4.9 Rank Nullity Theorem

Consider $Ax = 0$ where $A_{m \times n}$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

equivalent to

$\Leftrightarrow$ $RREF(A)\,x = 0$

$$A^{\#} = \begin{array}{c} \text{rank}(A) \left[\begin{array}{c|c} \text{non zero rows} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \text{zero rows} & \begin{matrix} \vdots \\ 0 \end{matrix} \end{array} \right] \end{array} \Bigg\} \text{Total of } n \text{ rows.}$$

Lets solve for $Ax = 0$ and remember

$$\text{nullity}(A) = \dim(\underbrace{\text{nullspace of } A}_{\ker(A)})$$

$$\ker = \text{kernel} = \underbrace{\{x \mid Ax = 0\}}_{\text{solution space.}}$$

For the zero rows, choose free variables.

You can solve for the variables in the nonzero rows.

Turns out: # of free variables = dim(ker(A))

Recall # of free variables = $n - r$,
where n = # of unknown variables = # of columns of A

You pick as a free variable all those that ARE NOT ASSOCIATED WITH A leading 1.

# of leading 1 = rank(A)

# of vars not associated with a leading 1
= # total vars − rank(A)
= n − rank(A)

rank(A) + null(A) = n = total # of cols

Ex $A^{\#} = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & \\ 3 & 4 & -1 & 2 & 0 \\ -1 & -2 & 5 & 4 & \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & \\ 0 & 1 & -7 & -7 & 0 \\ 0 & 0 & 0 & 0 & \end{array}\right]$

$\underbrace{}_{A}$

free variables $x_2 = t_3$ $x_3 = t_4$

$x_2 = 7t_3 + 7t_4$ $x_1 = -9t_3 - 10t_4$

$$S = \left\{ \begin{pmatrix} -9t_3 - 10t_4 \\ 7t_3 + 7t_4 \\ t_3 \\ t_4 \end{pmatrix} \right\}$$

$$= \left\{ t_3 \begin{pmatrix} -9 \\ 7 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -10 \\ 7 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} -9 \\ 7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -10 \\ 7 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\dim(\ker(A)) = 2$

rank(A) = 2 rank(A) + null(A) = 2+2

= 4 = n.

Rem: If $\operatorname{rank}(A) = n$, unique sol to $Ax = 0$ is $x = \vec{0}$.

Rem For $Ax = b$, a solution exists iff $b \in \operatorname{colspace}(A)$

It has a UNIQUE sol: $Ax_1 = Ax_2 = b$
$\Rightarrow A(x_1 - x_2) = 0$. The homo. sys. has a unique sol $x_1 - x_2 = 0$ if $\operatorname{rank}(A) = n$. Thus $x_1 = x_2$.

If $\operatorname{rank}(A) < n$ Then you can again say that if x_0 and y are 2 solutions $x_0 - y \in \ker(A)$

So ANY solution can be written as $y = x_0 + v$ where $v \in \ker(A)$.

Ex: $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{bmatrix} = A \qquad b = \begin{bmatrix} 3 \\ 10 \\ -4 \end{bmatrix}$

Find the GENERAL SOLUTION.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 3 \\ 3 & 4 & -1 & 2 & 10 \\ -1 & -2 & 5 & 4 & -4 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 0 & 1 & -7 & -7 & 1 \\ 0 & -1 & 7 & 7 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 & 3 \\ 0 & 1 & -7 & -7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + 2x_3 + 3x_4 = 3$

$x_2 - 7x_3 - 7x_4 = 1$

Simply set $x_3 = 0$

$x_4 = 0$

$x_1 + x_2 = 3$

$x_2 = 1$

$\Rightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ~~is~~ x_p is a

PARTICULAR solution.

To find a general solution we know

If $Av = 0$ then $v = t_3 \begin{pmatrix} -9 \\ 7 \\ 1 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} -10 \\ 7 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ $y = x_p + v$ is the GENERAL SOL.

17. Show that a 3×8 matrix A must have $5 \leq \text{null}(A) \leq 8$. Find an ex. of a 3×8 matrix A with $\text{null}(A) = 5$ and one with $\text{null}(A)=8$

$\begin{bmatrix} & & & \\ \end{bmatrix}$ $0 \leq \text{rank}(A) \leq 3$

$\Rightarrow 5 \leq \text{null}(A) \leq 8$

$\begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 \cdots \end{bmatrix}$ If $\text{null}(A) = 5$ $\text{rank}(A) = 3$

If $\text{null}(A) = 8$ $\text{rank}(A) = 0$?

Ok, from now on we're going to really accelerate, so be prepared for a wild ride