lec15 MTH165

Start with how

1) Polynomials form a vector space

2) Solutions DEs form a vector space

There are in lec 14. Continuing with spans of vectors.

Ex: let $M_2(\mathbb{R})$ be the space of Real 2×2 matrices

Show span $(A_1\ A_2\ A_3\ A_4) = M_2$

where $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$A_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

We have to show that a general matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as a LINEAR COMBINATION of $A_1 \cdots A_4$

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Have to find

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_4 & x_2 + x_3 + x_4 \\ x_3 + x_4 & x_5 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Written as a matrix

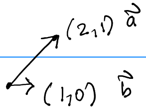
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= a \\ x_2 + x_3 + x_4 &= b \\ x_3 + x_4 &= c \\ x_4 &= d \end{aligned}$$

Has a solution $\forall \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ iff $\det \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}}_{C} \neq 0.$

But C is upper $\Delta \Rightarrow \det(C) = 1$

Look at example 4.4.6 about polynomials from the text book.

Linear Span



We showed that any vector in $\mathbb{R}$ can be written as

$(x, y) = c_1 \vec{a} + c_2 \vec{b}$ since $\det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \neq 0$

We say that $\text{span}(\vec{a}, \vec{b}) = \mathbb{R}^2$

What if you have the vectors $\vec{a} = (1,0,0)$ and $\vec{b} = (0,1,0)$ in $\mathbb{R}^3$. There is no way that $\text{span}(\vec{a}, \vec{b}) = \mathbb{R}^3$.

So $\text{span}(\vec{a}, \vec{b}) \subset \mathbb{R}^3$

In general in a vector space V

$\text{span}\{v_1, \dots, v_k\} = \{ a_1 \vec{v}_1 + \dots + a_k v_k : a_1, \dots a_k \in \mathbb{R} \}$

$=$ set of all linear combinations of $\{v_1, \dots, v_k\}$

<u>Ex</u>: DE: $y'' + y = 0$ has 2 solutions: $y_1 = \cos(x)$ and $y_2 = \sin(x)$.

Let $V = \text{span}\{y_1, y_2\}$ In fact $V =$ set of all solutions

and we have seen before that V is a subspace of the set of functions.

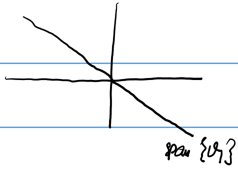
Theorem Let $v_1, v_2, \dots, v_k \in V$. Then $\text{span}\{v_1, \dots, v_k\}$ is a subspace of V.

Pf: Easy to check closure under addition and scalar mult.

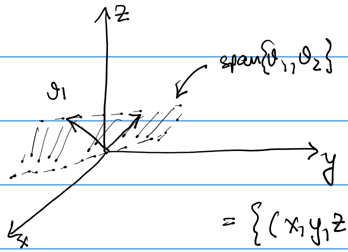
Ex: $V = \mathbb{R}^2$ and $v_1 = (-1, 1)$

$$\text{span}\{v_1\} = \{c(-1,1) : c \in \mathbb{R}\}$$

$$= \{(x,y) : -x = y\}$$



Ex: $V = \mathbb{R}^3$ $v_1 = (1\ 0\ 1)$ $v_2 = (0, 1, 1)$



$$\text{span}\{v_1, v_2\} = \{a(1\ 0\ 1) + b(0\ 1\ 1) : a, b \in \mathbb{R}\}$$

$$= \{(a, b, a+b) : a, b \in \mathbb{R}\}$$

$$= \{(x, y, z) : z = x + y\}$$

This is the eqn of a plane.

Let P_2 = polynomials of degree 2.

$p_1 = 1 + 3x \quad p_2 = x + x^2$

find span $\{p_1, p_2\} = \{ a + 3ax + bx + bx^2 : (a,b) \in \mathbb{R}^2 \}$

$= \{ a + x(3a+b) + bx^2$

Recall that $P_2 = \text{span}\{1, x, x^2\}$ so we definitely

must have span $\{p_1, p_2\} \subsetneq P_2$

strict subset

For example $1 + x^2 + 6x \notin \text{span}\{p_1, p_2\}$

4.5 Linear dependence and independence.

Minimal spanning sets:

$\{(0,1), (1,0)\}$ and $\{(0,1), (1,0), (1,1)\}$

both span $\mathbb{R}^2$ but $\{(0,1), (1,0)\}$ is minimal.

We cannot have a single vector span $\mathbb{R}^2$ so 2 vectors is the minimal # of vectors needed to span $\mathbb{R}^2$.

Consider $v_1 = (1,1,1)$

$v_2 = (2,3,4)$

$v_3 = 2v_1 + 3v_2$

Then $\det\left(\begin{bmatrix} v_1 \\ v_2 \\ 2v_1 + 3v_2 \end{bmatrix}\right) = \begin{vmatrix} v_1 \\ v_2 \\ 3v_2 \end{vmatrix}$

$\sim \begin{vmatrix} v_1 \\ v_2 \\ \vec{0} \end{vmatrix} = 0 \Rightarrow$ Coplanar! Volume of parallelepiped is 0

Here we had $\{v_1, v_2, v_3\}$ where v_3 was a linear combination of v_1 and v_2

v_1 and v_2 are not <u>colinear</u> $\Leftrightarrow$ $v_1 \neq c v_2$

So v_1 and v_2 span a plane.

So $\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3\}$ and

$\{v_1, v_2\}$ are minimal.

<u>Theorem</u>: If $\{v_1, \dots, v_k\}$ st v_k can be written as a linear combination of the others

$$v_k = a_1 v_1 + a_2 v_2 + \cdots \; a_{k-1} v_{k-1}$$

$$\Leftrightarrow \quad a_1 v_1 + a_2 v_2 + \cdots \; a_{k-1} v_{k-1} - v_k = 0$$

Then $\text{span}\{v_1, \dots, v_{k-1}\}$

$= \text{span}\{v_1, \dots, v_k\}$

<u>LINEAR DEPENDENCE</u>

So we know can reduce the # of vectors in the span of $\{v_1, \dots, v_k\}$ if $\exists \{a_i\}$

st $\quad a_1 v_1 + \cdots + a_k v_k = 0 \quad$ st

not all a_k are 0

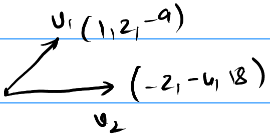
Suppose $a_1 \neq 0$ then

$$v_1 = \frac{-a_2 v_2 \cdots - a_k v_k}{a_1}$$

Or v_1 written as linear combination of others!

LINEAR INDEPENDENCE

Ex 4.5.8



Are v_1 and v_2 linearly independent?

NO!

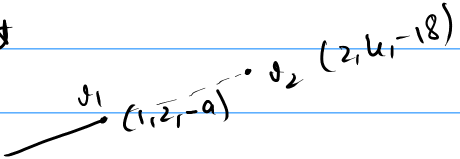
If not, $\exists\, a,b$ $a v_1 + b v_2 = 0$, suppose $a \neq 0$

then $v_1 = \underbrace{-\frac{b}{a}}_{\lambda} v_2$ $\quad (1,2,-9) = (-2\lambda, -4\lambda, 18\lambda)$

$1 = -2\lambda$ AND $2 = -4\lambda$ AND $-9 = 18\lambda$

Achieved with $\lambda = -\frac{1}{2}$

In fact



Lie on same line.

4.5.11 Example about matrices.

4.5.13 $v_1 = (1,2,3)$ $v_2 = (-1,1,4)$

$v_3 = (3,3,2)$ $v_4 = (-2,-4,-6)$

Determine a set of vectors that spans the same subspace of $\mathbb{R}^3$ as span$\{v_1, v_2, v_3, v_4\}$