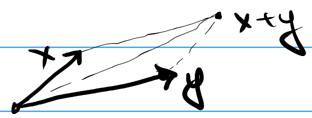
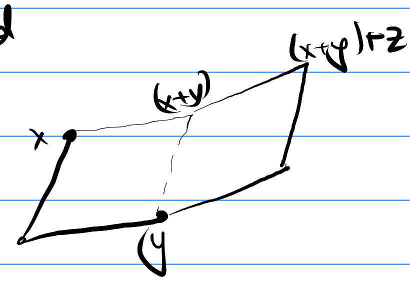
Lec 13 Vector spaces.



Vector addition defined using parallelogram law.

Obviously $x+y = y+x$ and



Similarly $x+\vec{0} = x$ and $x - x = \vec{0}$

where $\vec{0}$ is the vector having 0 magnitude and arbitrary direction.

Props : 1) Assoc.
2) Comm.
3) $\vec{0}$ vector
4) existence of unique inverse.

<u>Multiplication</u> : Easy to define integer multiples of $\vec{x}$.

$1\vec{x} = \vec{x}$, $s(t\vec{x}) = (st)\vec{x}$, $r(\vec{x}+\vec{y}) = r\vec{x}+r\vec{y}$

unit or mult. identity (mult. assoc.)

$(s+t)\vec{x} = s\vec{x} + t\vec{x}$

<u>Precise definition of vectors</u>

$\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$. = {set of vectors}

Define: $v + w = (v_1 + w_1, v_2 + w_2)$

$kv = (kv_1, kv_2)$ ↑ component of vector

$0 = (0, 0)$ obviously

Above properties satisfied

In $\mathbb{R}^3$, we write $v = (v_1, v_2, v_3)$

Define $\hat{i} = (1,0,0)$ $\hat{j} = (0,1,0)$ $\hat{k} = (0,0,1)$

(Unit vectors) $v = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

All of these ideas generalize to $\mathbb{R}^n$, $n \geq 1$

$\mathbb{R}^1, \mathbb{R}^2, \ldots$ all examples of VECTOR SPACES

(collections of vectors)

General vector spaces: Need V and F a field ($\mathbb{R}$ or $\mathbb{C}$)

1) If u, v vectors in V, $u+v$ also in V] Closure
2) $u, v \in V$, $kv \in V$ for all k
3) $u+v = v+u$
4) $u+(v+z) = (u+v)+z$
4a) $u+0 = u$
4b) $u-u = 0$

5) $1u = u$
6) $s(tu) = (st)u$
7) $r(\vec{x}+\vec{y}) = r\vec{x}+r\vec{y}$
8) $(s+t)\vec{x} = s\vec{x}+t\vec{x}$

Ex: $V = \{A_{2\times 2} \text{ matrices}\}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & 0 \end{bmatrix}$$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$$

Ex: Space of functions on $[0,1]$.

$$V = \{f : [0,1] \longrightarrow \mathbb{R}\}$$

$(f+g)(x) = f(x)+g(x)$ $\quad (\sin+\cos)(x) = \sin(x)+\cos(x)$

$\vec{0}$ vector $\vec{0}(x) = 0 \quad \forall x \in I$

"Inverse" function $(-f)(x) = -f(x) \Rightarrow f(x)+(-f)(x) = 0$

Unit $(1f)(x) = 1f(x)$

Distribution $[r(f+g)](x)$

$$= r[(f+g)(x)] = r(f(x)+g(x))$$

$$= rf(x) + rg(x)$$

and so on.

General Properties of Vectors

1) 0 vector is unique.

Pf: Suppose $\exists$ 0_1 and 0_2 then $v+0_1 = v+0_2 = v$

Use $v=0_2$ = $0_2+0_1 = 0_2 \Leftrightarrow \underbrace{0_1+0_2}_{\text{commutation}} = 0_2$

and use $v+0_2 = v$ to get $0_1+0_2 = 0_1 \Rightarrow 0_1 = 0_2$

2) $0\vec{u} = \vec{0}$ $\forall$ $\vec{u} \in V$.

$$0\vec{u} = (0+0)\vec{u} \quad \text{using distribution}$$
$$= (0\vec{u}) + (0\vec{u})$$

There is a $-(0\vec{u})$ so we have

$$-0\vec{u} + 0\vec{u} = \vec{0} = \underbrace{(-0\vec{u} + (0\vec{u}))}_{\text{association}} + 0\vec{u}$$

$\vec{0}$ vector

Therefore