3.3 (Cofactor expansion)

Another example: $\begin{bmatrix} 0 & 3 & -1 & 0 \\ 5 & 0 & 8 & 2 \\ 7 & 2 & 5 & 4 \\ 6 & 1 & 7 & 0 \end{bmatrix} = A$

$$\det(A) = 0 \begin{vmatrix} 0 & 8 & 2 \\ 2 & 5 & 4 \\ 1 & 7 & 0 \end{vmatrix} - 5 \begin{vmatrix} 3 & -1 & 0 \\ 2 & 5 & 4 \\ 1 & 7 & 0 \end{vmatrix}$$

$$+ 7 \begin{vmatrix} 3 & -1 & 0 \\ 0 & 8 & 2 \\ 1 & 7 & 0 \end{vmatrix} - 6 \begin{vmatrix} 3 & -1 & 0 \\ 0 & 8 & 2 \\ 2 & 5 & 4 \end{vmatrix}$$

Cofactor expansion Thm proof

$$\det(A) = \sum_{\sigma} a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n} P(\sigma)$$

Which terms in the sum above contain a_{11}?

Clearly $\sigma_1 = 1$. Thus we are looking at terms of the form

$$\sum_{\sigma_1 = 1} P(\sigma) a_{11} a_{2\sigma_2} \cdots a_{n\sigma_n} = a_{11} \sum a_{2\sigma_2} \cdots a_{n\sigma_n} P(\sigma)$$

where $(\sigma_2, \dots, \sigma_n)$ is a permutation of $\{2, \dots, n\}$

But THIS sum is simply the determinant of the minor M_{11}

It picks up an extra sign when you consider a_{k1} because if $\sigma_k = 1$ then we remove the inversions $(\sigma_1 1)$ $(\sigma_2 1) \cdots$ $(\sigma_{k-1} 1)$ and hence we pick up a sign if $k-1$ is odd; or in other words a factor of the form

$$(-1)^{k+1}$$

Ex: Use of the cofactor expansion: easily deal with rows containing many zeros.

$$\left.\begin{aligned} 10x_1 + kx_2 - x_3 &= 0 \\ kx_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \end{aligned}\right]$$ Determine k st we have unique solutions.

$$\begin{vmatrix} 10 & k & -1 \\ k & 1 & -1 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 10 & k & -1 \\ k-10 & 1-k & 0 \\ 2-30 & 1-3k & 0 \end{vmatrix}$$

$$= (-1)^{3+1} \begin{vmatrix} k-10 & 1-k \\ -28 & 1-3k \end{vmatrix}$$

$$= (k-10)(1-3k) + 28(1-k)$$

$$= -3k^2 + 30k + k - 10 - 28k + 28 = 0$$

$= -3k^2 + 3k + 18 = 0 \Rightarrow -3(k^2 - k - 6) = 0$

$\Rightarrow (k-3)(k+2) = 0 \quad k = 3$ or $k = -2$

$\Rightarrow \det(A) = 0$ So for $k \in \mathbb{R} \setminus \{3, 2\}$

we have a unique solution

Matrix Adjoints

$$A = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 5 & 4 \\ 3 & -2 & 0 \end{bmatrix}$$

M_C = Matrix of Cofactors

$$= \begin{bmatrix} C_{11} & C_{12} & \\ - & - & - \\ - & - & - \end{bmatrix} = \begin{bmatrix} -8 & 12 & -13 \\ 6 & 9 & 4 \\ 15 & -5 & 10 \end{bmatrix}$$

Adjoint = Transpose of cofactor matrix.

$$\operatorname{adj}(A) = \begin{bmatrix} -8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}$$

Theorem If $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$\det(A) = -8 \cdot 2 + (-3)(-13) = +39 - 16 = 23$

So $A^{-1} = \frac{1}{23}\begin{bmatrix} -8 & 6 & 15 \\ 12 & 9 & -5 \\ -13 & 4 & 10 \end{bmatrix}$

Proof: We need a couple of interesting results first

Notice that if I do $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$

$= (-8)(3) + 12 \cdot 2 = 0$ (weird huh?)

Theorem: $\sum a_{ik} C_{jk} = 0$ with $i \neq j$

Let B be the matrix obtained by

$R_j = R_i + R_j$

Then $\det(B) = \det(A)$ but $\det(B)$

$= \sum (a_{jk} + a_{ik}) C_{jk} = \sum a_{jk} C_{jk}$

$+ \sum a_{ik} C_{jk}$

$\det(A) = \det(A) + \sum a_{ik} C_{jk}$

Theorem: $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$A^{-1}A = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots \\ & & \\ & & \end{bmatrix}$

$\cdots$

$$(A^{-1}A)_{ij} = \frac{1}{\det(A)} \sum C_{ki}\, a_{kj} = \frac{\det(A)}{\det(A)} = 1 \text{ if } j = i$$

and $= 0$ if $j \neq i$

Of course for the purposes of explanation it's better to do

$$A A^{-1}$$

Cramer's Rule

Want to solve $Ax = b$.

$$\begin{aligned} 3x_1 + 2x_2 - 1 &= 4 \\ x_1 + x_2 - 5x_3 &= -3 \\ -2x_1 - x_2 - 4x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & -5 \\ -2 & -1 & -4 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 4 & 2 & -1 \\ -3 & 1 & -5 \\ 0 & 1 & -4 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & -5 \\ -2 & 0 & -4 \end{bmatrix} \qquad B_3 = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & -3 \\ -2 & -1 & 0 \end{bmatrix}$$

$$x_1 = \frac{\det(B_1)}{\det(A)} \qquad x_2 = \frac{\det(B_2)}{\det(A)} \qquad x_3 = \frac{\det(B_3)}{\det(A)}$$

This follows directly from

$$x = A^{-1} b \qquad x_k = \frac{1}{\det(A)} \left[\operatorname{adj}(A) b \right]_k$$

$$\left[\operatorname{adj}(A) b \right]_k = \sum_{j=1}^{n} \operatorname{adj}(A)_{kj} b_j$$

$$= \sum_{j=1}^{n} C_{jk} b_j = \det(B_k)$$