb{Z}^d$, where γ_i are the vertices along path. Suppose the weights $\tilde{\xi}$ in the polymer match the first k moments of the standard Gaussian such that

$$\mathbb{E}\,[\tilde{\xi}_{ij}^{k+1}] < \infty \quad k > \frac{d+1}{\alpha}.$$

For $n \in \mathbb{N}$, let γ^a, $a = 1,\ldots,n$ be n directed paths from the origin to $N(1,\ldots,1)$.

Theorem 3.1 (Auffinger-Chen, personal communication)**.** *Let L be a function depending on n paths $(\gamma^a)_{1\le a\le n}$ where n is fixed, and suppose $\|L\|_\infty \le 1$. Then,*

$$|\mathbb{E}_{\tilde{\xi}}\,[\langle L(\gamma)\rangle] - \mathbb{E}_{\xi}\,[\langle L(\gamma)\rangle]\,| \to 0 \quad \textit{as } N \to \infty.$$

Here, $\langle\cdot\rangle$ represents the average over the Gibbs measure on paths, and $\mathbb{E}_{\cdot}$ represents expectation over the corresponding set of weights.

This allowed them to show that various quantities of interest were universal, including the transversal fluctuation exponent of the path measure that is defined as follows. Let $\gamma_{N/2}$ be the midpoint of a path γ sampled from the Gibbs measure. The polymer has transversal fluctuation exponent α if for any $\alpha' < \alpha < \alpha''$ and $C > 0$,

$$\mathbb{P}_{\tilde{\xi}}\left(\gamma_{N/2} \le CN^{\alpha'}\right) \to 0, \text{ and } \mathbb{P}_{\tilde{\xi}}\left(\gamma_{N/2} \le CN^{\alpha''}\right) \to 1 \quad \text{as} \quad N \to \infty.$$

The transversal exponent has a relationship with χ, the fluctuation exponent of $\mathrm{Var}\left(\log Z^N(\beta)\right) \sim N^{2\chi}$. Under certain strong hypotheses on the *existence* of these exponents —they're not known to exist for any standard polymer— it's known that $\chi = 2\alpha - 1$ [6, 15]. Therefore, [4] also indicates that polymers ought to have the same fluctuation exponents as the Gaussian polymer under a moment matching condition.

However, there are a few problems with Guerra's interpolation and approximate integration by parts method. *In its current form, it only allows you to match moments with the Gaussian.* Hence one can only look at polymers where the weights are of the form $\xi_{i,j}(\beta) = \beta\xi_{i,j}$, and only match moments with the Gaussian distribution. This by itself is not a very serious shortcoming and may be overcome with some work. But for the Gaussian polymer, it's not known whether the fluctuation exponents discussed above exist; it's not known whether the fluctuations of $\log Z^N(\beta)$ are in the GUE Tracy-Widom universality class; and in fact, very little is known about the Gaussian polymer other than the fact that the limiting free energy exists.

On the other hand, a lot more is known about the log-gamma polymer. The scale of the variance [24], and the limiting fluctuations are known in a large parameter range [11]. Moreover, the Lindeberg replacement technique is well-suited to comparing other polymers with the log-gamma polymer. Since we need more terms in the Taylor expansion (than [14], e.g.), and since Guerra's Gaussian integration by parts technique doesn't apply directly to prove Lemma 2.3, we reproduce the fairly standard Lindeberg argument for completeness.

For a fixed vertex $x = (i, j)$, define

$$Z(y) = Z_{x^c} + Z_x e^y, \tag{16}$$

where Z_{x^c} represents the sum in (2) over paths that do not pass through x, and Z_x is the sum over paths that do pass through x, but do not include weight at x. Let $h(y) = N^{-1/3}\sigma^{-1}(\log Z(y) - NF)$. $Z(y)$ and $h(y)$ do indeed depend on all the other weights for $z \neq x$, but the dependence is suppressed in the notation because we want to isolate the effect of replacing ξ_x by $\tilde{\xi}_x$. For any function $\varphi \in C^k$ we will show that

$$|\mathbb{E}\,[\varphi(h(\xi_x))] - \mathbb{E}\,[\varphi(h(\tilde{\xi}_x))]| \leq C(\sigma N^{1/3})^{-1}\beta^k. \tag{17}$$

where the expectation is over the entire disorder. We obtain (13) by replacing ξ_x by $\tilde{\xi}_x$ N^2 times for each $x \in \{1, \dots, N\}^2$.

Fix all the other weights in the disorder, and write Taylor's theorem for $\varphi(h(\xi_x))$:

$$\varphi(h(\xi_x)) = \sum_{j=0}^{k-1} \frac{\partial_y^j \varphi(h(0))}{j!}\xi_x^j + \frac{\partial_y^k \varphi(h(\zeta))}{k!}\xi_x^k,$$

with ζ between 0 and ξ_x. Taking expectation and using the independence of $\{\xi_z\}_{z\in\mathbb{R}^2}$, we get

$$\mathbb{E}\,[\varphi(h(\xi_x))] = \sum_{j=0}^{k-1} \frac{a_j}{j!}\mathbb{E}\,[\xi_x^j] + \frac{a_k}{k!}\mathbb{E}\,[\xi_x^k], \tag{18}$$

where $a_j = \mathbb{E}\,[\partial_y^j \varphi(h)(0)]$, $j = 1, \dots, k-1$ and $a_k = \mathbb{E}\,[\partial_y^k \varphi(h)(\zeta)]$. One has an analogous expression for $\mathbb{E}\,[\varphi(h(\tilde{\xi}_x))]$, but note that in fact $a_j = \tilde{a}_j$ for $j = 1, \dots, k-1$ since all the other weights are the same in the two expressions at each stage of the switching. Hence, from the moment matching condition (12),

$$|\mathbb{E}\,[\varphi(h(\xi_x))] - \mathbb{E}\,[\varphi(h(\tilde{\xi}_x))]| \leq \left(\sum_{j<k} |a_j| + [|a_k| + |\tilde{a}_k|]\right) C\beta^k, \tag{19}$$

To control the error term, we will show that for any $k \geq 1$, and all $y \in \mathbb{R}$,

$$|\partial_y^k \varphi(h(y))| \leq C_{k,\varphi}(\sigma N^{1/3})^{-1}, \tag{20}$$

where $C_{k,\varphi}$ is a constant dependent only on φ, k and the constant from the moment matching condition. This estimate (20), the Taylor expansion (18) and the moment matching condition in Definition 2.2 together imply (17).

To prove (20), we expand the derivative of a composition (à la Faa di Bruno)

$$\partial^k \varphi(h) = \sum_{\sum s m_s = k} C_{m_1 \dots m_k} \partial^{\sum m_s}\varphi \prod_{r=1}^{k} (\partial^r h)^{m_r},$$

where the $C_{m_1 \dots m_k}$ are multinomial coefficients, and $m_s \geq 0$ for $s = 1, \dots, k$. Since φ is smooth with bounded derivatives up to order k, we only need to control $\partial^r h(0)$ for $r \geq 1$. Computing derivatives in (16),

$$\partial_y \log Z(y) = \frac{Z_x e^y}{Z_{x^c} + Z_x e^y} =: p(y),$$
$$\partial_y^i \log Z(y) = \mathcal{P}_i(p(y)), \quad i > 1$$

where $\mathcal{P}_i$ is the polynomial given by the recurrence

$$\mathcal{P}_{i+1}(p) = \mathcal{P}_i'(p)p(1-p), \quad \mathcal{P}_1(p) = p, \quad i \geq 1.$$

The recurrence follows from the chain rule and $p'(y) = p(y)(1 - p(y))$. Since $0 \leq p(y) \leq 1$ for all $y \in \mathbb{R}$, we can bound each of the polynomials $\mathcal{P}_i$ by constants for $i = 1, \dots, k$. Putting the last few observations together, we get (20) for $k \geq 1$.

In Corollary 2.5, we verify that there is a large class of distributions that can satisfy the moment matching condition in Definition 2.2 with the exp-gamma distribution.

Proof of Corollary 2.5. Since $\theta = \beta^{-2}$, weights distributed as in (15) satisfy the moment matching condition, as long as

$$\mathbb{E}\left[(\log X - \mathbb{E}[\log X])^k\right] \leq \frac{1}{\theta^{\lceil k/2 \rceil}}, \tag{21}$$

Recall that the k^{th} cumulant of the exp-gamma distribution is given by the $k - 1^{\text{th}}$ derivative of the digamma function (9). It's clear from (47) that

$$\kappa_k := \Psi^{(k-1)}(\theta) = O\left(\frac{1}{\theta^{k-1}}\right) \quad k > 1.$$

For any random variable X, the moments μ_n are related to the cumulants κ_n via the following combinatorial expansion (see Speed [25] for this formula and its famous Möbius inversion). If π is a (set) partition of $\{1, \dots, k\}$, then we represent π as the union of disjoint sets $\pi = \{B_i\}_{i=1}^{n(\pi)}$ where $B_i \subset \{1, \dots, k\}$, and $\cup B_i = \{1, \dots, k\}$. Then,

$$\mu_k = \sum_{\pi \in \mathcal{L}} \prod_{B \in \pi} \kappa_{|B|}$$

where $|B|$ represents the cardinality of the set B, and $\mathcal{L}$ is the set of all partitions of $\{1, \dots, k\}$. Since we're considering centered random variables, we simply set the first cumulant to zero ($\kappa_1 = 0$)

and therefore we get

$$\begin{aligned}
\mu_k &= \sum_{\pi\in\mathcal{L}} \prod_{i=1}^{n(\pi)} \kappa_{|B_i|} 1_{|B_i|\neq 1} \\
&\leq \sum_{\pi\in\mathcal{L}} \prod_{i=1}^{n(\pi)} \frac{C_{|B_i|}}{\theta^{|B_i|-1}} 1_{|B_i|\neq 1}, \\
&= \sum_{\pi\in\mathcal{L}} \frac{C_\pi}{\theta^{\sum_{i=1}^{n(\pi)}|B_i|-n(\pi)}} 1_{|B_i|\neq 1}, \\
&\leq \frac{C_k}{\theta^{k-\max_{\pi,|B_i|\neq 1} n(\pi)}} \\
&= \frac{C_k}{\theta^{\lceil k/2\rceil}},
\end{aligned}$$

where $C_\pi = \prod_{i=1}^{n(\pi)} C_{|B_i|}$ and $C_k = 2^k \max_\pi C_\pi$. In the final inequality, we used the following observation: when π contains no sets with only one element, it follows that $n(\pi) \leq \lfloor k/2 \rfloor$. This proves (21). □

4 Tracy-Widom fluctuations for the log-gamma polymer

4.1 Fredholm determinant formula

Theorem 4.1. *For $N \geq 9$, let Z_β^N be the partition function of the log-gamma polymer with $\theta = \beta^{-2}$. Then for $\operatorname{Re} u > 0$,*

$$\mathbb{E}\,[e^{-uZ_\beta^N}] = \det(I + K_u^N)_{L^2(\mathcal{C}_\varphi)} \tag{22}$$

where

$$K_u^N(v,v') = \frac{1}{2\pi \mathrm{i}} \int_{l_{z_{\text{crit}}+\delta}} -\frac{\pi}{\sin(\pi(w-v))} \left(\frac{\Gamma(v)}{\Gamma(w)} \frac{\Gamma(\theta-w)}{\Gamma(\theta-v)}\right)^N \frac{u^{w-v}}{w-v'}\, dw + \sum_{i=1}^{q(v)} \operatorname{Res}_{u,i}(v,v') \tag{23}$$

where,

$$\operatorname{Res}_{u,j}(v,v') = (-1)^j \left(\frac{\Gamma(v)}{\Gamma(v+j)} \frac{\Gamma(\theta-v-j)}{\Gamma(\theta-v)}\right)^N \frac{u^j}{v+j-v'}, \quad 1 \leq j \leq q(v) \tag{24}$$

where

$$q(v) = \lfloor z_{\text{crit}} + \delta - \operatorname{Re}(v) \rfloor, \quad z_{\text{crit}} = \theta/2. \tag{25}$$

and $0 < \delta \leq \frac{z_{\text{crit}}}{2}$. The contours are defined as follows: For any $\varphi \in (0, \pi/4]$, the $\mathcal{C}_\varphi$ contour is given by $\{z_{\text{crit}} + e^{\mathrm{i}(\pi+\varphi)} y\}_{y\in\mathbb{R}_+} \cup \{z_{\text{crit}} + e^{\mathrm{i}(\pi-\varphi)} y\}_{y\in\mathbb{R}_+}$. The ℓ_x contour is a vertical straight-line with real part x (see Figure 1). They're both oriented to have increasing imaginary parts.

Remark 3. Theorem 4.1 is proved by setting $\tau = 0$ in [12, Theorem 2.1], as suggested in Remark 2.9 of the same paper. This requires a new estimate, and this is done in Section A.

We'll see in the next section that critical point of the integrand of K_u^N in (23) is at z_{crit}. The contours $\ell_{z_{\text{crit}}+\delta}$ (as $\delta \to 0$) and $\mathcal{C}_\varphi$ are located at the critical point.

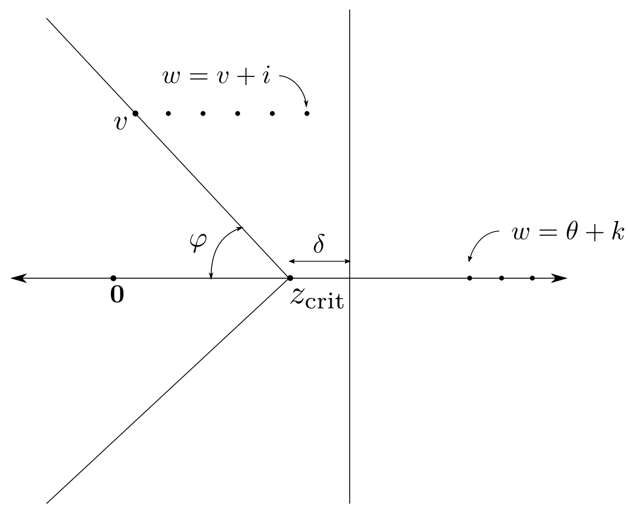

Figure 1: Contours in Theorem 4.1. The critical point is at $z_{\text{crit}} = \frac{\theta}{2}$. The triangular contour is C_φ, and the vertical contour $l_{z_{\text{crit}}+\delta}$ has real part $z_{\text{crit}} + \delta$. There are two sets of poles that one needs to watch out for. The poles of the sine are shown as dots to the right of v and are of the form $w = v+i$. The poles of $\Gamma(\theta - w)$ are of the form $w = \theta + k$, $k = 0, 1 \ldots$.

4.2 Estimates along the contours

We are interested in the asymptotic probability distribution of (3). The trick is to rewrite the left hand side of (22) as

$$\mathbb{E}\left[\exp\left(-e^{\sigma N^{1/3}(h_N - t)}\right)\right] \tag{26}$$

by taking

$$u = e^{-NF - t\sigma N^{1/3}}. \tag{27}$$

As $N \nearrow \infty$, by (5) $\sigma N^{1/3} \nearrow \infty$, and (26) becomes $\lim_{N\nearrow\infty} \mathbb{P}(h_N < t)$. Now we consider the same limit of the right hand side of (22). We start with a formal critical point analysis of the integral in (23), which can be rewritten as

$$-\frac{1}{2\pi \mathrm{i}} \int_{\ell_{z_{\text{crit}}}} \frac{\pi}{\sin(\pi(w - v))} e^{N[G(v)-G(w)]+t\sigma N^{1/3}(v-w)} \frac{dw}{w - v'} \tag{28}$$

where we've ignored the residues, dropped the subscript u in the kernel, and let

$$G(z) = \log \Gamma(z) - \log \Gamma(\theta - z) + F(\beta) z. \tag{29}$$

We have $G'(z) = \Psi(z) + \Psi(\theta - z) + F(\beta)$. From (11), it follows that the critical point, i.e. $G'(z_{\text{crit}}) = 0$ is at $z_{\text{crit}} = \theta/2$, and G'' vanishes there as well. Therefore the exponent is cubic near the critical point and it is natural to define

$$\tilde{v} = \sigma N^{1/3}(v - z_{\text{crit}}), \qquad \tilde{w} = \sigma N^{1/3}(w - z_{\text{crit}}), \tag{30}$$

and let $\tilde{K}^N(\tilde{v}, \tilde{v}') = K^N(v, v')$ in (23). The change of variable introduces a Jacobian factor of $(\sigma N^{1/3})^{-1}$ into the Fredholm expansion (**??**). Then, using (11), it is easy to prove the following lemma.

Lemma 4.2.

$$\lim_{N\to\infty} \frac{1}{(\sigma N^{1/3})} \tilde{K}^N(\tilde{v}, \tilde{v}') = K_{\mathrm{Ai}}(\tilde{v}, \tilde{v}'), \tag{31}$$

where the Airy kernel is defined as

$$K_{\mathrm{Ai}}(\tilde{v}, \tilde{v}') := \frac{1}{2\pi\mathrm{i}} \int \frac{\exp\left\{-\frac{1}{3}\tilde{v}^3 + t\tilde{v}\right\}}{\exp\left\{\frac{1}{3}\tilde{w}^3 + t\tilde{w}\right\}} \frac{d\tilde{w}}{(\tilde{v} - \tilde{w})(\tilde{w} - \tilde{v}')}. \tag{32}$$

The Airy kernel acts on the contour $e^{-2\pi\mathrm{i}/3}\mathbb{R}_+ \cup e^{2\pi\mathrm{i}/3}\mathbb{R}_+$ *and the integral in* $\tilde{w}$ *is on the contour* $\{e^{-\pi\mathrm{i}/3}\mathbb{R}_+ + \delta\} \cup \{e^{\pi\mathrm{i}/3}\mathbb{R}_+ + \delta\}$ *for any horizontal shift* $\delta > 0$*. They're both oriented to have increasing imaginary part.*

The Fredholm determinant of the right hand side of (32) is $F_{\mathrm{GUE}}(t)$ [11]. We prove Lemma 4.2 and flesh out the details of this sketch in the next few sections.

We prove that $\det(\mathbf{1} + K^N) \to \det(\mathbf{1} + K_{\mathrm{Ai}})$ rigorously in this section. Recall the kernel (23)

$$K^N(v, v') = \frac{1}{2\pi\mathrm{i}} \int_{\ell_{z_{\mathrm{crit}} + \delta(\sigma N)^{-1/3}}} I(v, v', w - v) dw + \sum_{i=1}^{q(v)} \mathrm{Res}_i(v, v') \tag{33}$$

where $I(v, v', w - v)$ is the integrand in (28). We drop the subscript u on K^N, I and Res_i in this section to indicate that we've set u as in (27). The kernel acts on the $\mathcal{C}_\varphi$ contour as before, and we set $\varphi = \pi/4$. The little extra displacement of the $\ell_{z_{\mathrm{crit}} + \delta(\sigma N)^{-1/3}}$ is a necessary technicality that we will address in due course. Therefore we will henceforth simply write $\ell_{z_{\mathrm{crit}}}$ as a shorthand.

For (v, v') on the $\mathcal{C}_{\pi/4}$ contour, we show that

$$|K^N(v, v')| \le f(v, N), \tag{34}$$

where $f(v, N)$ is defined in Lemma 4.8. Then, from the Hadamard inequality for determinants, we get for $m > 1$,

$$\left|\det(K^N(v_i, v_j))_{1\le i,j\le m}\right| \le \prod_{i=1}^m f(v_i, N) m^{m/2}. \tag{35}$$

$f(v, N)$ depends on v and N in such a way that it is integrates over $\mathcal{C}_{\pi/4}$ to a quantity that is bounded above by a constant independent of N. It follows that the Fredholm expansion of the determinant

$$\det(I + K^N)_{L^2(\mathcal{C}_{\pi/4})} = \sum_{m=0}^\infty \frac{1}{m!} \int_{\mathcal{C}_{\pi/4}} dv_1 \cdots \int_{\mathcal{C}_{\pi/4}} dv_m \det(K^N(v_i, v_j))_{1\le i,j\le m},$$

is absolutely summable uniformly in N (see (52)). Thus, we can take the $N \to \infty$ limit inside the series and integrals and replace K^N by its pointwise limit, the Airy kernel. This is similar to what was done in Borodin et al. [11], but now the constant in (35) must be bounded uniformly in θ as well as N, since θ can go to infinity with N (see Theorem 2.1). The rigorous estimates are shown in Lemma 4.8.

Recall the function $G(z)$ defined in (29)

$$G(z) = \log\Gamma(z) - \log\Gamma(\theta - z) - 2\Psi(z_{\text{crit}})z. \tag{36}$$

The bound in (34) will follow from an analysis of this function along the contours $\mathcal{C}_{\pi/4}$ and $\ell_{z_{\text{crit}}}$, and an estimate on the residues $\text{Res}_i(v, v')$. The analysis will be performed in several steps. The constants in the estimates are independent of N, but they do depend on θ. As long as $\theta \geq \theta_0 > 0$, the constants are well-behaved; this is guaranteed by $\sigma(\beta)N^{1/3} \to \infty$.

In the following, we set $\tilde{\sigma} = \sigma(\beta)N^{1/3}$, and note that

$$\tilde{\sigma} = \Theta((Nz_{\text{crit}}^{-2})^{1/3}). \tag{37}$$

1. We first show that the Taylor approximation is effective in the region $|z - z_{\text{crit}}| \leq c(\sigma N^{1/3})^{-1}$. Although G is analytic, one must be careful because the derivatives of G are a function of θ, which is allowed to go to infinity with N.

   Using the Taylor expansion, we may also arrange for an estimate of the form (recall $z_{\text{crit}} = \theta/2$)

   $$\text{Re}(G(z) - G(z_{\text{crit}})) \leq -\frac{C}{z_{\text{crit}}^2}|z - z_{\text{crit}}|^3, \quad |z - z_{\text{crit}}| \leq \frac{z_{\text{crit}}}{2}, \tag{38}$$

   where $C > 0$ can be explicitly chosen. This is needed to show that the pointwise limit of K^N is K_{Ai}.

2. Next we show that the real part of G decreases sufficiently rapidly away from the critical point on the $\mathcal{C}_{\pi/4}$ contour. The upper and lower halves of the $\mathcal{C}_{\pi/4}$ contour are parametrized as

   $$z(r) = z_{\text{crit}} + r\hat{e}_{\pm}, \quad r \geq 0. \tag{39}$$

   where $\hat{e}_{\pm} = -1 \pm \text{i}$. We show in Lemma 4.5 that the derivative of G satisfies

   $$\frac{d}{dr}\,\text{Re}(G(z(r)) - G(z_{\text{crit}})) \leq -\frac{2r^2}{(1 + z_{\text{crit}} + 2r)^2}. \tag{40}$$

   This captures the cubic behavior of G near the critical point, and the linear decay for large r: for some constants C and r_0 independent of z_{crit} and N,

   $$\text{Re}(G(z(r)) - G(z_{\text{crit}})) \leq -H(r) := \begin{cases} Cz_{\text{crit}}^{-2}r^3 & r \leq r_0, \\ Cz_{\text{crit}}^{-2}r_0^3 + z_{\text{crit}}^{-2}(r - r_0) & r > r_0. \end{cases} \tag{41}$$

3. On the upper and lower halves of the $\ell_{z_{\text{crit}}+\delta}$ contour, we use the parametrization

   $$w(r) = z_{\text{crit}} + r\hat{e}_{\pm} + \delta(\sigma N^{1/3})^{-1}, \quad r \geq 0, \tag{42}$$

   where $\hat{e}_{\pm} = \pm\text{i}$. We show that $\text{Re}(G(w(r)) - G(z_{\text{crit}}))$ is non-decreasing in r. Since $\ell_{z_{\text{crit}}+\delta}$ is not a steep-descent contour, we can't show that the derivative of $\text{Re}(G)$ is strictly positive (c.f. (40)). However, this is sufficient for our purposes since we use Taylor expansion close to the critical point to get a better estimate.

4. Finally, we estimate the contribution of the residues to the bound in (34). This is shown in Lemma 4.7. Using Lemma 4.7 and steps 1-3, we show that the m^{th} term of Fredholm series in (**??**) can be uniformly bounded in N. The bound in Lemma 4.7 also shows that the residues vanish in the limit (5).

Proof of Lemma 4.2. Using the estimates in steps 1-4, we first show that the pointwise limit of the integral in (33) is the Airy kernel. We first consider the integral term, and split the integral over the top half of the contour into three parts

$$\int_{r<M\tilde{\sigma}^{-1}} + \int_{M\tilde{\sigma}^{-1}}^{r_0} + \int_{r_0}^{\infty} \tilde{\sigma}^{-1} I(v, v', w(r) - v) dw(r) := I_1 + I_2 + I_3, \tag{43}$$

where $w(r)$ is parametrization in (42) of the $\ell_{z_{\text{crit}}}$ contour, and $\tilde{\sigma} = \sigma N^{1/3}$. Since the integrand is analytic in a tiny ball of size $M\tilde{\sigma}^{-1}$, we modify the contours so that they're locally aligned with the Airy contours when $r \leq M\tilde{\sigma}^{-1}$.

The following bound is implied by the Taylor expansion Prop. 4.3 and Lemma 4.6: there exist constants C and $r_0 > 0$ such that

$$\begin{aligned} G(w(r)) - G(z_{\text{crit}}) \geq C z_{\text{crit}}^{-2} r^3 \quad & r \leq r_0, \\ G(w(r)) \geq G(w(r_0)) \quad & r > r_0. \end{aligned}$$

We estimate the absolute value of the third integral as follows.

$$\begin{aligned} |I_3| &= \left| -\frac{1}{2\pi \mathrm{i}} \int_{r_0}^{\infty} \frac{\pi \tilde{\sigma}^{-1}}{\sin(\pi(w-v))} e^{N[G(v)-G(w)] + t\tilde{\sigma}(v-w)} \frac{dw}{w - v'} \right| \\ &\leq C\delta^{-2} e^{-NH(\operatorname{Im}|\tilde{v}|)} e^{-CM^3}, \end{aligned}$$

where $H(r)$ is defined in (41), and we've used (37)

$$|\sin(\pi(w(r) - v))|^{-1} \leq \frac{c}{\delta\tilde{\sigma}^{-1}} e^{-\pi r}, \text{ and } |w(r) - v'| \geq \delta\tilde{\sigma}^{-1}. \tag{44}$$

We then make the change of variable in (30) in I_1 and I_2. To estimate the second integral, we use the bound $e^{-\pi r} \leq 1$, and the fact that there is a constant c' independent of M such that $\tilde{r} \leq C\tilde{r}^3 - |t|\tilde{r}$ for all $\tilde{r} \geq M \geq c'$. Thus, we obtain:

$$|I_2| \leq C\delta^{-2} e^{-NH(\operatorname{Im}|\tilde{v}|)} \int_{M}^{r_0\tilde{\sigma}} e^{-\tilde{r}} d\tilde{r} \leq C\delta^{-2} e^{-H(\operatorname{Im}|\tilde{v}|)} e^{-M}.$$

In I_1, we make the change of variable in (30), introduce the Jacobian factor of $\tilde{\sigma}$ and take $N \to \infty$. By dominated convergence, the limit can be taken inside the integral, and by the argument in Section 4.2, the first integral goes to the integrand of K_{Ai} in the rescaled variables $(\tilde{v}, \tilde{v}')$. Letting $M \to \infty$ shows that the integral term in (33) goes to K_{Ai}.

From Lemma 4.7, it follows that residues converge pointwise to 0 as well. □

The pointwise convergence of $K^N(v, v')$ to K_{Ai} and the estimate on K^N in Lemma 4.8 shows that the Fredholm determinants converge. This proves Theorem 2.1.

Step 1. **Taylor expansion near the critical point.** We've already seen in Section 4.2 that the first two derivatives of G vanish at the critical point. If the third and fourth derivatives of G were well-behaved, the Taylor expansion for $G(z)$ is an effective approximation when z is close to the critical point:

$$G(z) - G(z_{\text{crit}}) = \frac{G^{(3)}}{3!}(z_{\text{crit}})(z - z_{\text{crit}})^3 + \frac{G^{(4)}}{4!}(\xi)(z - z_{\text{crit}})^4,$$

for $\xi \in \{y\colon |y - z_{\text{crit}}| < |z - z_{\text{crit}}|\}$. Since G is an analytic function, it's clear that $G^{(3)}$ and $G^{(4)}$ are well-behaved for fixed z_{crit}. However, we allow $z_{\text{crit}} \to \infty$; Prop. 4.3 shows roughly that $\frac{G^{(4)}(z)}{G^{(3)}(z_{\text{crit}})} \approx \frac{C}{z_{\text{crit}}}$ for a constant $C > 0$, when $|z(r) - z_{\text{crit}}| \leq z_{\text{crit}}/2$ and $z_{\text{crit}} \to \infty$.

Proposition 4.3. *When* $|z - z_{\text{crit}}| \leq z_{\text{crit}}/2$*,*

$$\frac{2}{(2 + z_{\text{crit}})^2} \leq -G^{(3)}(z_{\text{crit}}) - \frac{4}{z_{\text{crit}}^3} \leq \frac{2}{z_{\text{crit}}^2} \tag{45}$$

$$\left|G^{(4)}(z)\right| \leq \frac{96}{z_{\text{crit}}^4} + \frac{32}{z_{\text{crit}}^3}. \tag{46}$$

Proof of Prop. 4.3. The Digamma function can be written as [1, 6.3.16]

$$\Psi(z) = -\gamma_{EM} + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right), \tag{47}$$

where γ_{EM} is the Euler-Mascheroni constant. Note that the series is absolutely convergent when z is bounded away from the nonpositive integers. Differentiating (36) thrice we obtain

$$-G^{(3)}(z_{\text{crit}}) = -2\Psi^{(2)}(z_{\text{crit}}) = \frac{4}{z_{\text{crit}}^3} + 4\sum_{n=1}^{\infty}\frac{1}{(n + z_{\text{crit}})^3}.$$

Estimating the sum by an integral, we get

$$4\int_1^{\infty}\frac{1}{(x + z_{\text{crit}})^3}dx \leq -G^{(3)}(z_{\text{crit}}) - \frac{4}{z_{\text{crit}}^3} \leq 4\int_0^{\infty}\frac{1}{(x + z_{\text{crit}})^3}dx$$

which proves (45). We estimate $G^{(4)}(z)$ similarly: To apply the integral test as before, we first show that $|x + z|$ is increasing with $x \in \mathbb{R}^+$. It's clear that if $|z - z_{\text{crit}}| \leq z_{\text{crit}}/2$, then z has positive real part and consequently, so does $x + z$ for all $x > 0$. It follows that $|x + z|$ increases with x. Then, using (47) and $|x + z| \geq 2^{-1}(x + z_{\text{crit}})$,

$$\left|\Psi^{(3)}(z)\right| \leq \sum_{n=0}^{\infty}\frac{6}{|n + z|^4} \leq \frac{96}{|z_{\text{crit}}|^4} + \int_0^{\infty}\frac{96}{|x + z_{\text{crit}}|^4}dx,$$

which proves (46). □

Step 2. **Decay of G along the $\mathcal{C}_{\pi/4}$ contour.** In the following lemma, we first compute the derivative of G along a general contour. We will use this computation repeatedly to estimate G along the $\mathcal{C}_{\pi/4}$ and $\ell_{z_{\text{crit}}}$ contours (Lemma 4.6), and to estimate the residues in Lemma 4.7.

Lemma 4.4. *Let $z(r) = z_{\text{crit}} + v(r)$ be a contour. Then, the derivative of* $\text{Re}(G)$ *is*

$$\frac{d}{dr}\,\text{Re}(G(z(r))) = 2\sum_{n=0}^{\infty} \frac{-\,\text{Re}(v'(r)v^2)(n+z_{\text{crit}})^2 + \text{Re}(v'(r))|v(r)|^4}{(n+z_{\text{crit}})|(n+z_{\text{crit}})^2 - v^2|^2}$$

Proof. From (36),

$$\frac{d}{dr}G(z(r)) = z'(r)\left(\Psi(z_{\text{crit}} + v(r)) - \Psi(z_{\text{crit}})\right) + z'(r)\left(\Psi(z_{\text{crit}} - v(r)) - \Psi(z_{\text{crit}})\right).$$

Using (47),

$$\begin{aligned}\frac{d}{dr}G(z(r)) &= z'(r)\sum_{n=0}^{\infty} \frac{2}{n+z_{\text{crit}}} - \frac{1}{n+z(r)} - \frac{1}{n+\theta - z(r)} \\ &= v'(r)\sum_{n=0}^{\infty} \frac{2}{n+z_{\text{crit}}} - \frac{2(n+z_{\text{crit}})}{(n+z_{\text{crit}}+v(r))(n+z_{\text{crit}}-v(r))} \\ &= 2\sum_{n=0}^{\infty} \frac{-v'(r)v^2(n+z_{\text{crit}})^2 + v'(r)|v(r)|^4}{(n+z_{\text{crit}})|(n+z_{\text{crit}})^2 - v^2|^2}.\end{aligned}$$

□

Lemma 4.5. $\text{Re}(G)$ *in* (36) *satisfies the following derivative bound:*

$$\frac{d}{dr}\,\text{Re}(G(z(r))) \leq -\frac{2r^2}{(1+z_{\text{crit}}+2r)^2}, \quad r > 0,$$

where $z(r)$ is the parametrization of $\mathcal{C}_{\pi/4}$ given in (39).

Proof. We parametrize the upper-half of the $C_{\pi/4}$ contour as in Lemma 4.4 with $v(r) = r\hat{e}$ where $\hat{e} = -1 + \mathrm{i}$. Then,

$$\begin{aligned}\frac{d}{dr}\,\text{Re}(G(z(r))) &= 2\sum_{n=0}^{\infty} \frac{-2r^2(n+z_{\text{crit}})^2 - 4r^4}{(n+z_{\text{crit}})|(n+z_{\text{crit}})^2 - 2r^2\mathrm{i}|^2} \\ &\leq -4\sum_{n=0}^{\infty} \frac{r^2}{(n+z_{\text{crit}}+2r)^3} \leq -4\int_1^{\infty} \frac{r^2}{(x+z_{\text{crit}}+2r)^3}\,dx \\ &= -2\frac{r^2}{(1+z_{\text{crit}}+2r)^2}.\end{aligned}$$

This captures the behavior of G along the steep-descent contours rather well: Cubic near the critical point, and then linear decay for $r \geq Cz_{\text{crit}}$. G behaves symmetrically in the lower half plane, and hence satisfies the same estimates on the lower half of the $\mathcal{C}_{\pi/4}$ contour. □

Step 3. **Decay along the $\ell_{z_{\text{crit}}}$ contour.**

Lemma 4.6. $\mathrm{Re}(G)$ *in* (36) *increases away from the critical point along the* $\ell_{z_{\text{crit}}}$ *contour.*

Proof. Recall that the $\ell_{z_{\text{crit}}}$ contour starts off at a distance $\hat{\delta}_N = \delta\tilde{\sigma}^{-1}$ away from z_{crit}. Let $w(r)$ be the parametrization of $\ell_{z_{\text{crit}}}$ in (42); as before we focus on the upper half of the contour. Using Lemma 4.4 and $v(r) = r\mathrm{i} + \hat{\delta}_N$, we get

$$\frac{d}{dr}\,\mathrm{Re}(G(w(r))) = 2\sum_{n=0}^{\infty} \frac{-\,\mathrm{Re}\left(\mathrm{i}(\hat{\delta}_N - r^2 + 2\hat{\delta}_N r\mathrm{i})\right)(n+z_{\text{crit}})^2 + \mathrm{Re}(\mathrm{i})|v|^4}{(n+z_{\text{crit}})|(n+z_{\text{crit}})^2 - v^2|^2} \geq 0.$$

□

Step 4. **Triviality of the residues**

Lemma 4.7. *There exist constants* $c_1, C > 0$ *independent of* N *and* z_{crit} *such that for* $\ell = 1, \dots \lfloor |\,\mathrm{Im}(v)| \rfloor$, *the residues in* (33) *satisfy*

$$\log|\mathrm{Res}_\ell(v,v')| \leq \begin{cases} -c_1 N\ell^2 \dfrac{|\,\mathrm{Im}(v)|}{z_{\text{crit}}^2} & 1 \leq |\,\mathrm{Im}(v)| \leq C z_{\text{crit}} \\ -c_1 N\ell \log\left(1 + \dfrac{|\,\mathrm{Im}(v)|}{z_{\text{crit}}}\right) & |\,\mathrm{Im}(v)| > C z_{\text{crit}} \end{cases}$$

when $v, v' \in \mathcal{C}_{\pi/4}$. *If* $C z_{\text{crit}} < 1$, *the first bound holds vacuously.*

Lemma 4.7 helps show the estimate on the kernel in (34) that's used in the Hadamard bound. It also shows that the residues go to 0 as $\tilde{\sigma} \to \infty$.

Proof. From (24) and (27), we can write the residues in the form

$$\begin{aligned} |\,\mathrm{Res}_\ell(v,v')| &= \left| \left(\frac{\Gamma(v)}{\Gamma(v+\ell)} \frac{\Gamma(\theta - v - \ell)}{\Gamma(\theta - v)} \right)^N \frac{e^{2\Psi(z_{\text{crit}})N\ell - t\ell\tilde{\sigma}}}{v + \ell - v'} \right|, \\ &\leq e^{N(G(v) - G(v+\ell)) + |t|\ell\tilde{\sigma}}, \end{aligned} \tag{48}$$

since $\ell \geq 1$. We estimate $G(v) - G(v+\ell)$ using Lemma 4.4 again. Let $v(r) = k\hat{e}_+ + r$ in the parametrization of the contour in Lemma 4.4, where $\hat{e}_+ = -1 + \mathrm{i}$. Since $k = |\,\mathrm{Im}(v)|$, (25) implies that we only need to consider r in the range $0 \leq r \leq k$ (assuming δ is small enough). We interpolate between $G(v)$ and $G(v+l)$ by computing the following derivative:

$$\begin{aligned} &\frac{d}{dr}\,\mathrm{Re}(G(z(r))) \\ &= 2\sum_{n=0}^{\infty} \frac{r(2k-r)(n+z_{\text{crit}})^2 + ((r-k)^2 + k^2)^2}{(n+z_{\text{crit}})|(n+z_{\text{crit}}) + v|^2|(n+z_{\text{crit}}) - v|^2} \\ &\geq 2\sum_{n=0}^{\infty} \frac{r(2k-r)(n+z_{\text{crit}})^2 + ((r-k)^2 + k^2)^2}{(n+z_{\text{crit}})(n+z_{\text{crit}} + 2k - r)^4} \\ &\geq 2\int_{1+\max(k,z_{\text{crit}})}^{\infty} \frac{r(2k-r)x}{(x+2k-r)^4}\,dx + 2\int_{1+z_{\text{crit}}}^{\infty} \frac{((r-k)^2+k^2)^2}{x(x+2k-r)^4}\,dx := I_1 + I_2. \end{aligned}$$

The limits of integration in I_1 have been obtained as follows. Notice that $f(x) = x/(x+2k-r)^4$ is decreasing for $x \geq \max(k, z_{\text{crit}})$. This lets us use the integral test to estimate the sum. Our bounds on the integrals I_j, $j = 1, 2$ will ensure:

$$G(v+\ell) - G(v) \geq \begin{cases} c_1 \ell^2 \frac{|\operatorname{Im}(v)|}{z_{\text{crit}}^2} & |\operatorname{Im}(v)| \leq C z_{\text{crit}} \quad (49) \\ c_1 \ell \log\left(1 + \frac{|\operatorname{Im}(v)|}{z_{\text{crit}}}\right) & |\operatorname{Im}(v)| > C z_{\text{crit}} \quad (50) \end{cases}$$

1. The first bound (49) ensures that the exponent in (48) contains a negative term of order at least $N z_{\text{crit}}^{-2}$, and thus overwhelms $\tilde{\sigma} = \Theta((N z_{\text{crit}}^{-2})^{1/3})$ as they both go to infinity (by (5)).
2. The second bound (50) ensures that the residue is integrable in v on the contour $\mathcal{C}_{\pi/4}$ over the range $|\operatorname{Im}(v)| \in [C z_{\text{crit}}, \infty)$.

Explicitly computing I_1, we get

$$I_1 = \frac{r(2k-r)(3(1+\max(k, z_{\text{crit}})) + 2k - r)}{3(1+\max(k, z_{\text{crit}}) + 2k - r)^3} \geq \frac{rk}{3(1+\max(k, z_{\text{crit}}) + 2k)^2},$$

for $r \leq k$. For the second integral,

$$I_2 \geq k^4 \left(\frac{1}{(2k-r)^4} \log\left(1 + \frac{2k-r}{1+z_{\text{crit}}}\right) - \frac{2}{(2k-r)^3(1+z_{\text{crit}} + (2k-r))} \right),$$

where we've used

$$\int_a^\infty \frac{dx}{x(x+c)^4} = -\frac{6a^2 + 15ac + 11c^2}{6c^3(a+c)^3} + \frac{1}{c^4} \log\left(1 + \frac{c}{a}\right)$$

Here, for $k \geq C z_{\text{crit}}$ where C is some constant, the first term in I_2 dominates the second for all $r \leq k$. Therefore, integrating over r, we get (49) and (50) for some constants c_1, C. □

Finally, we prove the inequality used in the Hadamard bound in (34). We look at the kernel in the rescaled coordinates $\tilde{v}, \tilde{v'}$ in (30).

Lemma 4.8. *There exist constants $c_1, c_2, C > 0$ that are independent of N and z_{crit} such that for all N large enough,*

$$|K^N(v, v')| \leq \begin{cases} c_1 \tilde{\sigma} \exp\left(-c_2 N z_{\text{crit}}^{-2} |\operatorname{Im}(v)|^3\right) & |\operatorname{Im}(v)| < 1 \\ c_1 \tilde{\sigma} \exp\left(-c_2 N z_{\text{crit}}^{-2} |\operatorname{Im}(v)|\right) & 1 \leq |\operatorname{Im}(v)| \leq C z_{\text{crit}} \\ c_1 \left(1 + \frac{|\operatorname{Im}(v)|}{z_{\text{crit}}}\right)^{-c_2 N} & |\operatorname{Im}(v)| > C z_{\text{crit}} \end{cases}$$
$$=: f(v, N) \quad (51)$$

*on the contour $\mathcal{C}_{\pi/4}$. Consequently the m^{th} term of the Fredholm series for K^N in (**??**) satisfies*

$$\frac{1}{m!} \int_{\mathcal{C}_{\pi/4}} dv_1 \cdots \int_{\mathcal{C}_{\pi/4}} dv_m \det(K^N(v_i, v_j))_{1 \leq i,j \leq m} \leq \frac{C}{m^{(m-1)/2}}. \quad (52)$$

Proof. Using the technique in of splitting up the integral term as in (43), and from the estimates in Lemma 4.5 and Lemma 4.6, it follows that the integral in (33) has two regimes of behavior: for constants $c_1, c_2, C > 0$

$$\int_{\ell_{z_{\text{crit}}}} I(v, v', \omega) dw \leq \begin{cases} c_1 \tilde{\sigma} \exp\left(-c_2 N z_{\text{crit}}^{-2} |\operatorname{Im}(v)|^3\right) & |\operatorname{Im}(v)| \leq C z_{\text{crit}} \\ c_1 \tilde{\sigma} \exp\left(-c_2 N |\operatorname{Im}(v)|\right) & |\operatorname{Im}(v)| > C z_{\text{crit}} \end{cases}$$

The $\ell_{z_{\text{crit}}}$ contour is a small distance $\delta(\sigma N^{1/3})^{-1}$ away from z_{crit}. From (25), it follows there are about $|\operatorname{Im}(v)|$ residues, at least when δ is small. Hence, when $|\operatorname{Im}(v)| < 1$ and when N large enough, there are no residues. We estimate the contribution of the residues to K^N when $\operatorname{Im}(v) > 1$. For N large enough, and $1 \leq |\operatorname{Im}(v)| \leq C z_{\text{crit}}$, Lemma 4.7 implies

$$\sum_{\ell=1}^{\lfloor |\operatorname{Im}(v)| \rfloor} \operatorname{Res}_\ell(v, v') \leq \sum_{\ell=1}^{\lfloor |\operatorname{Im}(v)| \rfloor} c_1 \exp\left(-c_1 N \ell \frac{|\operatorname{Im}(v)|}{z_{\text{crit}}^2}\right) \leq c_1 \exp\left(-c_2 N z_{\text{crit}}^{-2} |\operatorname{Im}(v)|\right).$$

When $|\operatorname{Im}(v)| > C z_{\text{crit}}$,

$$\sum_{\ell=1}^{\lfloor |\operatorname{Im}(v)| \rfloor} \operatorname{Res}_\ell(v, v') \leq \sum_{\ell=1}^{\lfloor |\operatorname{Im}(v)| \rfloor} c_1 \left(1 + \frac{|\operatorname{Im}(v)|}{z_{\text{crit}}}\right)^{-c_2 N l} \leq c_1 \frac{1}{(1 + |\operatorname{Im}(v)|/z_{\text{crit}})^{c_2 N}}.$$

In the previous two displays, c_1 was allowed to change between inequalities. Thus, (51) follows, and we can integrate the bound over the $\mathcal{C}_{\pi/4}$ contour to obtain

$$\begin{aligned} \int_{\mathcal{C}_{\pi/4}} f(v, N) dv &\leq C \left(\tilde{\sigma} \int_0^1 e^{-c_2 N z_{\text{crit}}^{-2} x^3} \, dx + \tilde{\sigma} \int_1^{C z_{\text{crit}}} e^{-c_2 N z_{\text{crit}}^{-2} x} \, dx + \int_{C z_{\text{crit}}}^\infty \left(1 + \frac{x}{z_{\text{crit}}}\right)^{-c_2 N} \right) \\ &\leq C \left(\frac{\tilde{\sigma}}{(N z_{\text{crit}}^{-2})^{1/3}} + \tilde{\sigma} e^{-c_2 N z_{\text{crit}}^{-2}} + \frac{(1+C)^{-c_2 N}}{N z_{\text{crit}}^{-1}} \right). \end{aligned}$$

Since $\tilde{\sigma} = \Theta((N z_{\text{crit}}^{-2})^{1/3})$, it's clear that the integral is bounded above by a constant independent of N and z_{crit}. The Hadamard inequality now implies (52).

□

A Fredholm determinant as a limit of the formula of Borodin-Corwin-Ferrari-Veto

A.1 The BCFV theorem

Borodin et al. [12] consider a mixed polymer that consists of Seppäläinen's log-gamma polymer [24] and the O'Connell-Yor semi-discrete polymer [23]. For $N \geq 1$, the up-right paths $\mathbf{x}$ consist of a discrete portion $\mathbf{x}^d$ adjoined to a semi-discrete portion $\mathbf{x}^{sd}$. The discrete portion is an up-right nearest-neighbor path on $\mathbb{Z}^2$ that goes from $(-N,1)$ to $(-1,n)$ for some $1 \le n \le N$. For $0 \le s_n < \cdots < s_{N-1} \le \tau$, the semi-discrete path consists of horizontal segments on (s_i, s_{i+1}) for $i = n, \ldots, N-2$ and a final interval (s_{N-1}, τ) connected by vertical jumps of size 1 at each s_i. For $1 \le m, n \le N$ let $\xi_{-m,n}$ be independent exp-gamma random variables with parameter θ, and for all $1 \le n \le N$ let B_n be independent Brownian motions. The paths have energy

$$H_\beta(\mathbf{x}) = -\sum_{(i,j)\in\mathbf{x}^d} \xi_{i,j} + B_n(s_n) + \big(B_{n+1}(s_{n+1}) - B_{n+1}(s_n)\big) + \ldots + \big(B_N(\tau) - B_N(s_{N-1})\big). \tag{53}$$

The partition function is given by

$$\mathbf{Z}^N(\tau) = \sum_{i=1}^{N} \sum_{\mathbf{x}^d:(-N,1)\nearrow(-1,i)} \int_{\mathbf{x}^{sd}:(0,i)\nearrow(\tau,N)} e^{-H_\beta(\mathbf{x})} d\mathbf{x}^{sd}$$

where $d\mathbf{x}^{sd}$ represents the Lebesgue measure on the simplex $0 \le s_n < s_{n+1} < \cdots < s_{N-1} \le \tau$.

Remark 4. When $\tau = 0$, the polymer is simply the standard log-gamma polymer. There is no semi-discrete part.

Theorem A.1 (Borodin et al. [11], Theorem 2.1)**.** *Fix $N \ge 9$, $\tau \ge 0$ and $\theta > 0$. For all $u \in \mathbb{C}$ with positive real part*

$$\mathbb{E}\left[e^{-u\mathbf{Z}^N(\tau)}\right] = \det(\mathbf{1} + K^N_{u,\tau})_{L^2(\mathcal{C}_\varphi)} \tag{54}$$

where

$$\begin{aligned} K^N_{u,\tau}(v,v') &= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{D}_v} \frac{1}{\sin(\pi s)} \left(\frac{\Gamma(v)}{\Gamma(s+v)} \frac{\Gamma(\theta - v - s)}{\Gamma(\theta - v)}\right)^N \frac{e^{\tau(sv+s^2/2)}}{v+s-v'} u^s \, ds \\ &=: \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{D}_v} I_{u,\tau}(v,v',s)\, ds\,. \end{aligned}$$

The $\mathcal{C}_\varphi$ contour is the wedge-shaped contour defined in Theorem 4.1, and depends on the angle φ and the parameter θ. The $\mathcal{D}_v$ contour depends on v and parameters R and d. For every $v \in \mathcal{C}_\varphi$ we choose $R = -\operatorname{Re}(v) + 3\theta/4$, $d > 0$, and let $\mathcal{D}_v$ consist of straight lines from $R - \mathrm{i}\infty$ to $R - \mathrm{i}d$ to $\theta/8 - \mathrm{i}d$ to $\theta/8 + \mathrm{i}d$ to $R + \mathrm{i}d$ to $R + \mathrm{i}\infty$. The parameter d must be taken small enough so that $v + \mathcal{D}_v$ does not intersect $\mathcal{C}_\varphi$. Both contours are oriented to have increasing imaginary part.

Remark 5. In Borodin et al. [12], the $\mathcal{D}_v$ contour consisted of straight lines from $R - \mathrm{i}\infty$ to $R - \mathrm{i}d$ to $1/2 - \mathrm{i}d$ to $1/2 + \mathrm{i}d$ to $R + \mathrm{i}d$. The formula only holds if the poles in s of $\Gamma(\theta - v - s)$ lie strictly to the right of the contour $\mathcal{D}_v$, and this imposes a lower bound $\theta > 1$. We remove the restriction $\theta > 1$ as follows.

Note first of all that both sides of (54), with $\mathcal{D}_v = \mathcal{D}_{v,1/2}$ are analytic functions of θ in some region containing the ray $(1,\infty)$, on which they coincide, by Borodin et al. [12]: The left hand side is actually analytic in a region containing the ray $\theta \in (0,\infty)$ because the expectation is an N^2-fold integral of a function $e^{-u\mathbf{Z}^{N,N}(\tau)}$ of the variables $\xi_{i,j}$, $i,j = 1,\ldots,N$ with

$$\mathbb{E}[e^{-u\mathbf{Z}^N(\tau)}] = \int F(\xi_{ij}) \prod_{i,j} \frac{e^{-\xi_{i,j}} \xi_{i,j}^{\theta-1}}{\Gamma(\theta)} d\xi_{i,j} \qquad F(\xi_{ij}) = \mathbb{E}[e^{-u\mathbf{Z}^N(\tau)} \mid \xi_{i,j}].$$

The right hand side is analytic because the Fredholm determinant of a kernel analytic in θ is analytic in θ, as long as one has a Hadamard bound uniformly in that region.

Call $\mathcal{D}_{v,\eta}$ the contour consisted of straight lines from $R-\mathrm{i}\infty$ to $R-\mathrm{i}d$ to $\eta-\mathrm{i}d$ to $\eta+\mathrm{i}d$ to $R+\mathrm{i}d$. Fix $\theta^*>0$. We can use Cauchy's theorem to deform the contour in (A.1) to $\mathcal{D}_{v,\theta^*/4}$ without changing the kernel, since the only obstacle is the zero of the sin in the denominator, which is at 0. Prop. A.2 gives us a uniform Hadamard bound on the kernel in the region, say $[\theta^*/16, 2]$, so using this new representation of the kernel, the determinant in (54) is an analytic function of θ, now in a region containing $[\theta^*-\gamma, 1+\gamma]$ for some $\gamma>0$. Because the kernel is unchanged, this coincides with the old determinant for $\theta\in(1,1+\gamma)$. By the formula from Borodin et al. [12], this coincides with the left hand side of (54) for $\theta\in(1,1+\gamma)$. But the left hand side is analytic in a region containing the ray $(0,\infty)$, hence the determinant with the new, deformed kernel is equal to the left hand side on a region containing the interval $[\theta^*-\gamma,1+\gamma]$, including at the value θ^* which is what we wanted to prove.

Remark 6. We write $\mathcal{D}_v$ contour as a union of the vertical contour $\ell_{-Re(v)+R}$ defined in Theorem 4.1 and the "sausage" $\mathcal{D}_{v,\square}$ (which consists of $(\mathcal{D}_v\setminus\ell_{-Re(v)+R})\cup H(R)$ and forms an anticlockwise loop). The integral of the kernel $K^N_{u,\tau}$ over $\mathcal{D}_{v,\square}$ consists of residues due to the sine that we will estimate separately. For each v, there is some wiggle room in the R parameter that allows the vertical contour $\ell_{-\operatorname{Re}(v)+R}$ to avoid the singularity of the sine function in $I_{u,\tau}(v,v',s)$.

A.2 Proof of Theorem 4.1

In this section, we obtain Theorem 4.1 by letting $\tau\searrow 0$ in Theorem A.1. We may do so if we can truncate the series that defines the Fredholm determinant of $\mathbf{Z}^N(\tau)$ uniformly in τ. This is done by proving an estimate on $K^N_{u,\tau}(v,v')$ that depends favorably with τ, and then using Hadamard's inequality in the usual way (see Section 4.2). The constants in the following propositions may depend on the the angle of the contour $\varphi\in(0,\pi/4]$.

Proposition A.2 (BCFV kernel estimate)**.** *When* $|\operatorname{Im}(v)|>\max((2e^{\tau\theta/2}|u|)^{1/2N}, c_3\theta))$*, for some* $c_1,c_2,c_3>0$*, the kernel in* (A.1) *satisfies the bound*

$$|K^N_{u,\tau}(v,v')|\le 2\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}}e^{-c_1\tau|\operatorname{Im}(v)|}+e^{C(\theta,N,\tau)}e^{-c_2N|\operatorname{Im}(v)|(\log|v|-c_3\theta)},$$

where $C(\theta,N,\tau)=\mathcal{O}(N(\theta+|\log\theta|+\theta^{-1}+1)+\tau\theta)$.

Proposition A.2 is proved by estimating I_u on the closed rectangular contour $\mathcal{D}_{v,\square}$, and the vertical line ℓ_R.

Proposition A.3 (Integral over the $\mathcal{D}_{v,\square}$ contour)**.** *When* $|\operatorname{Im}(v)|>\max((2e^{\tau\theta/2}|u|)^{\frac{1}{2N}},c_3\theta)$,

$$\left|\int_{\mathcal{D}_{v,\square}}I_{u,\tau}(v,v',s)\,ds\right|\le\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}}e^{-c\tau|\operatorname{Im}(v)|},$$

where $c>0$ *is some constant.*

Proposition A.4 (Integral over the ℓ_R contour)**.** *For some constants* $c_1, c_2, c_3 > 0$*,* $|\operatorname{Im}(v)| \geq c_3\theta$

$$\int_{\ell_R} I_{u,\tau}(v, v', s)\, ds \leq e^{C(\theta,N,\tau)} e^{-c_1 N |\operatorname{Im}(v)|(\log|v| - c_2\theta)}$$

where $C(\theta, N, \tau)$ *is the constant in Prop. A.2.*

Before proving the propositions, we complete the proof of Theorem 4.1.

Proof of Theorem 4.1. In Theorem A.1, the vertical contour ℓ_R is at $R = -\operatorname{Re}(v) + 3\theta/4$. This must be moved to the critical point, so that $R = -\operatorname{Re}(v) + z_{\text{crit}} + \delta$ for small δ. This is done by using the bound on $I_{u,\tau}(v, v', s)$ in (68). This implies that we can truncate the vertical contour at large $|\operatorname{Im}(s)|$, and use Cauchy's theorem to move over the vertical contour.

Next, we have to take a limit $\tau \searrow 0$ in (54). Since u has positive real part, $e^{-uZ_\beta^N}$ is absolutely bounded, and we can take the limit inside the integral by bounded convergence. For the right hand side, we can use the Hadamard inequality argument in Section 4.2 and the bounds in Prop. A.3 and Prop. A.4, to show that it converges to the Fredholm determinant of the pointwise limit of the kernel $K_{u,\tau}^N$ as $\tau \searrow 0$. This proves Theorem 4.1. □

Proof of Proposition A.3. The integral over $\mathcal{D}_{v,\square}$ simply collects residues from the poles of $\sin^{-1}(\pi s)$ (up to signs). Then,

$$\begin{aligned}\int_{\mathcal{D}_{v,\square}} I_{u,\tau}(v, v', s)\, ds &= \sum_{i=1}^{q(v)} \left(\frac{\Gamma(v)\Gamma(\theta - v - i)}{\Gamma(v+i)\Gamma(\theta - v)}\right)^N u^i \frac{e^{\tau(\operatorname{Re}(v)i + i^2/2)}}{|v + i - v'|}(-1)^i \\ &=: \sum_{i=1}^{q(v)} \operatorname{Res}_{u,i}(v, v'),\end{aligned}$$

where $q(v) \leq R$ is the number of zeros of the sine caught inside the sausage (25). Since $v, v' \in \mathcal{C}_\varphi$, we have

$$\operatorname{Re}(v) = \frac{\theta}{2} - \cot(\varphi)|\operatorname{Im}(v)| \tag{55}$$

where $0 < \delta \leq \frac{\theta}{4}$. Then, we may estimate $q(v)$ as follows:

$$q(v) \leq R = -\operatorname{Re}(v) + \frac{\theta}{2} + \delta = \cot(\varphi)|\operatorname{Im}(v)| + \delta.$$

For our bound, the number of residues doesn't matter, and the contribution of the first residue dominates. The ratio of gamma functions in the residues become $\Gamma(v)/\Gamma(v+i) = \prod_{j=0}^{i-1}(v+j)^{-1}$ and $\Gamma(\theta - v - i)/\Gamma(\theta - v) = \prod_{j=1}^{i}(\theta - v - j)^{-1}$. It's clear that $|v + i| \geq |\operatorname{Im}(v)|$ and $|\theta - v - i| \geq |\operatorname{Im}(v)|$.

The $|v-v'+i|^{-1}$ term can be bounded above by a constant. Therefore for $|\operatorname{Im}(v)| > (e^{\tau\theta/2}|u|/2)^{\frac{1}{2N}}$,

$$
\begin{aligned}
\sum_{i=1}^{q(v)} \operatorname{Res}_{u,i}(v,v) &\le \sum_{i=1}^{q(v)} \frac{1}{|\operatorname{Im}(v)|^{2Ni}} |u|^i e^{i\tau\theta/2} e^{-\tau i(\cot(\varphi)|\operatorname{Im}(v)|-i/2)} \\
&\le \sum_{i=1}^{q(v)} \left(\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}}\right)^i e^{-\tau c|\operatorname{Im}(v)|}, \\
&\le 2\frac{e^{\tau\theta/2}|u|}{|\operatorname{Im}(v)|^{2N}} e^{-\tau c|\operatorname{Im}(v)|},
\end{aligned}
\tag{56}
$$

where c is a φ-dependent constant that comes from bounding the $i(\cot(\varphi)|\operatorname{Im}(v)| - i/2)$ term on the interval $1 \le i \le \cot(\varphi)|\operatorname{Im}(v)| + \delta$. We choose the constant c_3 to ensure that $\cot(\varphi)|\operatorname{Im}(v)| \ge \cot(\varphi)c_3\theta > \theta/4 \ge \delta$. The same constant c_3 appears in the Prop. A.4. □

Proof of Prop. A.4. We will focus first on estimating the product of Gamma functions in $I_{u,\tau}(v, v', s)$. For $s \in \ell_{-\operatorname{Re}(v)+R}$,

$$\operatorname{Re}(s) = \delta + \cot(\varphi)|\operatorname{Im}(v)|. \tag{57}$$

Stirling's formula holds whenever $\arg(z)$ remains bounded away from $\pm\pi$ (see Abramowitz and Stegun [1]),

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + \mathcal{O}\left(\frac{1}{|z|}\right),$$

and

$$\operatorname{Re}(\log\Gamma(z)) = -\operatorname{Im}(z)\arg(z) + \operatorname{Re}(z)\left(\log|z| - 1\right) - \frac{1}{2}\log|z| + \mathcal{O}\left(\frac{1}{|z|}\right).$$

This gives

$$\log\left(\frac{\Gamma(v)}{\Gamma(v+s)}\right) + \log\left(\frac{\Gamma(\theta-v-s)}{\Gamma(\theta-v)}\right) =$$

$$-\operatorname{Im}(v)\arg(v) + \operatorname{Im}(\theta-v)\arg(\theta-v) \tag{58}$$

$$+\operatorname{Re}(v)(\log|v|-1) - \operatorname{Re}(\theta-v)(\log|\theta-v|-1) \tag{59}$$

$$+\operatorname{Im}(v+s)\arg(v+s) - \operatorname{Im}(\theta-v-s)\arg(\theta-v-s) \tag{60}$$

$$-\operatorname{Re}(v+s)(\log|v+s|-1) + \operatorname{Re}(\theta-v-s)(\log|\theta-v-s|-1) \tag{61}$$

$$-\frac{1}{2}\log\left|\frac{v(\theta-v-s)}{(v+s)(\theta-v)}\right| + \mathcal{O}(|\theta|^{-1}) \tag{62}$$

since $|v|, |\theta-v|, |\theta-v-s|, |v+s| \ge c\theta$ for some constant $c > 0$. We will estimate the numbered terms in the above display one-by-one.

We first estimate (59):

$$\operatorname{Re}(v)(\log|v|-1) - \operatorname{Re}(\theta-v)(\log|\theta-v|-1) = \frac{\theta}{2}\log\frac{|v|}{|\theta-v|} - \cot(\varphi)|\operatorname{Im}(v)|\left(\log(|v||\theta-v|) - 2\right).$$

Since the ratio inside the logarithm is $\mathcal{O}(1)$ for all $v \in \mathcal{C}_\varphi$ we have for some φ-dependent constant c,

$$(59) \le \mathcal{O}(\theta) - c|\operatorname{Im}(v)|\log(|v||\theta-v|). \tag{63}$$

Thus, the terms in (59) dominate the terms in (58). This gives us the exponential decay in v that we need.

Since $\operatorname{Im}(\theta - v - s) = -\operatorname{Im}(v+s)$, (60) becomes $\operatorname{Im}(v+s)(\arg(v+s) + \arg(\theta - v - s))$. From (55) and (57), we get $\theta/2 + \delta = \operatorname{Re}(v+s) \geq \operatorname{Re}(\theta - (v+s)) = \theta/2 - \delta$. It follows that $\operatorname{Im}(v+s)$ and $\arg(v+s) + \arg(\theta - v - s)$ have opposite signs, and hence

$$(60) \leq \operatorname{Im}(v+s)(\arg(v+s) + \arg(\theta - v - s)) \leq 0. \tag{64}$$

For some constant $c > 0$,

$$\begin{aligned}(61) &= -\frac{\theta}{2}\log\frac{|v+s|}{|\theta - v - s|} - \delta\log|v+s||\theta - v - s| \\ &\leq -\frac{\theta}{2}c - \delta\log\left(\frac{\theta}{2} - \delta\right)\left(\frac{\theta}{2} + \delta\right) \\ &= \mathcal{O}(\theta) + \mathcal{O}(|\log(\theta)|).\end{aligned} \tag{65}$$

Equation (62) should be split up into two terms: the first term is $-\log|v/(\theta - v)|$ that is $\mathcal{O}(|\log\theta|)$ for small v, and $\mathcal{O}(1)$ for $|v| \geq c\theta$. The second term $-\log|(\theta - v - s)/(v+s)|$ behaves similarly, and we get

$$(62) = \mathcal{O}(1) + \mathcal{O}(|\log\theta|). \tag{66}$$

From (55) and (57), we get $|v + s - v'|^{-1} \leq \frac{2}{\theta}$. Finally, to analyze $\exp(\tau(sv + s^2/2))$, we look at the real part of $sv + s^2/2$:

$$\begin{aligned}&\operatorname{Re}(sv + s^2/2) \\ &\quad= \operatorname{Re}(s)\operatorname{Re}(v) - \operatorname{Im}(s)\operatorname{Im}(v) + \frac{\operatorname{Re}(s)^2 - \operatorname{Im}(s)^2}{2} \\ &\quad= \operatorname{Re}(s)\operatorname{Re}(v) + \frac{\operatorname{Re}(s)^2}{2} + \frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2} \\ &\quad= (\delta + \cot(\varphi)|\operatorname{Im}(v)|)(\frac{\theta}{2} - \cot(\varphi)|\operatorname{Im}(v)|) + \frac{(\delta + \cot(\varphi)|\operatorname{Im}(v)|)^2}{2} + \frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2} \\ &\quad= \frac{\theta\delta + \delta^2}{2} + \cot(\varphi)\frac{\theta}{2}|\operatorname{Im}(v)| + (1 - \cot(\varphi)^2)\frac{\operatorname{Im}(v)^2}{2} - \frac{(\operatorname{Im}(s) + \operatorname{Im}(v))^2}{2} \\ &\quad\leq C\theta + \cot(\varphi)\frac{\theta}{2}|\operatorname{Im}(v)|,\end{aligned} \tag{67}$$

using $\cot(\varphi) \geq 1$.

Putting (63), (64), (65), (66), (67) together with $\frac{1}{|\sin(\pi s)|} \leq Ce^{-\pi|\operatorname{Im}(s)|}$, we get

$$I_{u,\tau}(v, v', s) \leq e^{C(\theta,N,\tau)}e^{-N|\operatorname{Im}(v)|(\log|v| - c\tau\cot(\varphi)\theta)}e^{-\pi|\operatorname{Im}(s)|} \tag{68}$$

where

$$C(\theta, N, \tau) = \mathcal{O}(N(\theta + |\log\theta| + \theta^{-1} + 1) + \tau\theta).$$

Integrating this over s completes the proof. □

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