lec16 Wronskian

$$W(\underbrace{f_1, f_2, f_3}_{3 \text{ fns}})(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}$$

If $W(f_1 f_2 f_3)(x) \neq 0$ at even one point then $\{f_1, f_2, f_3\}$ are lin independent.

If $W(f_1, f_2, f_3)(x) = 0 \quad \forall \; x \in I$

Then NOTHING can be said about lin. dep. or indep.

Prof. Gelsa did 2 examples: one where $W(x) = 0$ and f_1, f_2, f_3 are independent and one where f_1 and f_2 are dependent.

Wronskian example:

$$f_1 = x^3 \qquad f_2 = \begin{cases} 2x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$$

Show that f_1 and f_2 are linearly indep.

If $a f_1(x) + b f_2(x) = 0 \Rightarrow a = b = 0$

($\Rightarrow$ lin. independence). Suppose $a \neq 0$

$$\Rightarrow f_1(x) = \frac{-b f_2}{a} = k f_2(x) \quad k \in \mathbb{R}.$$

$\forall x \in I$

$x=1$

$$\Rightarrow 1^3 = k\,2(1)^3 \Rightarrow k = \frac{1}{2}$$

$$x=-1 \Rightarrow (-1)^3 = k\,-(-1)^3 \Rightarrow k=-1$$

$\Rightarrow$ contradiction. $\Rightarrow$ f_1, f_2 are independent.

☆ Ex. 4.5.22

[a) $f_1 = e^x$ $f_2 = x^2 e^x$

b) $f_1 = x$ $f_2 = x + x^2$ $f_3 = 2x - x^2$

[c) $f_1 = x^2$ $f_2 = \begin{cases} 2x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$

Determine whether or not functions are independent on $I = (-\infty, \infty)$

$$w(f_1, f_2) = \det \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \det \begin{vmatrix} e^x & x^2 e^x \\ e^x & 2xe^x + x^2 e^x \end{vmatrix} = e^x e^x (2x + x^2) - e^x e^x x^2$$

$$= e^{2x} [2x]$$

At $x = 1$, $w(1) = e^2 \cdot 2 \neq 0$

$\Rightarrow$ f_1 and f_2 ARE INDEPENDENT

IMPORTANT: when $\{f_1, \dots, f_n\}$ are solutions of a DE then $W(x) = 0$
$\Rightarrow f_1, \dots, f_n$ are DEPENDENT.

(Partial converse to the theorem we just used)

4.6 Basis and dimension.

Ex: $\mathbb{R}^2$. $\{\underset{e_2}{(0,1)}, \underset{e_1}{(1,0)}\} = S$. Then

$\text{span}(S) = \mathbb{R}^2$, S is called a <u>spanning set.</u>

S is independent $\Rightarrow$ S is a basis.

<u>Basis</u>: Is a set S st S spans the vector space <u>and</u> S is independent.

$\dim(V)$ = # of vectors in a basis

(finite dim V)

space of polynomials

<u>Std basis for P_2</u> $= \{ax^2+bx+c : a,b,c \in \mathbb{R}\}$

$p_1 = 1, \quad p_2 = x, \quad p_3 = x^2$

Of course $\text{span}\underbrace{\{p_1, p_2, p_3\}}_{S} = P_2$

But why is S a basis? B'cos S is independent!

$$W(p_1\ p_2\ p_3) = \det\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix}$$

$= 2$ (upper Δ)

$\neq 0 \Rightarrow$ INDEP.

1) Bases do not need to be finite.
2) All bases have same # of vectors if there exists a basis with a finite # of vecs
3) Any spanning set must have at least the same # of vectors as a basis.

Ex: 1.2.16 More in the textbook.

DE: $y'' + \omega^2 y = 0$.

will show ALL solutions are of the form $y(x) = c_1 \cos \omega x + c_2 \sin \omega x$.

Recall uniqueness thm: let

$$\underbrace{a_0(x)}_{\text{coefficients.}} y^{(n)} + y^{(n-1)} a_1(x) + \cdots \; a_n(x) y = F(x)$$

on an interval I. Suppose $\{a_0, a_1, \ldots a_n\}$ are continuous. Then the initial value problem.

$y(0) = y_0$, $y^{(1)}(0) = y_1$, $\ldots$ $y^{(n-1)}(0) = y_{n-1}$

has a UNIQUE sol.

$\Rightarrow$ $y'' + \omega^2 y = 0$ has a UNIQUE soln to any initial value problem

$y(0) = a$ $\quad y'(0) = b$ $\quad (a,b) \in \mathbb{R}$

So suppose $y(x) = f(x)$ is any soln.

We must show that $f(x)$ of the form

$a\cos\omega x + b\sin\omega x$

But consider the IV problem

$$y(0) = f(0) \qquad y'(0) = f'(0)$$

$$y(x) = f(0)\cos\omega x + \frac{f'(0)}{\omega}\sin(\omega x) \quad \star 1$$

is the unique soln on I

$$\Rightarrow f(x) = f(0)\cos(\omega x) + \frac{f'(0)}{\omega}\sin(\omega x)$$

$\Rightarrow \star 1$ is a solution. But so is $f(x)$

But $f(x) = \star 1$ by the uniqueness theorem.

$$S = \{ a\cos(\omega x) + b\sin(\omega x) : a, b \in \mathbb{R} \}$$
$$= \operatorname{span}\{ \cos\omega x, \sin\omega x \}$$

$$W(x) = \det \begin{bmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{bmatrix} = \omega(\cos^2\omega x + \sin^2\omega x)$$

$(\cos^2\omega x + \sin^2\omega x = 1)$ TRIG Identity. $\quad = \omega \neq 0$

$\Rightarrow S$ is independent

$\Rightarrow \dim(\text{solution space}) = 2$

with continuous coefficients

Thm: n^{th} order LINEAR DE on $\mathbb{R}$ has n dimensional solution space lec 16 (linear independence)

Notes on video

Linear dependency. Dan does an example with 3 matrices in $M_{2\times 2}$ and discovers that they are linearly dependent.

2 shortcuts: put $v_1, \dots, v_k$ in a matrix and find the rank of $A = [v_1 \dots v_k]$

suppose $v_i \in \mathbb{R}^n$

If rank(A) $< k$ $\Rightarrow$ linearly dep.

rank(A) $= k$ and $k \leq n$ $\Rightarrow$ linearly indep

rank(A) $= k$ and $k > n$ $=$ lin. dep.

rank(A) $\leq n < k$ in the third case.

Since rank(A) $\leq$ min(rows, columns)

If $k = n$ (same # of vectors as the dimension of the space) then

$\text{rank}(A) = n \iff \det(A) \neq 0$ $\quad$ ★1

(We showed $\text{rank}(A) = n$ iff A is invertible)

and ★ follows from that fact.

If $k < n$ then go back to original method:

compute the rank(A) and see if $\text{rank}(A) = k$.

Then Dan does an example with 3 vectors in $\mathbb{R}^3$ and finds $\det(A)$.

<u>2nd shortcut</u> $V = \{f : I \to \mathbb{R}\}$ (space of fns)

Suppose have 3 functions $\{\sin x, \cos x, \tan x\}$

and ask, are they <u>LI</u> (linearly indep. on)

a $[-\pi, \pi]$.

Need them to be at least twice differentiable.

and $f^{(2)}$ to be continuous

Wronskian

$W(f_1, f_2, f_3)(x)$

$$= \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix}$$

$$= \det \begin{bmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ -\sin x & -\cos x & 2\tan x \sec^2 x \end{bmatrix}$$

Try $W(0) = \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$= 0$ (No good.

Try $x = \pi/4$ $\sin(x) = \frac{1}{\sqrt{2}}$ $\cos x = \frac{1}{\sqrt{2}}$ $\tan x = 1$

$\sec x = \sqrt{2}$

$$W(\pi/4) = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 2 \\ -1/\sqrt{2} & -1/\sqrt{2} & 4 \end{vmatrix}$$

$$= 1\left(\frac{-1}{2}-\frac{1}{2}\right) - 2\underbrace{\left(-\cancel{\frac{1}{2}}+\frac{1}{2}\right)}_{0} + 4\left(\frac{-1}{2}-\frac{1}{2}\right)$$

$= -1-4 = -5 \neq 0 \Rightarrow$ Not linearly dependent.

What if $W(x) = 0 \ \forall\, x \in I$? Then NO conclusion. Cannot say linearly dependent.

Ex: $\{f_1 = 1, f_2 = x, f_3 = 2x^2 - 1\}$

$$W(x) = \begin{vmatrix} 1 & x & 2x^2-1 \\ 0 & 1 & 4x \\ 0 & 0 & 4 \end{vmatrix} = 4 \neq 0$$

★ Could do proof of Wronskian theorem.

Lec3 in Prof Gebas video picks up at the Wronskian.

Ex = $\{ f_1 = 1, f_2 = x, f_3 = -1+2x \}$.

Dan does an example with $f_1 = x^3$, $f_2 = \begin{cases} 2x^3 & x \geq 0 \\ -x^3 & x < 0. \end{cases}$

something like that and get the Wronskian = 0

everywhere.

Lesson: Wronskian = 0 everywhere does not

mean anything

Dan asks us to show that f_1 and f_2 are

linearly independent.

Suppose not; then $\exists k$ st

$$f_1(x) = k f_2(x)$$

$$x^3 = k \begin{cases} 2x^3 \\ -x^3 \end{cases} \quad \forall x$$

$5^3 = k\, 2(5^3)$ and $-5^3 = k(-5)^3$

$\Rightarrow k = \frac{1}{2}$ and $k = 1$

Thus this is a contradiction.

Basis

If span $\{v_1, \cdots v_n\} = V$ then $\{v_1, \cdots v_n\}$ called a spanning set. If $\{v_1, \cdots v_n\}$ are also independent then $\{v_1, \cdots v_n\}$ called BASIS.

Any spanning set can be trimmed to form a basis

Ex: $V = \{f : I \longrightarrow \mathbb{R},\ f'' \text{ exists and is continuous}\}$

Then $\exists$ S which is an ∞ basis.

Three more examples: $P_3(\mathbb{R})$, $M_{3\times 2}$ matrices and $\mathbb{R}^4$. Dan covers all 3 of these and writes their basis down.

Prob: If $\{v_1, \cdots v_n\}$ basis for V then any subset of V with more than n vectors must be lin. dependent.

$\{u_1, \dots u_m\}$ $m > n$ span V. wlog $u_i \neq \vec{0}$.

$a_1 u_1 \dots a_m u_m = v_1$

$a_1 (b_{1i} v_1 \dots b_{in} v_n) + \dots = v_1$

$\Rightarrow (\sum a_i b_{i1} - 1) v_1 + \dots (\sum_i a_i b_{mi}) v_m = 0$

$$\underbrace{\begin{bmatrix} b_{11} & \dots & b_{1n} \\ b_{m1} & & b_{mn} \end{bmatrix}^T}_{\text{rank } \leq n.} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So can repeat this argument for $v_1, \dots v_n$.

In RREF B has at least 1 row of zeros. Thus for at least one choice of $v_1, \dots v_n$ we will get an impossible equation.

It's easy to see this with some thought. A proof is little messier.

<u>Cor</u> All bases in a finite dim vector space have the same # of elements.] defn of dim.