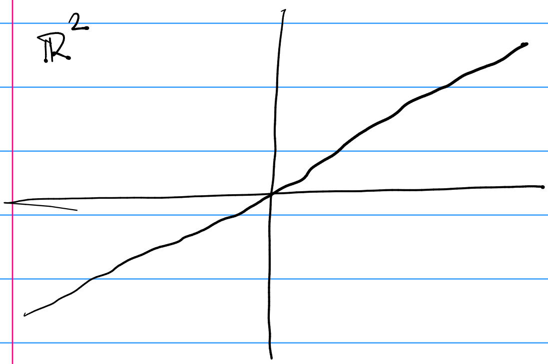
lec14

Subspaces: Theorem that says enough to check closure under addition and scalar multiplication.

Example of a linear system.

$\mathbb{R}^2$



$S = \{(x,y) \mid y = kx\}$ for some fixed k

Note

1) $(x,y), (m,n) \in S$ then

$(x+m, y+n)$ also in S since

$$\frac{y+n}{x+m} = \frac{kx+km}{x+m} = k$$

2) $r(x,y) = (rx, ry) \in S$ $\forall r$, including 0

1), 2) are called closure under addition and scalar multiplication.

Example: $S = \{x \in \mathbb{R}^2 \mid x = (r, -3r+1, r) \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}$.

Example: subspace of real symmetric matrices.

$S = \{ A \in M_n : A^T = A\}$

Ex: $V = \{ f : [a,b] \to \mathbb{R}\}$

$S = \{ f \in V : f(a) = 0\}$

Example: $S = \{0\}$ the trivial subspace.

Thm: $Ax = 0$ Solution set S is always a subspace

Pf: suppose $A_{m \times n}$ and x_1, x_2 are solutions

then $A(kx_1) = k(Ax_1) = \vec{0}$ $\forall k$

$A(x_1 + x_2) = Ax_1 + Ax_2 = \vec{0} + \vec{0} = \vec{0}$

Def: Nullspace of a Matrix. $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

$=$ Solution space of $A_{m \times n}$

Ex: Suppose $A_{2\times 2}$ and we consider

$Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Is S a subspace

if $\det(A) \neq 0$

No since $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin S$.

<u>Thm</u>: $V = \{ f : [a,b] \longrightarrow \mathbb{R} \}$ $\quad I = [a,b]$

$C^k(I) = \{ f \in V \mid f^{(1)}, \dots, f^{(k)}$ exist and are continuous on $I \}$

$\vec{0}(x) \in C^k$, $f+g \in C^k$ since $(f+g)^{(i)} = f^{(i)} + g^{(i)}$

$(kf)^{(i)} = k f^{(i)}$

Ex: $V = \{f : I \to \mathbb{R}\}$

$C^k(I) = \{f \in V : f^{(1)}, \dots, f^{(k)}$ all exist$\}$

Thm: $y'' + a_1(x)y' + a_2(x)y = 0$

S = solution set on I

S is a vector space.

In fact it turns out that $\dim(S) = 2$

and $\exists$ 2 functions $y_1(x), y_2(x)$ st

any solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$y_1(x)$ and $y_2(x)$ are NONproportional.

Pf: If y_1 and y_2 are solutions:

to show $y_1 + y_2$ are also solutions.

$(y_1 + y_2)^{(k)} = y_1^{(k)} + y_2^{(k)}$ (Linearity)

$\Rightarrow$ $(y_1+y_2)'' + a_1(y_1+y_2)' + a_2(y_1+y_2)$

$= y_1'' + a_1 y_1' + a_2 y_1$

$+ y_2'' + a_1 y_2' + a_2 y_2 = 0 + 0$

Similarly if $y \in S$, then so is ky for any $k \in \mathbb{R}$

The space of polynomials of degree k

$V_2 = \{ p(x) : p(x) = ax^2 + bx + c \text{ for any } a, b, c \}$

V_2 is a vector space where

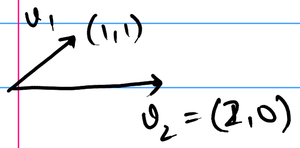
$$(p_1 + p_2)(x) := p_1(x) + p_2(x)$$

$$(kp_1)(x) := k p_1(x)$$

Ex: $p_1(x) = 2x^2 \quad p_2(x) = 3x$

$$(p_1 + p_2)(x) = 2x^2 + 3x \in V_2$$

4.4 Spanning set.



what set of vectors can I express as

$$a v_1 + b v_2 \qquad a, b \in \mathbb{R} \,?$$

Ex: $a=1, b=1, \quad (1,1) + (2,0) = (3,1)$

Turns out that you can express any vector in $\mathbb{R}^2$ this way.

Let $x = (x_1, x_2) \in \mathbb{R}^2$. To show $\exists\, a, b$ st

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If $\det(A) \neq 0$ then $\begin{pmatrix} a \\ b \end{pmatrix} = A^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\det(A) = -2$

$$C_M = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} \qquad \operatorname{adj}(A) = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1/2 & -1/2 \end{bmatrix}$$

Ex: determine whether

$$v_1 = (1,-1,4) \quad v_2 = (-2,1,3) \quad v_3\ (4,-3,5)$$

span $\mathbb{R}^3$.

As before we only need to check if

$$\det([v_1\ v_2\ v_3]) \neq 0.$$

$$\begin{bmatrix} 1 & -1 & 4 \\ -2 & 1 & 3 \\ 4 & -3 & 5 \end{bmatrix} = 1(5+9) + 1(-10-12) + 4(6-4)$$

$$= 14 - 22 + 8 = 0$$

Another way to see this

$$\begin{bmatrix} 1 & -2 & 4 \\ -1 & 1 & -3 \\ 4 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix}$$

$$A^{\#} = \begin{bmatrix} 1 & -2 & 4 & x_1 \\ -1 & 1 & -3 & x_2 \\ 4 & 3 & 5 & x_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 4 & x_1 \\ 0 & -1 & 1 & x_1+x_2 \\ 0 & 11 & -11 & x_3+4x_1 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 4 & x_1 \\ 0 & 1 & -1 & -x_1 - x_2 \\ 0 & 11 & -11 & x_3 + 4x_1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 4 & x_1 \\ 0 & 1 & -1 & -x_1 \\ 0 & 0 & 0 & x_3 + 4x_1 + 11x_1 + 11x_2 \end{array}\right]$$

For this to be consistent need

$$15x_1 + 11x_2 + x_3 = 0$$

So we can only solve for the system when

we are in $\hat{S} = \{ (x_1, x_2, x_3) \mid 15x_1 + 11x_2 + x_3 = 0 \}$

This is a SUBSPACE (equation of a plane)

$\Rightarrow$ v_1, v_2, v_3 lie in a plane.

<u>Thm</u>: span $(v_1, v_2, \ldots v_n)$ ~~is~~ $\mathbb{R}^n$ for $v_i \in \mathbb{R}^n$

iff $\det([v_1 \ldots\ldots v_n]) \neq 0$