Lec24 6.3 (Undetermined Coeffs / Annihilators)

Solve: $\overbrace{(D+1)(D-1)}^{L} y = 16e^{3x}$ $(Ly = F)$

$L = P(D)$

$y(x) = y_p(x) + y_h(x)$

Particular HOMO. Solution $Ly_h = 0$

How to find PARTICULAR SOLUTIONS?

First:

$P(D) = D+3$ $Q(D) = D-1$

$P(D)y = \frac{dy}{dx} + 3y$

$P(D)Q(D)y = (D+3)(D-1)y = (D+3)(Dy-y)$

$= D^2y - Dy + 3Dy - 3y$

$= (D^2 - 4D - 3)y$

$Q(D)P(D)y$ ⅃

In otherwords $P(D)Q(D) = Q(D)P(D)$.

Polynomial differential operators COMMUTE

Annihilator: $A(D)$. $(P(D)y = F)$ ↙ polynomial

$A(D)F = 0$

Ex: $F = 16e^{3x}$ $A(D) = (D-3)$, $A(D)e^{3x} = 0$

Then if

$P(D)y = F$ WE MUST HAVE

$\underbrace{A(D)P(D)}y = A(D)F = 0$ $[S(D)y = 0]$

We know all solutions to this equation

$\overbrace{(D-3)}^{A(D)}\overbrace{(D+1)(D-1)}^{P(D)}y = 0$

$\Rightarrow y(x) = Ae^{x} + Be^{-x} + \underbrace{Ce^{3x}}$

Thus for some A, B, C we can find a particular solution of $P(D)y = F = 16e^{3x}$ $P(D)e^{x} = 0$ $P(D)e^{-x} = 0$

$(D+1)(D-1)y = (D+1)(D-1)Ce^{3x}$

$= C(D+1)(3e^{3x} - e^{3x}) = C(D+1)2e^{3x}$

$= C(6e^{3x} + 2e^{3x}) = C8e^{3x} (= 16e^{3x} = F)$ $\Rightarrow C8 = 16$

$\Rightarrow C = 2$.

THUS $\boxed{y_p(x) = 2e^{3x}}$

GENERAL solution: $2e^{3x} + Ae^{-x} + Be^{x}$

To find particular solution find $A(D)F = 0$.

$(D-3)e^{3x} = 3e^{3x} - 3e^{3x} = 0$

Once we find a particular solution we know it solves $A(D)P(D)y = 0$

Aux: $S(x) = (x-3)^1(x+1)^1(x-1)^1$

Solving for C is the "undetermined coefficient" part.

Ex: Find an annihilator for $F(x) = 5\cos 2x$

Recall that if we have $y'' + \omega^2 y = 0$ its general solution is $\overset{\text{5}}{A}\cos\omega x + B\sin\omega x$.

So we should choose $A(D) = D^2 + 2^2$

Need $A(D)F = 0$

$A(D) = (D^2 + \omega^2)$

Note: what kinds of F can I find annihilators for?

Then $A(D)F = 0 \Rightarrow$ F is a solution to a polynomial differential operator.

$\Rightarrow$ F must be a linear comb of

1) $x^r e^{kx}$ $\quad r \geq 0. \quad k \in \mathbb{R}$

2) $x^j e^{ax}\cos bx$, $x^j e^{ax}\sin bx$ $\quad j \geq 0, b \in \mathbb{R}$

We have solved linear homo. eqs with constant coefficients.

$x^2 e^{3x}$ $\quad A(D) = (D-3)^3$

e^{3x} $\quad x e^{3x}$ $\quad x^2 e^{3x}$

$A(D) = (D-k)^m$ annihilates for $m > r$ $(m = r+1)$

$A(D) = (D - \underbrace{a+bi}_{\lambda})^m (D - \underbrace{a-bi}_{\lambda})^m$ for $m > j$ $(m = j+1)$

$A(D) = \underbrace{(D-\lambda)^m (D-\bar{\lambda})^m}$

$e^{ax}\cos bx$ $\quad e^{ax}\sin bx$

$xe^{ax}\cos bx$ $\quad \vdots$

$\vdots$ $\quad \vdots$

$x^{m-1} e^{ax}\cos x$

Ex: $(D-4)(D+1)y = 15e^{4x}$ $\quad (D-4)e^{4x} = 0$

$A(D) = (D-4) \qquad A(D)P(D) = (D-4)^2(D+1)$

has sol. $y = \cancel{Ae^{4x}} + Bxe^{4x} + \cancel{Ce^{-x}}$

$P(D)y = B(D+1)(e^{4x}) = B(5e^{4x}) = 15e^{4x}$

$\Rightarrow B = 3. \qquad y_p = 3xe^{4x}.$

$A(D)$ and $P(D)$ share an eigenvalue.

$P(D)y = F$ for appropriate A, B, C

$P(D)y = B(D-4)(D+1)(xe^{4x})$

$= B(D+1)\left[e^{4x} + \cancel{4xe^{4x}} - \cancel{4xe^{4x}}\right]$

Ex: $y'' - y' - 2y = 10\sin x \qquad y(0) = 0 \quad y'(0) = 1$

IVP

$(D^2 - D - 2) = P(D) \qquad A(D) = (D^2+1)$

$= (D-2)(D+1)$

$A(D) = D^2 + \omega^2$

$\left[A(D)P(D)y = (D^2+1)(D-2)(D+1)y = 0\right.$

has general solution

$y = A\cos x + B\sin x + \cancel{C}e^{2x} + \cancel{D}e^{-x}$

$P(D)\cos x = (D^2 - D - 2)\cos x$

$= -\cos x + \sin x - 2\cos x$

$P(D)\sin x = -\sin x - \cos x - 2\sin x$

$P(D)y = A(-3\cos x + \sin x) + B(-3\sin x - \cos x)$

$= F = 10\sin x$

$\cos x(-3A - B) + \sin x(A - 3B)$

$-3A - B = 0 \qquad B = -3A$

$A - 3B = 10 \qquad A + 9A = 10 \Rightarrow A = 1$

$B = -3$

$y_p = \cos x - 3\sin x$

Ex: $F = \boxed{2e^x} - \boxed{3x}$ Inhomogeneities that are linear combinations.

$A_1 = D-1$ annihilates $2e^x$

$A_2 = D^2$ " $-3x$

$\Rightarrow A = A_1 A_2 = (D-1)D^2$

$= D^3 - D^2$

xe^{ax} $(D-\overset{0}{a})^2 = D^2$

$(D-1)D^2\left[2e^x - 3x\right] = D^2\cancel{(D-1)2e^x}^{\,0}$

$- (D-1)D^2\cancel{3x}^{\,0}$

Ex: $F = x^{\overset{4}{3}} e^{7x} + 5 \sin \overset{\omega}{4} x$ $(D^2 + \omega^2)$

$A = (D-7)^4 (D^2+4^2)$

and so on.

Summary: (to find solutions to linear inhomo. differential eqs with constant coeffs)

0) See if F is in the right form

1) Find an appropriate annihilator $A(r)$ for F

2) Factor both $P(r)$ and $A(r)P(r)$

3) Find a general soln to $A(D)P(D)y = 0$

4) Remove solution terms that are common with $P(D)y = 0$ Obtain y_t $\rightarrow$ candidate particular solution.

5) Find constants in y_t by solving $P(D)y_t = F$

Ex

$$(D^2-4D+5)\,y = 8xe^{2x}\cos x \qquad 2+i$$

$P(D) = (D-\lambda)(D-\bar\lambda) \qquad \lambda = 2+i$

$A(D) = (D-2-i)^2(D-2+i)^2$

$A(D) = (D-\lambda)^2(D-\bar\lambda)^2$

lets solve

$A(D)P(D)y = 0 \qquad (D-\lambda)^3(D-\bar\lambda)^3 y = 0$

$e^{2x}\cos x \qquad e^{2x}\sin x$

$xe^{2x}\cos x \qquad xe^{2x}\sin x$

$x^2e^{2x}\cos x \qquad x^2e^{2x}\sin x$

Next step: Find trial solution.

$P(D)y_t = F$

$y_t = Axe^{2x}\cos x + Bx^2e^{2x}\cos x$

$\quad Cxe^{2x}\sin x \qquad Fx^2e^{2x}\sin x$

$P(x) = x^2-4x+5 \qquad \dfrac{4\pm\sqrt{16-20}}{2} = \dfrac{4\pm\sqrt{-4}}{2} = 2\pm i$

$P(D)y = 0$

$e^{2x}\cos x \qquad e^{2x}\sin x$

$(D-\lambda)(D-\bar\lambda)\,xe^{2x}\cos x$

$= (D^2-(\lambda+\bar\lambda)D+|\lambda|^2)\,xe^{2x}\cos x$

$= (D^2-4D+5)\,xe^{2x}\cos x$

$D(xe^{2x}\cos x) = e^{2x}\cos x + 2xe^{2x}\cos x - xe^{2x}\sin x$

$D^2(xe^{2x}\cos x) = 2e^{2x}\cos x - \cancel{e^{2x}\sin x} + 4xe^{2x}\cos x$

$\quad \cancel{+2e^{2x}\cos x} - 2xe^{2x}\sin x - e^{2x}\sin x$

$\quad -2xe^{2x}\sin x - xe^{2x}\cos x$

$= 4e^{2x}\cos x - 2e^{2x}\sin x - 4xe^{2x}\sin x$

$\quad 3xe^{2x}\cos x$

$$D^2-4D+5 = \begin{aligned} & 4e^{2x}\cos x - 2e^{2x}\sin x - 4xe^{2x}\sin x + 3xe^{2x}\cos x \\ & -4e^{2x}\cos x - 8xe^{2x}\cos x + 4xe^{2x}\sin x \\ & 5xe^{2x}\cos x \end{aligned} = -2e^{2x}\sin x$$

Find $P(D)y_t = A(-2e^{2x}\sin x) + B(\cdots) + C(\cdots) + F(\cdots) = 8xe^{2x}\cos x$

Ex: $\underbrace{(D+1)(D^2+9)}_{P(D)}y = 4xe^{-x} + 5e^{2x}\cos x]$ $\qquad A_2(D) = (D-2-i)(D-2+i)$

$A_1(D) = (D+1)^2 \qquad\qquad = (D^2-4D+5)$

$A(D) = (D+1)^2(D^2-4D+5)$

$A(D)P(D)y = 0$ and $P(D)y = 0$

$(D+1)^3(D-\lambda)(D+\lambda)(D^2+9)y = 0$

e^{-x} $\qquad \cos 3x \qquad \sin 3x$

$\cancel{e^{-x}} \quad xe^{-x} \quad x^2e^{-x} \qquad e^{2x}\cos x \quad e^{2x}\sin x \qquad \cancel{\cos 3x} \qquad \cancel{\sin 3x}$

$$y_t = A\,xe^{-x} + B\,x^2e^{-x} + C\,e^{2x}\cos x + F\,e^{2x}\sin x$$

$P(D)y_t = F$

$$(D+1)(D^2+9)(xe^{-x}) = (D^2+9)(e^{-x} - \cancel{xe^{-x}} + \cancel{xe^{-x}})$$
$$= e^{-x} + 9e^{-x} = 10e^{-x}$$

$$(D+1)(D^2+9)(x^2e^{-x}) = (D^2+9)(2xe^{-x} - x^2e^{-x} + x^2e^{-x}) = (D^2+9)(2xe^{-x})$$
$$= (D(2e^{-x} - 2xe^{-x}) + 18xe^{-x})$$
$$= (-2e^{-x} - 2e^{-x} + 2xe^{-x} + 18xe^{-x})$$
$$= -4e^{-x} + 20xe^{-x}$$

$(D^2+9)(D+1)\ e^{2x}\cos x = (\cdots)$

$\qquad\qquad\qquad\quad e^{2x}\sin x = (\cdots\cdot)$

$$A(10e^{-x}) + B(-4e^{-x} + 20xe^{-x}) + C(\cdots) + F(\cdots) = 4xe^{-x} + 5e^{2x}\cos x$$

$20B + \cdots = 4$

My notes on Dan's lecture.

Dan does a couple of computations

1) $(D-s)^m (x^k e^{sx}) = 0$ for $k < m$

and

2) $P(D) = Q(D)(D-s)^m = (D-s)^m Q(D)$

commutation.

Then he talks about annihilators.

$Ly = F$ $L = P(D)$ polynomial differential operator.

$A(D)F = 0$.

$\underbrace{A(D)P(D)}_{S(D)} y = A(D)F = 0$

Any solution to $\underbrace{A(D)P(D)}_{S(D)y} y = 0$ can be found.

We can write a general solution. Thus we can write

$y = Ae^{-\lambda_1 x} + Be^{-\lambda_2 x} + \cdots$

Any $P(D)y = F$ must solve $S(D)y = 0$

Thus for SOME constants $A, B, \ldots$ we can find

a particular solution to $P(D)y = F$.

Then we can use the methods of 6.2 to solve
for the general solution