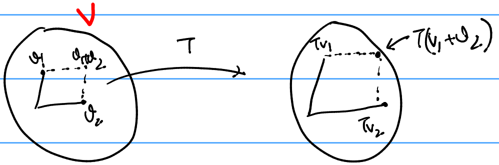
Dans lecture

lec19

$T: \overbrace{V}^{\mathbb{R}^2} \longrightarrow \overbrace{W}^{\mathbb{R}^2}$ $\quad (V,W)$ vector spaces

$\underbrace{T(\overbrace{v_1+v_2}^{V})}_{W} = \overbrace{Tv_1}^{W} + \overbrace{Tv_2}^{W}$] linearity (additivity)



(2) (scaling)

$T(\lambda v) = \lambda Tv$ $\quad$ set $\lambda = 0$ to get

$T(\underbrace{0_v}_{\text{zero vector } V}) = 0_w$.] 0 vector in W.

Equivalent

(1),(2) $\Longleftrightarrow$ $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$

$\forall\, \overbrace{a,b}^{\mathbb{R}},\ \overbrace{v_1,v_2}^{\in V}$.

Induction $\boxed{T(a_1v_1 + \cdots + a_nv_n) = a\,Tv_1 + \cdots + a_nTv_n.}$

Suppose $\{e_1 \cdots e_n\}$ basis for V.

Suppose $\{Te_1 \cdots Te_n\} = \{u_1 \cdots u_n\} \in W$.

Then $T(\overbrace{a_1e_1 + \cdots + a_ne_n}^{\text{expand any } v \text{ in this basis.}}) = a_1Te_1 + \cdots a_nTe_n$

$= \underbrace{a_1u_1 + \cdots + a_nu_n}_{\text{image can be written in terms of } u_i}$

Lesson: enough to know how basis transforms.
(Only need to know images of $Te_1, \ldots, Te_n$).

Checking whether or not something is a linear transformation.

1) $T(A) = A^T$
2) $T(p) = p'$
3) $Tx = \underbrace{A}_{m \times n} \underbrace{x}_{\in \mathbb{R}^n}$.
4) $TA = A^2$ (for matrices)
5) $T(p) = p + p'' - 2$
6) Given $T(1) = x+1$, $T(x) = x^2 - 1$, $T(x^2) = 3x+2$

Find T. (Recall enough to know how basis transforms).

$u_1 = x+1 \qquad u_2 = x^2 - 1 \qquad u_3 = 3x+2$

$$T(a + bx + cx^2) = a u_1 + b u_2 + c u_3$$
$$= a(x+1) + b(x^2-1) + c(3x+2)$$

IMPORTANT: Given any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ we can FIND D st

$Tx = Dx$ where $D_{m \times n}$.

(Matrix representation of Linear transformation)

Relies on

$$T(a_1 \underset{\text{standard basis}}{e_1} + \cdots + a_n e_n) = a_1 Te_1 + \cdots a_n Te_n$$

$$= a_1 u_1 + \cdots + a_n u_n$$ linear comb of vectors.

$$\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\text{represents } a_1 e_1 + \cdots + a_n e_n} = a_1 u_1 + \cdots + a_n u_n$$

So you simply choose $c_1 = u_1, c_2 = u_2 \cdots$!

D CONSISTS OF $\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$ arranged in columns!

Ex: $T(\overbrace{x_1, x_2, x_3}^{\mathbb{R}^3}) = \overbrace{(x_3 - x_1, -x_1, 3x_1 + 2x_3, 0)}^{\mathbb{R}^4}$

$(x_1, x_2, x_3) \in \mathbb{R}^3$ D is a 4×3 matrix.

$u_1 = Te_1$ $u_2 = Te_2$ $u_3 = Te_3$

$T\underbrace{(1,0,0)}_{e_1} = (-1, -1, 3, 0)$ $T\underbrace{(0,0,1)}_{e_3} = (1, 0, 2, 0)$

$T\underbrace{(0,1,0)}_{e_2} = (0, 0, 0, 0)$

$$D = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 1,\ x_2 = 0\ x_3 = 0$

$T(1,0,0) = (0-1, -1, 3+0, 0)$

$= (-1, -1, 3, 0)$

$T: \mathbb{R}^3 \to \mathbb{R}^4$ is represented by

$Dx \quad , x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_3 \\ -x_1 \\ 3x_1 + 2x_3 \\ 0 \end{bmatrix}$$

<u>Ex 5.15</u> $C^k(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : \overset{f \text{ has}}{k}$ continuous derivatives, f is also continuous$\}$.

$T : C^2 \to C^0$ defined by $Ty = \underbrace{y''}_{\substack{\uparrow \\ \text{continuous}}} + \underbrace{y}_{\substack{\uparrow \\ \text{cont.}}} (= 0)$

Show T is linear:

$y_1, y_2 \in C^2 \quad \alpha, \beta \in \mathbb{R}$

$T(\alpha y_1 + \beta y_2) = (\alpha y_1 + \beta y_2)'' + (\alpha y_1 + \beta y_2)$

$= \alpha(y_1'' + y_1) + \beta(y_2'' + y_2) = 0$

$= \alpha T(y_1) + \beta T y_2$

<u>5.1.6</u> $S : M_{2\times2} \to M_{2\times2}$ with $S\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a - 2b & 0 \\ 3a + 4d & a + b - c \end{bmatrix}$. Show that S is linear.

$S\left(\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right)$

Matrix addition.

$= S\left(\begin{bmatrix} \alpha a + \beta p & \alpha b + \beta q \\ \cdot & \cdot \end{bmatrix}\right)$

definition of S

$= \begin{bmatrix} \alpha a + \beta p - 2(\alpha b + \beta q) & 0 \\ 3(\alpha a + \beta b) + 4(\alpha d + \beta s) & \cdots \end{bmatrix}$

Matrix addition

$$= \begin{bmatrix} \alpha a - 2\alpha b & 0 \\ 3\alpha a + 4\alpha d & \cdots \end{bmatrix}$$

$$+ \begin{bmatrix} \beta p - 2\beta q & 0 \\ \cdots & \cdots \end{bmatrix}$$

definition of S

$$= \alpha S \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \beta S \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

Ex 5.1.15 Let T be defined as $T: \mathbb{R}^4 \to \mathbb{R}^3$

$T(x_1, \dots x_4) = (2x_1 + 3x_2 + x_4, \; 5x_1 + 9x_3 - x_4,$
$4x_1 + 2x_2 - x_3 + 7x_4)$

Enough to look at action of T on unit vectors

$e_1, \dots e_4$ since

$$T(a_1 e_1 + \dots + a_4 e_4) = a_1 T e_1 + \dots + a_4 T e_4$$

As matrix multiplication

$$= \left.\begin{array}{c} \\ \\ \end{array}\right\}\underbrace{\begin{bmatrix} Te_1 & \cdots & Te_4 \\ & & \end{bmatrix}}_{\text{4 columns}} \begin{bmatrix} a_1 \\ \vdots \\ a_4 \end{bmatrix}$$

3 rows → dim of the image space of T

4 columns] dim of the domain of T.

$$D = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 5 & 0 & 9 & -1 \\ 4 & 2 & -1 & 7 \end{bmatrix}$$

Ex $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$T(\underbrace{1,0,0}_{e_1}) = (4,5)$ $T(\underbrace{0,1,0}_{e_2}) = (-1,1)$

$T(\underbrace{2,1,-3}_{u_3}) = (7,-1)$

Q: Represent T as a matrix multiplication.

I know Te_1, Te_2. To figure out Te_3

You sort of need the basis to be the unit basis

If I could find $a,b,c \in \mathbb{R}$ st

$$e_3 = ae_1 + be_2 + cu_3 \quad \bigstar 2$$

$$\underbrace{Te_3} = a\underbrace{Te_1}_{\text{given.}} + b\underbrace{Te_2}_{\text{given}} + c\underbrace{Tu_3}_{\text{known}}$$

follows from linearity

I can figure out Te_3

to write T as a matrix. So lets find

$$T(0,0,1) = T\left(a(1\,0\,0) + b(0\,1\,0) + c(2,1,-3)\right)$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ \underbrace{0}_{e_1} & \underbrace{0}_{e_2} & \underbrace{-3}_{u_3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{e_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right] \Rightarrow \quad c = -\frac{1}{3} \quad b = \frac{1}{3}$$

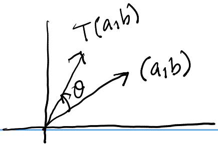
$$a = \frac{2}{3}$$

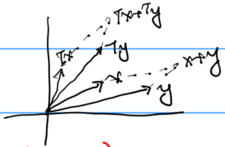
$$Te_3 = \frac{2}{3}\begin{bmatrix}4\\5\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}7\\-1\end{bmatrix}$$

$$= \begin{bmatrix}0\\ \frac{10}{3}\end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix}4 & -1 & 0\\ 5 & 1 & \frac{10}{3}\end{bmatrix}$$

Rotation in the plane by θ



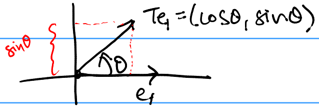


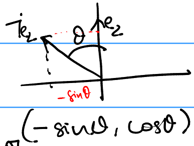
rotation by a fixed angle θ.

$T(x+y) = Tx + Ty$

We haven't defined rotation in the plane mathematically as yet or shown that it is linear.

We know it's enough to define it on e_1 and e_2.





Then $\left[\, Te_1 \;\vdots\; Te_2 \,\right] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

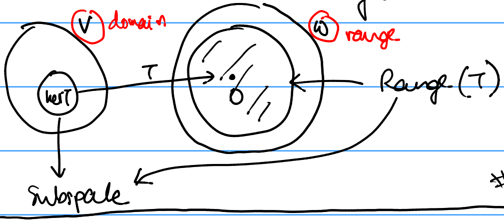
(Anticlockwise rotation)

(T is linear, enough to specify T on a basis $\{e_1, e_2\}$)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cos\theta - 3\sin\theta \\ 2\sin\theta + 3\cos\theta \end{bmatrix}$$

If $\theta = \frac{\pi}{4}$ $= \begin{bmatrix} -1/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}$

2nd lecture: kernel and range.



\# of cols of A.

$$\boxed{\underbrace{\dim(\ker(T))}_{\text{null}(A)} + \underbrace{\dim(\text{Ran}(T))}_{\dim(\text{colspace}(A)) = \text{ran}(A)} = \boxed{\dim V}}$$

Rank-Nullity Theorem

If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad T = Ax$

$\text{Range}(T) = \text{colspace}(A)$

$\ker(T) = \text{nullspace}(A)$

Then Dan does proofs of these theorems Not just for $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, but for any finite dim vector space.

General Problem type: Given $T: V \longrightarrow W$

Find
1) $\ker(T)$ (Find a basis)
2) $\text{Ran}(T)$ (Find a basis)

T is linear $T0_V = 0_W$

$\ker(T) = \{x \in V \mid Tx = 0\}$

If $T: \mathbb{R}^n \to \mathbb{R}^m$ Then $Tx = D_{[m \times n]} \, x_{[n \times 1]}$

$\ker(T) = \{x \in \mathbb{R}^n \mid Dx = 0\}$ = solution space to the homo. eqns

$\operatorname{ran}(T) = \{Ty \mid y \in V\}$

In step 3: Start with a basis for $\ker(V) = \{v_1, \dots v_k\}$

Then COMPLETE IT TO get a basis for V

$\{v_1 \dots v_k, v_{k+1} \dots, v_n\}$

THEN. $Tv_{k+1}, \dots Tv_n$ is a basis for RANGE (T).

Another option: Start with any basis $\{v_1, \dots v_n\}$

$\text{span}\{Tv_1, \dots Tv_n\} = W \to$ reduce $\{Tv_1 \dots Tv_n\}$ to a basis

__Ex:__ $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{(a-b+d)}_{\text{linear}} + \underbrace{(-a+b-d)}_{\text{linear}} x^2$

$T: M_{2\times 2} \longrightarrow \mathbb{P}_2$ easy to check that T is linear

$\ker(T) = \{v : Tv = 0\} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} a-b+d = 0 \\ -a+b-d = 0 \end{matrix} \right\}$

Choose b, c, d free $\boxed{a = b-d}$.

linearly independent

$\ker(T) = \left\{ b\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

split

$= \left\{ \begin{pmatrix} b-d & b \\ c & d \end{pmatrix} : b, c, d \in \mathbb{R} \right\}$

COMPLETE THE Basis: find a vector V that's not in this span. Just pick

$v = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (a \neq b-d)$

$$\begin{pmatrix} b-d & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} -d & 0 \\ 0 & d \end{pmatrix}$$

$$\ker(T) = \text{span}\underbrace{\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}}_{\text{basis.}}$$

$$\dim(\ker(T)) = 3$$

$$\dim(\ker(T)) + \underbrace{\dim(\text{ran}(T))}_{1} = \dim M_{2\times 2} = 4$$

If $\{a_1\ a_2\ a_3\ a_4\}$ is a basis for V, then

any vector $T(x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4)$

$$= \underbrace{x_1 \overset{0}{\overline{Ta_1}} + x_2 \overset{0}{\overline{Ta_2}} + x_3 \overset{0}{\overline{Ta_3}} + x_4 Ta_4}$$

$$\text{span}\{Ta_1, Ta_2, Ta_3, Ta_4\}$$

$$= \text{ran}(T)$$

$a_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $a_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Find a_4 that is indep. of $a_1\ a_2\ a_3$ and then $a_1\ a_2\ a_3\ a_4$ will form a basis.

$$a_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} b-d & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\in \text{span}\{a_1\ a_2\ a_3\}$$

Ta_1 Ta_2 Ta_3 all must map to the 0 polynomial since a_1 a_2 a_3 $\in$ ker (T).

$$Ta_4 = T\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = (2-1+1) + (-2+1-1)x^2$$
$$= 2-2x^2 = 2(1-x^2).$$

ran (T) = span $\{1-x^2\}$ dim(ran (T)) = 1

$Tv = 1 - x^2$ So $\text{span}\{(1-x^2)\} = \text{range}(T)$.

degree 2 polynomials

Ex: $T : P_2 \longrightarrow P_1(\mathbb{R})$

$P_2 = \text{span}\{1, x, x^2\}$
$\Rightarrow \text{ran}(T) = \text{span}\{T1, Tx, Tx^2\}$

$T(ax^2+bx+c) = (a+b) + (b-c)x$

Find $\text{ker}(T)$, $\text{range}(T)$.

In this Dan does it differently:
Take the standard basis $\{1, x, x^2\}$ and use the fact that $\text{span}\{T1, Tx, Tx^2\}$ $= \text{range}(T)$.

$\text{ker}(T) = \left\{ ax^2+bx+c \;\middle|\; \begin{array}{l} a+b=0 \\ b-c=0 \end{array} \right\}$

Choose c to be free $\Rightarrow$ $b = c$, $a = -b = -c$

1 dimensional.

$= \{c(-x^2+x+1) \mid c \in \mathbb{R}\}$ $\dim(\text{ker}(T)) = 1$

$\dim(\text{ker}(T)) + \underbrace{\dim(\text{ran}(T))}_{2} = \dim(P_2) = 3$

$a=0, b=0, c=1$ $\quad a=0, b=1, c=0$

$P_2 = \text{span}\{1, x, x^2\}$ $\text{ran}(T) = \text{span}\{T1, Tx, Tx^2\}$

$= \text{span}\{-x, 1+x, x\} = \text{span}\{x, 1+x\}$

continuous fns with 2 cont. derivatives

cont. fns.

Ex: $T : \underbrace{C^2(\mathbb{R})} \to \underbrace{C^0(\mathbb{R})}$ defined by

$T(f) = f'' + f$. Find ker(T).

$\ker(T) = \{ f : f'' + f(x) = 0 \ \forall x \}$

We know the general soln is $A\cos(x) + B\sin(x)$

so $\ker(T) = \operatorname{span}\{\cos x, \sin x\}$.

We know that the general solution to this

$T(\cos x) = (\cos x)'' + \cos x = (-\sin x)' + \cos x$

$= -\cos x + \cos x = 0$

$T(\sin x) = 0$.

We know that T is linear and so $A\cos x + B\sin x$ is a solution.

How do you show that all solutions are of the form $A\cos x + B\sin x$?

(This is in Chp 1 of the textbook and was UNIQUENESS OF THE INITIAL VALUE PROB.)

$\ker(T) = \operatorname{span}\{\cos x, \sin x\}$

Ex: $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \end{bmatrix}$

matrix corresponding to T

Find $\ker(T)$, $\operatorname{ran}(T)$

$$A^\# \sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 is free $= t$, $x_3 = s$

$x_1 = 2t - 5s$

$\ker T = \{ (2t-5s, t, s) : s,t \in \mathbb{R} \} = \{ (2t, t, 0) + (-5s, 0, s) \}$

$= \operatorname{span} \{ (2,1,0), (-5,0,1) \}$

Recall $\operatorname{ran}(T) = \operatorname{span}\{ Te_1 \ Te_2 \ Te_3 \}$ "complete e_1, e_2 to form a basis of $\mathbb{R}^3$"

But $A = [Te_1 \ Te_2 \ Te_3]$

$\Rightarrow \operatorname{ran}(T) = \operatorname{colspace}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$

$$\underbrace{\dim(\ker T)}_{2} + \underbrace{\dim(\operatorname{ran} T)}_{1} = \underbrace{3}_{\text{dim of the domain } (\mathbb{R}^3) \text{ of } T}$$

To find the $\ker(A)$: $Ax = 0$ (find all solutions) $\operatorname{RREF}(A^\#)$. $\operatorname{ran}(A)$: $\operatorname{colspace}(A)$ using $\operatorname{RREF}(A)$.

To find e_3 by "completing" e_1 and e_2 to a basis of $\mathbb{R}^3$. $\operatorname{span}(e_1, e_2) = \{ (2t-5s, t, s) \}$

$(2,1,0)$

$(5,1,0) \neq (2t-5s, t, s)$

$$\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5-2 \\ -10+4 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\operatorname{ran}(T) = \operatorname{span} \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Ex: Find $\ker(S)$, $\operatorname{ran}(S)$ for $S: M_{2\times2} \to M_{2\times2}$

$$S(A) = A - A^T$$

$$\ker(S) = \{A \in M_{2\times2} : A = A^T\}$$

$$= \operatorname{span}\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$$

$$S\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \qquad S\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

$$S\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \qquad S\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

$$= \underline{\operatorname{ran}(S)} = \operatorname{span}\left\{\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right\}$$

$\dim = 1$

$$\dim(\ker) = 3 \qquad \dim(M_{2\times2}) = 4.$$

Ex: $T(a+bx) = (2a-3b) + (b-5a)x + (a+b)x^2$

$T: \overset{\text{domain}}{P_1} \longrightarrow P_2$

Find $\ker(T)$

$$\begin{aligned} 2a - 3b &= 0 \\ b - 5a &= 0 \\ a + 3b &= 0 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 1 & -5 & 0 \\ 1 & 3 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 0 \\ 1 & -5 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix} = \text{RREF}$$

. 2 variables a and b

$\Rightarrow a = 0,\ b = 0 \qquad \ker(T) = \{0\}$ $\hookrightarrow$ 0 polynomial

$\dim(\ker(T)) = 0$

$0 + \dim(\operatorname{ran}(T)) = \dim(P_1) = 2$

$\Rightarrow \dim(\operatorname{ran}(T)) = 2$

$\operatorname{ran}(T) = \{2a - 5ax + ax^2 - 3b + bx + bx^2\}$

$= \operatorname{span}\{\overbrace{2 - 5x + x^2}, \overbrace{-3 + x + x^2}\}$

$P_2 = \operatorname{span}\{1, x\}$ $\quad a=1, b=0 \quad a=0, b=1$

$\operatorname{ran} T = \operatorname{span}\{T1, Tx\}$