Lec21

$\mathbb{C}^n$ = Complex vector space.

$= \{ (z_1, \dots z_n) \mid z_i \in \mathbb{C} \}$

Allow all scalars, real and complex.

$= \text{span}\{(1,0\cdots0),(0,1,\cdots)\cdots,(0,\cdots0,1)\}$

Ex: $\underset{2\times2}{Y' = AY}$ let $\boxed{Y(x) = e^{\lambda x}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$

Then

$Y' = \lambda e^{\lambda x}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A e^{\lambda x}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$= e^{\lambda x} A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Gives eigenvalue equation.

$\boxed{(A - \lambda I)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0}$ homogeneous system.

$Bx = \vec{0}$

Nontrivial solution requires $\det(A - \lambda I) = 0$

(otherwise $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the UNIQUE soln.)

(in other words $\text{rank}(A - \lambda I) \neq 2$)

So step 1 : find λ st $\det(A - \lambda I) = 0$

step 2 : Solve homogeneous system

$(A - \lambda I) \cdot C = \vec{0}$ $\rightarrow$ Solve for $\vec{c}$.

We know $\operatorname{rank}(A - \lambda I) < n$. why?

Note that if $\operatorname{rank}(A - \lambda I) = r$ Then

$\ker(A - \lambda I) = n - r$ (we can find this

many unique eigenvectors)

Ex: $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}$

The 2nd matrix has complex eigenvalues

and the lesson is that if v is an eigenvector

for λ, $\bar{v}$ is an eigenvector for $\bar{\lambda}$.

Ex: $\begin{bmatrix} 1 & 1 \\ -3 & 5 \end{bmatrix} = A$. (Find evalues and evectors)

$\det(A - \lambda I) = 0 \quad \det \begin{bmatrix} (1-\lambda) & 1 \\ -3 & 5-\lambda \end{bmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(5-\lambda) + 3 = 0$

$\Rightarrow \quad 5 - 6\lambda + \lambda^2 + 3 = 0$

$\Rightarrow \quad (\lambda - 4)(\lambda - 2) = 0 \quad \left(\text{use } \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$

$\lambda = 4$ and $\lambda = 2$.

Eigenvectors:

$(A - \lambda I)v = 0 \qquad \lambda = 4$

$\begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}_{2\times 2} v = \vec{0}$.

$B^{\#} = \left[\begin{array}{cc|c} -3 & 1 & 0 \\ -3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$-3v_1 + v_2 = 0 \qquad v_2 = t \Rightarrow v_1 = \frac{t}{3}$

$S = \{ t(\frac{1}{3}, 1) : t \in \mathbb{R} \}$

$(\frac{1}{3}, 1)$, $(\frac{5}{3}, 5)$ ... are evectors.

$\lambda=2$ case: $\begin{bmatrix}-1 & 1\\ -3 & 3\end{bmatrix}$

rank$(A-\lambda I) = 1 < 2$

$-v_1+v_2=0 \qquad v_2=t, \quad v_1= t$

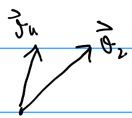
$S = \{t(1,1) : t\in \mathbb{R}\} = \text{span}\{(1,1)\}$

E_2 = eigenspace of $\lambda=2$

$= \text{span}\{(1,1)\}$

E_4 = ei . . . . . $\lambda=4$

$= \text{span}\{(\frac{1}{3},1)\}$



Notice that $\vec{v}_1$ and $\vec{v}_2$ are linearly indep.

$\left[\; Y_1(x) = e^{2x}\begin{pmatrix}1\\1\end{pmatrix} \qquad Y_2(x) = e^{4x}\begin{pmatrix}1/3\\1\end{pmatrix} \;\right]$

$Y'=AY$

Ex: $A = \begin{bmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 12 & -6 \\ -3 & -10-\lambda & 6 \\ -3 & -12 & 8-\lambda \end{bmatrix}$$

(cofactor expansion)

$$= (5-\lambda)\left[(10+\lambda)(\lambda-8) + 72\right]$$
$$- 12\left[3(\lambda-8) + 18\right]$$
$$- 6\left[36 - 3(\lambda+10)\right]$$
$$= (5-\lambda)\left[\lambda^2 + 2\lambda - 8\right] - 36\lambda + 72$$
$$- 36 + 18\lambda$$

↓ factorizing

$$= (5-\lambda)(\lambda+4)(\lambda-2) \underbrace{- 18\lambda + 36}_{-18(\lambda-2)}$$
$$= (\lambda-2)\left[(5-\lambda)(\lambda+4) - 18\right]$$
$$= (\lambda-2)\left[5\lambda + 20 - \lambda^2 - 4\lambda - 18\right]$$

$$= (\lambda-2)\left[2 + \lambda - \lambda^2\right] = (\lambda-2)\left[(2-\lambda)(\lambda+1)\right]$$

1, 2, 3 times

+1

$\lambda = -1$

multiplicity

$$= (-1)(\lambda-2)^2(\lambda+1)^1$$

$\lambda = 2 \quad m_2 = 2$

$\lambda = 1 \quad m_1 = 1$

→ Total degree of character polynomial

$$\begin{bmatrix}5 & 12 & -6\\ -3 & -10 & 6\\ -3 & -12 & 8\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = 2\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}3 & -12 & -6\\ -3 & -12 & 6\\ -3 & -12 & 8\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \vec{0}$$

If you do things right you will get

2 evectors for $\lambda_1 = 2$ and 1 for $\lambda_1 = 1$

Notice that $\dim(E_2) = 2 = m_2$, $\dim(E_1) = 1 = m_1$

$E_2 = \ker(A - 2I)$ $\quad$ $E_1 = \ker(A - I)$

eigenspace

A is thus NON DEFECTIVE

$\#$ of vectors in $M = n = 3$ $\quad (3 \times 3\ A)$

M is linearly independent

<u>Defective</u> : If $A_{n \times n}$ doesnot have n linearly indep

eigenvectors.

<u>Ex:</u>

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$$(\lambda - 1)^{2} = \overbrace{P(\lambda)}^{\text{characteristic polynomial}} . \qquad m_1 = 2 \quad (\lambda = 1)$$

$$(A - \overset{\lambda=1}{1} I)v = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} v = 0$$

$$\Rightarrow v_2 = 0 \quad \text{So} \quad \underbrace{E_1}_{\text{eigenspace of } \lambda=1} = \text{span}\{(1,0)\}$$

$$\boxed{\text{But} \quad \dim(E_1) = 1 < 2 = m_1}$$

A HAS ONLY 1 LIN. INDEP.

eigenvector But $n = 2$ $(A_{2 \times 2})$. So

A is DEFECTIVE.

Ex: Complex evalues)

$$A = \begin{bmatrix} -2 & -6 \\ 3 & 4 \end{bmatrix}$$ $\det(A - I\lambda) = \begin{vmatrix} -2-\lambda & -6 \\ 3 & 4-\lambda \end{vmatrix}$

$= P(\lambda) = (-2-\lambda)(4-\lambda) + 18 = 0$

$\Rightarrow \quad \lambda^2 - 2\lambda - 8 + 18 = 0$

$\Rightarrow \quad \lambda^2 - 2\lambda + 10 = 0$ $\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 10}}{2} = 1 \pm \sqrt{-9}$$

$\lambda = 1 + 3i \quad , \quad \bar{\lambda} = 1 - 3i$] complex roots occur in pairs.

Here will notice a useful TRICK

if v is an evec of λ then $\bar{v}$ is an eval of $\bar{\lambda}$. $Av = \lambda v \Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$

$(A - \lambda I)v = 0$ (eigenvalue equation)

$$\begin{bmatrix} -2-1-3i & -6 \\ 3 & 4-1-3i \end{bmatrix} = B = A - \lambda I$$

simplified above $=$ $\begin{bmatrix} -3(1+i) & -6 \\ 3 & 3-3i \end{bmatrix}$

$\underset{\sim}{\frac{R_1}{-3(1+i)}}$ $\begin{bmatrix} 1 & -6/_{-3(1+i)} \\ 3 & 3-3i \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 3 & 3-3i \end{bmatrix}$

$\frac{\cancel{-6}^{2}}{\cancel{-3}(1+i)} \frac{(1-i)}{(1-i)} = \frac{2(1-i)}{1+1} = 1-i$

$\underset{\sim}{R_2 = R_2 - 3R_1}$ $\begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$ rank $= 1$

$v_1 + (1-i)v_2 = 0.$ $\Rightarrow v_1 = (-1+i)t$

$v_2 = t$ $S = \{(-(1-i)t, t) : t \in \mathbb{R}\}$

$S = \text{span}\{(-1+i, 1)\}$ $\to v$

Eigenvector of $1+3i \equiv (-1+i, 1)$

$\underline{\lambda = 1-3i}$. Trick $Av = \lambda v$ complex conjugate

$$\left.\begin{aligned} \overline{a_{11}v_1 + a_{12}v_2} &= \overline{\lambda v_1} \\ \overline{a_{21}v_1 + a_{22}v_2} &= \overline{\lambda v_2} \end{aligned}\right] Av = \lambda v$$

$$\Rightarrow \begin{aligned} \bar{a}_{11}\bar{v}_1 + \bar{a}_{12}\bar{v}_2 &= \bar{\lambda}\bar{v}_1 \\ \bar{a}_{21}\bar{v}_1 + \bar{a}_{22}\bar{v}_2 &= \bar{\lambda}\bar{v}_2 \end{aligned}$$

a_{11}, a_{12}
a_{21}, a_{22}
are all real

$$\Rightarrow \begin{aligned} a_{11}\bar{v}_1 + a_{12}\bar{v}_2 &= \bar{\lambda}\bar{v}_1 \\ a_{21}\bar{v}_1 + a_{22}\bar{v}_2 &= \bar{\lambda}\bar{v}_2 \end{aligned}$$

$A\bar{v} = \bar{\lambda}\bar{v} \implies \bar{v}$ is an evector of $\bar{\lambda}$

$E_\lambda = \operatorname{span}\{(\underbrace{-1+i}_{v},1)\}$ = eigenspace of λ $\begin{pmatrix}-1+i\\1\end{pmatrix}, \begin{pmatrix}-1-i\\1\end{pmatrix}$

$E_{\bar{\lambda}} = \operatorname{span}\{(\underbrace{-1-i}_{\bar{v}},1)\}$ = " of $\bar{\lambda}$

$E_\lambda \cup E_{\bar{\lambda}} = \operatorname{span}\{v,\bar{v}\} = \operatorname{span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}+\begin{pmatrix}i\\0\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}-\begin{pmatrix}i\\0\end{pmatrix}\right\}$

$= \operatorname{span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$ (with complex scalars)

(General theory 7.2)

Linear independence of evecs.

A has eigenvalues

$\lambda_1 \quad \lambda_2 \quad \cdots\cdots \quad \lambda_m$

$A_{n\times n}$ has m eigenvalues.

$m \leq n$.

$\{x \mid (A-\lambda_1 I)x=0\}$] $E_{\lambda_1} = \ker(A-\lambda_1 I)$, $E_{\lambda_2} = \ker(A-\lambda_2 I)\ldots$

$\underbrace{= \text{span}\{v_{11}\cdots v_{1k_1}\}}_{\text{Eigenspace of } \lambda_1}$ $\quad \text{span}\{v_{21}\cdots v_{2k_2}\} \quad \text{span}\{v_{m1}\cdots v_{km}\}$] associated eigenspace

For each eigenspace I find a basis of e-vectors.

all vectors are linearly indep. $\leftarrow$ $\{v_{11},\ldots v_{1k_1}, \; v_{21}\cdots \underset{M}{\overset{}{}} v_{2k_1},\ldots\ldots, v_{m1},\ldots v_{mk_m}\}$

$M = \{v_{11},\ldots v_{1k_1}, \; v_{21}\cdots v_{2k_1},\ldots\ldots, v_{m1},\ldots v_{mk_m}\}$

NOT DEFECTIVE if we can find a set of n lin. indep. eigenvectors for A.

$|M| = n$ if not defective.

<u>Prop</u>

The $\overset{\dim E_{\lambda_i}}{\overset{\shortparallel}{\text{nullity}(A-\lambda_i I)}} \leq$ multiplicity of λ_i in the char. polynomial.

$$\boxed{\dim E_{\lambda_i} \leq m_i}$$

$\shortparallel$

# of basis vectors in each eigenspace is $\leq$ than the multiplicity of the eigenvalue.

$\det(A-\lambda I) = P(\lambda) = (\lambda-\lambda_1)^{m_1}\cdots\cdots(\lambda-\lambda_m)^{m_m}$

n^{th} order polynomial λ^n

RHS

we must have

$m_1 + m_2 + \cdots\cdots + m_m = n$

Theorem:

A is nondef. iff for all its eigenvalues,
dim eigenspace = multiplicity of the eigenvalue
in the char polynomial

$$\dim E_{\lambda_i} = m_i \qquad i = 1, \dots, k$$

Dan's examples

Ex $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}$, Ex $A = \begin{bmatrix} -7 & 0 \\ -3 & -7 \end{bmatrix}$

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \quad \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2)+1 = 0$$

$$\lambda^2 - 6\lambda + 8 + 1 = 0 \qquad \lambda^2 - 6\lambda + 9 = 0 \qquad (\lambda-3)^{2} = 0 \quad m_3 = 2$$

$$E_3 = \ker(A-3I) = \{x : (A-3I)x = 0\} \qquad \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} x = \vec{0}$$

$$\Rightarrow \quad x_1 - \underbrace{x_2}_{\text{free}} = 0 \quad = \{t(1,1) : t \in \mathbb{R}\} = \text{span}\{(1,1)\}$$

$$\dim(E_3) = 1 < m_3 = 2 \quad \Rightarrow \quad A \text{ is DEFECTIVE}$$

A simple(r) criterion for indep evecs.

When do we have n independent eigenvector for $A_{n\times n}$?

One way for this to happen is if all eigenvalues have multiplicity 1

Prop: If $A_{n\times n}$ and all its eigenvalues have multiplicity 1 (SIMPLE) then its eigenvectors form a basis for $\mathbb{R}^n$ (NON DEFECTIVE)

Cor: If $A_{n\times n}$ has n distinct eigenvalues then is non defective.

$m_1 = m_2 \cdots m_n = 1$

$m_1 + m_2 + \cdots = n \Rightarrow m = n$.

$\Rightarrow$ n distinct eigenvalues.

$\det(A - I\lambda_i) = 0 \qquad i = 1, \dots, n$

$\Rightarrow (A - I\lambda_i)v = \vec{0}$ does not a unique solution.

$\Rightarrow \exists$ a nontrivial (non zero) v st

$(A - I\lambda_i)v = 0 \Rightarrow \dim(E_{\lambda_i}) \geq 1$

$1 \leq \dim(E_{\lambda_i}) \leq m_i = 1$

$\Rightarrow$ A has n independent evectors

$\Rightarrow$ A is non DEFECTIVE.