lec23: L = Linear Differential operator, order n

$= (D^n + a_1 D^{n-1} + \cdots a_n) = P(D)$

Auxiliary polynomial $P(\lambda) = (\lambda-\lambda_1)^{m_1} \cdots (\lambda-\lambda_k)^{m_k}$ multiplicities

$m_1 + \cdots + m_k = n.$

Then for each real root λ_i there are m_i

linearly indep solutions to Homo. equations:

$e^{\lambda_i x}, \; xe^{\lambda_i x}, \; \ldots, \; x^{m_i - 1} e^{\lambda_i x}$

$L = D^2 + 2D + 1$

$P(x) = x^2 + 2x + 1 = (x+1)^2$

Ex: $y'' + 4y' + 4y = 0 \quad y(0) = 1 \quad y'(0) = 4$

$P(D) = D^2 + 4D + 4 \qquad P(x) = x^2 + 4x + 4 = (x+2)^2$

$\lambda = -2$ has multiplicity 2

Linearly indep solns: $e^{-2x} \quad xe^{-2x}$

$(xe^{-2x})'' = (-2xe^{-2x} + e^{-2x})' = 4xe^{-2x} - 2e^{-2x} - 2e^{-2x})$

$= e^{-2x}(4x - 4)$

$(xe^{-2x})'' + 4(xe^{-2x})' + 4xe^{-2x} = e^{-2x}\left(4x - 4 - 8x + 4 + 4x\right) = 0$

General solution:

$y = Ae^{-2x} + Bxe^{-2x} = e^{-2x}(A + Bx)$

If $\lambda = \lambda_i$ is complex with multiplicity m_i

Then $\lambda_j = \overline{\lambda_j} = \bar{\lambda}$ with multiplicity $m_i = m$

By our theorem $\dim(\ker(L-\lambda)) = m$

$\dim(\ker(L-\bar{\lambda})) = m$

So we need $2m$ lin. indep solutions.

let $\lambda = a + ib$ Then there are:

$e^{ax} \cos bx$ $\quad$ $e^{ax} \sin bx$

$e^{ax} x \cos bx$ $\quad$ $e^{ax} x \sin bx$

$\vdots$ $\quad$ $\vdots$

$e^{a} x^{m-1} \cos bx$ $\quad$ $e^{a} x^{m-1} \sin bx$

Ex: $y'' + 6y' + 25y = 0$

$P(x) = x^2 + 6y + 25 = (x+3-4i)(x+3+4i)$

$m = 1$ for λ and 1 for $\bar{\lambda}$

$y(x) = e^{3x}(A\cos 4x + B\sin 4x)$

Ex: $y''' + 2y'' + 3y' + 2y = 0$

$P(x) = x^3 + 2x^2 + 3x + 2$

Try $x = -1$ $\quad P(-1) = -1 + 2 - 3 + 2 = 0$

$x+1$ is a factor. Do LONG DIVISION

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\ \hline
(x+1) & x^3 + 2x^2 + 3x + 2 \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 + 3x \\
 & \quad \underline{x^2 + x} \\
 & \qquad 2x + 2 \\
 & \qquad \underline{2x + 2} \\
 & \qquad\quad 0
\end{array}
$$

$P(x) = (x+1)(x^2 + x + 2)$

Ex: Determine the general solution to

$$(D-3)(D^2+2D+2)^2 y = 0$$

<u>Lessons</u>: To factor auxiliary polynomial

1) Guess roots
2) Look for common factors
3) Use long division.

(End of section titled: "Constant Coefficient Homo. Linear Differential Equations"