Lec 17

Example 4.6.16

Determine a basis for $\{ x \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \}$

$\left\{ \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \end{array}\right] \right.$ already in RREF

A

2 free variables α, β.

$[\, x_2 = \alpha \quad x_3 = \beta \qquad x_1 = -2\alpha + \beta$ (solving for x_1)

$$S = \left\{ \begin{pmatrix} -2\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -2\alpha \\ \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \\ 0 \\ \beta \end{pmatrix} \right\}$$

$$= \left\{ \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

$= \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ by defn.

Then $\dim(S) = 2$ since $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are indep.

(Finding a basis for a subspace)

4.6.18 $S = \{ A \in M_{2\times 2} \mid A^T = A \}$

Set of symmetric matrices.

Find dim(S). Ie find a basis for S.

$$\underbrace{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{general } 2\times 2 \text{ matrix}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^T \Rightarrow b = c$$

$$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Clear that these vectors are independent

$\Rightarrow \dim(S) = 3.$

$(\dim(M_{2\times 2}) = 4)$

Ex 6.6.19 $C^n(I) = \{ f : I \to \mathbb{R} \mid f, f^{(1)}, \cdots f^{(n)}$ are continuous$\}$

Show $C^n(I)$ is ∞ dimensional.

Enough to find functions $\{f_1, \cdots, f_k\} \in C^n$ that are linearly indep. for any k.

Why? Suppose $\dim(C^n) = M < \infty$. (FINITE)

Then any set of $M+1$ vectors MUST be lin. dependent. So if you find $M+1$ vectors that are lin. indep then $\Rightarrow \dim(C^n) \neq M$.

Take $S = \{1, x, x^2, \cdots, x^M\}$ $M+1$ vectors.

$$W(S) = \det \begin{bmatrix} 1 & x & x^2 & x^3 & \cdots & x^M \\ 0 & 1 & 2x & 3x^2 & \cdots & Mx^{M-1} \\ 0 & 0 & 2 & 6x & & \\ & & & & & \\ \cdots & & & & & M! \end{bmatrix} \neq 0 \quad \forall x$$

$\Rightarrow$ S is linearly independent.

$\Rightarrow$ In Prof. Geba's slides.

$\Rightarrow C^n$ is not finite dimensional.

Rem: Will eventually show that solutions of linear DEs will be subspaces of $\mathbb{C}^n$ that are FINITE DIM.

4.8 Row and Column space.

Take $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ want to find RREF of A. We do $R_1 = R_1 - R_2$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2-1 & 3-2 \\ 1 & 2 \end{bmatrix}$$

In general

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overbrace{\begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \end{bmatrix}}^{A}$$

rows of my matrix A.

$$= \begin{bmatrix} a\vec{r}_1 + b\vec{r}_2 \\ c\vec{r}_1 + d\vec{r}_2 \end{bmatrix}$$

Elementary row ops

ROW OPERATIONS $\Leftrightarrow$ Left Multiplication

So we can see the operation of reducing A to RREF(A) as a series of multiplication

$$\text{RREF}(A) = \overbrace{L_1 L_2 \cdots L_k}^{} A$$

where each matrix is invertible and SQUARE.

So L has an inverse

and we can write

$$\underbrace{L^{-1}}_{\substack{\downarrow \\ \text{linear}}} RREF(A) = A$$

linear combination of RREF(A) = rows of A

$\Rightarrow$ span (RREF) = rowspace (A)

(= span(rows of A))

Further we can show that the rows of RREF(A) are lin. indep. Thus they ~~are a basis for~~ the rowspace of A.

(span rowspace of A and are lin indep)

$\Rightarrow$ form a basis.

Ex 4.8.4 Find a basis for

span(S) , $S = \{(1,2,3,4),(4,5,6,7),(7,8,9,10)\}$

vectors in S "live in" $\mathbb{R}^4$ (4-tuples)

$S \subseteq \mathbb{R}^4$

skipped steps here

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\text{row space}(A) = \text{span}(S) = \text{span}\left((1,2,3,4),(0,1,2,3)\right\}}_{\text{statement of the theorem}}$$

span (RREF(A))

statement of the theorem

Why does our prescription for discovering a basis for colspace (A) work?

let $E = RREF(A)$ ← E is already in RREF

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(e_1 and e_2) span the colspace of E

e_1 e_3 e_2 e_4] It's clear that e_1 and e_2 are indep. and because of the form of the RREF,

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$E \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = c_1 e_1 + \cdots + c_4 e_4 = 0$$

RREF(A)

But the solution space of $E\vec{c} = \vec{0}$ and $A\vec{c} = 0$ are the same. This is because $E = RREF(A)$ and you're not really changing the eqs by performing elementary row ops.

$(c_1, 0, c_3, 0)$ is a solution to $Ec = 0 \Leftrightarrow (c_1, 0, c_3, 0)$ is a sol to $A\vec{c} = 0$

So if $c_1 \vec{e}_1 + c_3 \vec{e}_3 = 0 \Leftrightarrow c_1 \vec{a}_1 + c_3 \vec{a}_3 = 0$

So $\vec{a}_1$ and $\vec{a}_3$ must be linearly indep.

Since $e_3 \in \text{span}\{e_1, e_2\}$ thus $\exists$

c_1, c_2, c_3 st $c_1 e_1 + c_2 e_2 + c_3 e_3 = 0$

$\Rightarrow c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

So $a_3 \in \text{span}\{a_1, a_2\}$ and so on.

Thus colspace $(A) = \text{span}\{a_1, a_2\}$

Basis for colspace (A)

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} = A \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = RREF(A)$$

leading 1

$\Rightarrow$ colspace(A) $=$ span $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Note that colspace (A) $\neq$ span $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Important difference between colspace (A) and rowspace (A).

Find the subspace of $\mathbb{R}^3$ spanned by

$$\{ \underbrace{(1,-1,2)}_{v_1} \quad \underbrace{(5,-4,1)}_{v_2} \quad \underbrace{(7,-5,-4)}_{v_3} \}$$

We know $v_1\ v_2\ v_3$ are lin. indep if

$$\det\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq 0$$

$$\det \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 5 & -4 & 1 \\ 7 & -5 & -4 \end{bmatrix}}_{A} = \overbrace{1(21) + 1(-20-7) + 2(-25+28)}^{\text{cofactor exp}}$$

$= 21 - 27 + 6 = 0 \Rightarrow v_1, v_2, v_3$ not indep

Next, find RREF(A)

$$\begin{bmatrix} 1 & -1 & 2 \\ 5 & -4 & 1 \\ 7 & -5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -9 \\ 0 & 2 & -18 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

$S = \text{span}\{(1,0,-7), (0,1,-9)\}$.

My notes on Dan's lectures

Lec 17 If $V = \text{span}\{v_1, v_2, \ldots, v_n\}$ and $v_1, \ldots v_n$ are lin. DEP. Then you can remove at least one vector (say v_n) and still have $V = \text{span}\{v_1, \ldots v_{n-1}\}$

So you can reduce to $\{v_1 \ldots v_m\}$ that is linearly independent st $\text{span}\{v_1 \ldots v_m\} = V$

This is a basis, $\dim(V) = m$.

Suppose S has m vectors

Then S is lin. indep $\Leftrightarrow$ S spans V.

Consequences: • If S has $< m$ vectors, S is not a basis.

• Enough to check indep or spanning property for a collection of m vectors.

Ex: Is S a basis for $\mathbb{R}^3$?

$S = \{(1,-1,1), (2,5,-2), (3,11,-5)\}$

$|S| = 3$ so enough to check indep.

S indep $\Leftrightarrow$ rank $(S) = 3 \Leftrightarrow \det(S) \neq 0$

Thm: $W \subseteq V$ subspace $\Rightarrow \dim(W) \leq \dim V$ and any basis of W can be extended to basis for V.

Cor If $W \subseteq V$ and $\dim(W) = \dim V$ then $W = V$. (finite dim).

Ex: $V = P_2(\mathbb{R})$ $W = \{p \in V;\ p(1) = 0\}$

(Find a basis for W)

Row Space and Column Space

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mn} \end{bmatrix}$$

rowspace $(A) = \operatorname{span}(R_1 \cdots R_m)$

colspace $(A) = \operatorname{span}(C_1, \cdots C_n)$

$\dim(\text{rowspace}(A)) = \dim(\text{colspace}(A)) = \text{rank}(A)$

Thm: Basis of rowspace $(A) =$ non zero rows in RREF(A)

" colspace $(A) =$ columns of A with leading ones in RREF(A)

Ex: $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 6 & -3 & 5 \\ 1 & 2 & -1 & -1 \end{bmatrix}$

$\text{RREF}(A) \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \text{rank}(A) = 2$

and $\text{rowspace}(A) = \text{span}\{(1,2,-1,3),(0,0,0,1)\}$

$\text{colspace}(A) = \text{Span}\{(1,3,1),(3,5,-1)\}$

Important Theorem: Rank-Nullity

rank = dim(rowspace) = dim colspace

nullity = $\dim(\{x \mid Ax = 0\})$

rank (A) + nullity (A) = # of rows of A

Easy $\quad Ax = 0 \quad \begin{array}{r} \text{rank}\{ \\ \\ \end{array}\left[\begin{array}{c|c} \text{non zero rows} & 0 \\ \hline \text{zero rows} & \end{array}\right] \quad A^{\#}$ in RREF

# of zero rows = # of free variables

= dimension of null space of A.

$\underbrace{\#\text{ of nonzero rows}}_{\text{rank}} + \underbrace{\#\text{zero rows}}_{\text{nullity}} = n$

Important $Ax = b \iff b \in$ colspace of (A)

☆ Good for examples.

Rank-nullity can be used to show that

$\text{rank}(A) = n \iff \text{nullity}(A) = 0 \iff x = \vec{0}$ unique

solution of $Ax = \vec{0}$.

Q: If A is $m \times n$ and rowspace (A) $=$ colspace (A), then $m = n$. True or False?