lec22:

$$y^{(n)} + a_1(x)\, y^{(n-1)} + \cdots + a_n(x)\, y = F(x)$$

$a_1(x), \ldots, a_n(x), F(x)$ are GIVEN functions.

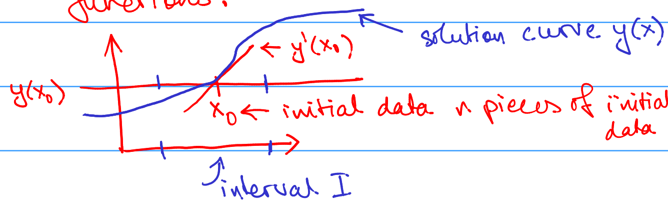
Initial data:

$y(x_0) = y_0$

$y^{(1)}(x_0) = y_1$

$\vdots$

$y^{(n-1)}(x_0) = y_n$



Thm: If $a_1, \ldots, a_n, f$ continuous, then initial value problem has a unique solution on $I \subseteq \mathbb{R}$. that contains x_0.

SOLUTION to the IVP exists and is UNIQUE.

Review: 1) Integrating factor ($n=1$) $\quad y' + p(x)\, y = q(x)$

$$I = e^{\int p(x)\,dx}$$

2) Electrical circuit ($n=2$): $\quad q'' + \frac{R}{L} q' + \frac{1}{LC} q = \frac{1}{L} E(t)$

$q(t)$ was the charge.

→ Full RLC circuit in his lecture (we did an RC circuit)

(We did not do this but apparently Prof Gebadid.) → constant coefficient linear inhomogeneous eqn of ORDER 2.

Derivative operator D:

$$D^n y + a_1 D^{n-1} y + \cdots \; a_n y = F$$

$$\underbrace{(D^n + a_1 D^{n-1} + \cdots + a_n)}_{\text{Linear operator (map)}} y = F$$

$$D = \frac{d}{dx} \qquad D^n = \frac{d^n}{dx^n}$$

$$D^n : \underbrace{C^n}_{n \text{ continuous derivatives.}}(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$$

$L : C^n \to C^0(\mathbb{R})$

$D^2(y_1+y_2) = D^2y_1 + D^2y_2$

$D^2(\lambda y) = \lambda D^2 y$

Ex: $D^2 + \overline{4x}D - \overline{3x} = L$ $4x, 3x$ are continuous fns.

$L : C^2 \to C^0$

Then $L(\cos x) = (\cos x)'' + 4x(\cos x)' - 3x\cos x$

$= -\cos x - 4x\sin x - 3x\cos x = -\cos x - 7x\cos x$

Main Theorem If L is order n of the form

$L = D^n + a_1 D^{n-1} + \cdots a_n(x)$ If $a_1 \ldots a_n, f$ are

continuous on I then $Ly = F(x)$ has a unique

soln for the initial value problem.

$y(x_0) = y_0, \; y'(x_0) = y_1, \ldots$

Homogeneous DE: Have $F=0$. Then, we solve $\boxed{Ly=0}$

$\rightarrow$ L is an n^{th} order linear differential operator.

Thm: $\ker(L)$ is a subspace of C^n of dimension n.

$\ker(L) = \{y \mid Ly = 0\}$ = solution space

$\dim(\ker(L))$ = order of L.

(Remember how we showed that ALL solns of $y''+y=0$ $y \in \operatorname{span}\{\cos x, \sin x\}$ using the UNIQUENESS Theorem.)

Ex: $L = D^2+1$ $\quad Ly=0 \quad y''+y=0$

L is order 2

$\dim(\ker(L)) = 2$ $\quad \operatorname{span}\{\cos x, \sin x\} = \ker(L)$

proved using existence/uniq.

Alternately

$\sin x, \cos x \in \ker L$ where $L = (D^2+1)$

Theorem says $\dim(\ker L) = 2$.

$$\left[W(x) = \det \begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix} = 1 \Rightarrow \sin x \text{ and } \cos x \text{ are independent.} \right.$$

Thus $\operatorname{span}\{\cos x, \sin x\} = \ker L$.

$y''+y=0$, $\cos x$, $\sin x$

$\{\cos x, \sin x\}$ form a basis (using the theorem as long as we show they're indep.

$W(x) = \cos^2 x + \sin^2 x = 1 \neq 0$

$\boxed{\text{Remark:}}$ If L order n and $Ly_1 = \cdots = Ly_n = 0$

Then $W(y_1, \ldots, y_n) \neq 0 \Rightarrow$ independence

AND $W(y_1, \ldots, y_n)(x_0) = 0$ at any point x_0 $\Rightarrow$ dependent.

Again this depends on the EXISTENCE and uniqueness theorem.

$L = D^2 + D - 6$ order 2 $\qquad \dim(\ker L) = 2$

Ex: $y'' + y' - 6y = 0$ Find all solns of the form e^{rx}.

$\dim(\ker(L)) = 2$.

Get $\quad r^2 e^{rx} + r e^{rx} - 6 e^{rx} = 0 \quad \Rightarrow \quad r^2 + r - 6 = 0$

$(r-3)(r+2) = 0$ So 2 possible solutions are e^{3x}, e^{-2x}. The theorem says that these are ALL the solutions of the form e^{rx} since they are linearly indep. (why?)

$\operatorname{span}\{e^{3x}, e^{-2x}\} = \ker(L)$

$y = e^{rx} \qquad y' = r e^{rx} \qquad y'' = r^2 e^{rx}$

$\underbrace{e^{rx}}_{\neq 0}(r^2 + r - 6) = 0$

$$\begin{vmatrix} e^{3x} & e^{-2x} \\ 3e^{3x} & -2e^{-2x} \end{vmatrix} = -2e^{x} - 3e^{x} = -5e^{x} \neq 0.$$

GENERAL SOLUTION is of the form

$y(x) = A e^{3x} + B e^{-2x}$

$y(0) = 1 \qquad y'(0) = 2$ for example

$$\boxed{\begin{aligned} 1 &= A + B \\ 2 &= 3A - 2B \end{aligned}}$$

$y' = 3Ae^{3x} - 2Be^{-2x}$

$y'(0) = 3A - 2B = 2$

$A = \frac{4}{5} \quad B = \frac{1}{5}$

$y = \frac{4}{5} e^{3x} + \frac{1}{5} e^{-2x}$

Thm:

If $y_1 \cdots y_n$ solve $Ly_i = 0$ on I.

and $W(y_1 \cdots y_n)(x_0) = 0$ for some $x_0 \in I$

$\Rightarrow$ $y_1 \cdots y_n$ are DEPENDENT.

Pf: Do it for $n=2$.

$$W(y_1,y_2)(x_0) = \det\begin{pmatrix} y_1(x_0) & y_2 \\ y_1' & y_2'(x_0) \end{pmatrix} = 0$$

$\Rightarrow$ $\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \vec{0}$ has nontrivial solutions (a,b).

Thus $u(x) = ay_1 + by_2$ is a SOLN to the

IVP $u(x_0) = a\,y_1(x_0) + by_2(x_0) = 0$

$u'(x_0) = 0$

BUT $u(x) = 0$ also solves this IVP.

By uniqueness $ay_1(x) + by_2(x) = 0$

$\Rightarrow$ y_1, y_2 dependent.

$W(y_1, \cdots y_n)(x_0) \neq 0$

$\Rightarrow$ $y_1 \cdots y_n$ are independent

BUT if $Ly_1 = Ly_2 \cdots = Ly_n = 0$

$W(y_1 \cdots y_n)(x_0) = 0$ at some x_0, then

they're dependent

Solve In-homogeneous linear DEs with constant coeffs

Recall $A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$] inhomogeneous linear eqs

This has a solution if $b \in$ colspan(A).

Suppose x_1 is one such and x_2 is another

Then $A(x_1 - x_2) = 0 \Rightarrow x_1 - x_2 \in \ker(A)$

So if any one solution is known: x_1 to

$Ax_1 = b$ Then ALL solutions must be of the

form $x_2 = \underbrace{x_1}_{\text{particular solution}} + k$, where $k \in \ker(A)$.

↳ kernel of A

$Ax_1 = b \quad Ax_2 = b$

$\Rightarrow A(x_1 - x_2) = 0$

$(Ay = 0)$ $\quad x_1 - x_2 = k,\ k \in \ker(A)$

$\Rightarrow$ KNOW x_1 ANY OTHER solution

$x_2 = x_1 + k \to$ I KNOW how to find all k.

$Ax_2 = Ax_1 + \cancel{Ak}^{0} = b$

Similarly we have:

If y_0 solves $Ly_0 = F$ ← inhomogeneity

then the GENERAL SOLN. IS OF THE FORM

$$y = \underbrace{y_0}_{\text{particular}} + y_1$$] ← ker L (homogeneous solution)

Ex: $y'' + y = e^x$ Find a general soln.

Try $y = Ae^x$ Then $Ae^x + Ae^x = e^x \Rightarrow A = \frac{1}{2}e^x$

Thus general solution is of the form

$$\boxed{y = \frac{1}{2}e^x + A\cos x + B\sin x}$$

$y'' + y = 0$ (homogeneous eqn)

Ex: $y'' - 2y' - 3y = e^{2x}$

Particular soln of the form Ae^{2x}.

$\frac{d}{dx} e^{2x} = 2e^{2x}$

$4Ae^{2x} - 4Ae^{2x} - 3Ae^{2x} = e^{2x}$

$\Rightarrow A = -\frac{1}{3}$ $\quad y = -\frac{1}{3} Ae^{2x}$ is a particular solution.

Then try to find general soln to

$Ly = 0$ where $L = D^2 - 2D - 3$

We know $\dim(\ker L) = 2$.

Aux. polynomial $P(r) = (r^2 - 2r - 3)$

Try $y = e^{rx}$. Get

$r^2 e^{rx} - 2re^{rx} - 3e^{rx} = 0 \Rightarrow$

$\underbrace{e^{rx}}_{\neq 0}(r^2 - 2r - 3) = 0$

$\Rightarrow r^2 - 2r - 3 = 0 \Rightarrow (r-3)(r+1) = 0$

$r = 3$ or $r = -1$ Give 2 homo. solns

$y = e^{3x}$ and $y = e^{-x}$ (Lin independent)

$\operatorname{span}\{e^{3x}, e^{-x}\} = \ker L$

$y = Be^{3x} + Ce^{-x}$

Thus the general solution is of the form:

$$\boxed{y = \underbrace{Ae^{2x}}_{\text{particular}} + \underbrace{Be^{3x} + Ce^{-x}}_{\text{general solution to homo. equation}}}$$