3.2 Lec 11

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \ddots & & \\ 0 & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Upper triangular matrix

$$\det(A) = \sum_{\sigma} a_{1\sigma_1} a_{2\sigma_2} \cdots a_{n\sigma_n} \quad \sigma \in S_n \text{ (perm).}$$

Thus if any row i has a term of the form

$a_{i\sigma_i}$ such that $\sigma_i < i$ (column less than row)

Then $a_{i\sigma_i} = 0$. So all of those terms vanish in the determinant.

So the only possibility is to have $\sigma_i \geq i$. Lets start $\sigma_n \geq n$. Since $\sigma_n \in \{1, \ldots, n\}$ we must have $\sigma_n = n$. Similarly $\sigma_{n-1} \geq n-1$ but n is taken by σ_n so $\sigma_{n-1} = n-1$. We proceed by induction to get

$\sigma_1 = 1, \sigma_2 = 2, \ldots, \sigma_n = n$. Therefore

$\det(A) = a_{11} a_{22} \cdots a_{nn}$ (product of terms along diag.)

<u>Ex:</u> $\begin{bmatrix} 2 & 5 & -1 & 3 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix} = A$ $\quad \det(A) = 2(-1)\,7\,5$

$= -70$

ERO and determinants

1) $A' \overset{P_{ij}}{\sim} A$ then $\det(A') = -\det(A)$

Permuting rows flips sign of determinant.

2) Multiply row by k $\quad R_j = k R_j$

$A \overset{R_j = kR_j}{\sim} A'$ $\quad \det(A') = k \det(A)$

3) $A \underset{R_j = R_j + kR_m}{\sim} A'$ add k times row$_j$ to row m.

$\det(A) = \det(A')$

$$B = \begin{bmatrix} -5 & 20 \\ 3 & -2 \end{bmatrix}$$

$$\det(B) = 5 \det \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} = 5 \left[2 - 12 \right] = -50$$

Ex:

$$\begin{bmatrix} 2 & -1 & 3 & 7 \\ 1 & -2 & 4 & 3 \\ 3 & 4 & 2 & -1 \\ 2 & -2 & 8 & -4 \end{bmatrix} \sim A$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & 4 & 3 \\ 3 & 4 & 2 & -1 \\ 2 & -2 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & -3 & 5 & -1 \\ 0 & 1 & 5 & -13 \\ 0 & -4 & 10 & -12 \end{bmatrix}$$

swap rows $\det(B) = -\det(A)$

$$B \sim \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 5 & -13 \\ 0 & -3 & 5 & -1 \\ 0 & -4 & 10 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 5 & -13 \\ 0 & 0 & 20 & -40 \\ 0 & 0 & 30 & -64 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 5 & -13 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 30 & -64 \end{bmatrix} \qquad \det(C) = \frac{1}{20}\det(B)$$

$$C \sim \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 5 & -13 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -64+60 \end{bmatrix}$$

$$\begin{array}{r} 2 \\ 19 \\ 3 \\ \hline 57 \end{array}$$

$$\det(C) = 1 \cdot 1 \cdot 1 \cdot \left(-64+60\right) = -4$$

$$\det(B) = -80, \quad \det(A) = 80$$

Recall: $Ax = b$ where $A_{n\times n}$ $b_{n\times 1}$

Then we have a unique solution iff

$$\operatorname{rank}(A) = n$$

Or $\operatorname{RREF}(A) = I$

In terms of determinants we have

$$\det(A) \neq 0 \Leftrightarrow \operatorname{rank}(A) = n$$

This is because the operations 1 and 2 in getting A to RREF only change A by multiplicative factors.

Thm $A_{n\times n}$ A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

Recall:

$$\left[\begin{array}{c|c} A & I \end{array}\right] \sim \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

A is invertible $\Leftrightarrow$ $RREF(A) = I$

$\Leftrightarrow \text{rank}(A) = n$

Cor: $Ax = 0$ has infinitely many solutions if $\det(A) = 0$ and has unique soln $x = 0$ if $\det(A) \neq 0$

$$A^{\#} = \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \end{array}\right] \Rightarrow \text{rank}(A) = \text{rank}(A^{\#})$$

$\Rightarrow$ ∞ly many solutions OR unique soln.

$\Updownarrow$

$\det(A) \neq 0$

So ∞ly many solutions

$\Leftrightarrow \det(A) = 0$

More properties:

1) $\det(A^T) = \det(A)$

2) $A = \begin{bmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}$ $B = \begin{bmatrix} a_1 \\ \vdots \\ b_i \\ \vdots \\ a_n \end{bmatrix}$ $C = \begin{bmatrix} a_1 \\ \vdots \\ c_i \\ \vdots \\ a_n \end{bmatrix}$

$a_i = b_i + c_i$

$\det(A) = \det(B) + \det(C)$

3) If A has a row (or column) of ZEROS

$\det(A) = 0$

4) If 2 rows of A are equal OR

2 columns " " " $\Rightarrow \det(A) = 0$

5) $\det(AB) = \det(A)\det(B)$

1) is important since anything true for rows must be true for columns.

2) $\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 & -3 & 1 \\ -2 & 4 & 6 & 2 \\ -3 & -6 & 9 & 3 \\ 2 & 11 & -6 & 4 \end{vmatrix} \underset{=(-3)}{C_3/(-3)} \begin{vmatrix} 1 & 2 & 1 & 1 \\ -2 & 4 & -2 & 2 \\ -3 & -6 & -3 & 3 \\ 2 & 11 & 2 & 4 \end{vmatrix} = 0$$

$C_1 = C_3$

Ex Find all solutions to

$$\begin{vmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0$$

$$\begin{matrix} x^2 & x & 1 & x^2 & x \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 4 & 2 \end{matrix}$$

LHS $x^2 + 4x + 2 - 4 - 2x^2 - x = -x^2 + 3x - 2 = 0$

$(x-2)(x-1) = 0 \Rightarrow x = 2, x = 1$

Another way

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0 \quad \text{when you set } x = 1$$

$$\begin{vmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0 \quad \text{when } x = 2$$

Let me prove two of these things

$$A = \begin{bmatrix} a_1 \\ \vdots \\ b_i + c_i \\ \vdots \\ a_n \end{bmatrix} \qquad \det(A) = \det(B) + \det(C)$$

$$\det(A) = \sum_{\sigma} a_{1\sigma_1} \cdots a_{i\sigma_i} \cdots a_{n\sigma_n} \, P(\sigma)$$

$$= \sum_{\sigma} a_{1\sigma_1} \cdots (b_{i\sigma_i} + c_{i\sigma_i}) \cdots a_{n\sigma_n} \, P(\sigma)$$

$$= \sum_{\sigma} a_{1\sigma_1} \cdots b_{i\sigma_i} \cdots a_{n\sigma_n} P(\sigma) + \sum a_{1\sigma_1} \cdots c_{i\sigma_i} \cdots a_{n\sigma_n} P(\sigma)$$

$= \det(B) + \det(C)$

$\det(A) = \det(A^T)$

$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $\det(A) = \sum_{\sigma} a_{1\sigma_1} \cdots a_{n\sigma_n} P(\sigma)$

$(A^T)_{ij} = A_{ji}$

$A^T = \begin{bmatrix} a_1^T \cdots & a_n^T \end{bmatrix}$ $\det(A^T) = \sum_{\sigma} a'_{1\sigma_1} \cdots a'_{n\sigma_n} P(\sigma)$

$= \sum_{\sigma} a_{\sigma_1 1} \cdots a_{\sigma_n n} P(\sigma) = \sum_{\sigma} a_{1q_1} \cdots a_{nq_n} P(q)$

By rearranging terms. So only thing to show

is $P(\sigma) = P(q)$

\# of interchanges to go from $(1, 2, \cdots, n)$

to $(\sigma_1 \cdots \sigma_n) = N$

By inverting the flips you go from $(\sigma_1 \cdots \sigma_n)$

to $(1, 2 \cdots, n)$

If you apply these flips to $(1, 2, \ldots n)$ you go to $(q_1, \ldots q_n)$ Thus $P(\sigma) = P(q)$

Anyway this is quite a complicated proof that is worth skipping

Section 3.3 Cofactor Expansion.

Another way to evaluate determinants.

Take $A_{n\times n}$ matrix. M_{ij} is the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 3 & 1 & 5 \end{bmatrix} \qquad M_{11} = \begin{vmatrix} 4 & -2 \\ 1 & 5 \end{vmatrix} = 20+2 = 22$$

$$M_{12} = \begin{vmatrix} -1 & -2 \\ 3 & 5 \end{vmatrix} = -5+6 = 1 \qquad M_{13} = \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} = -1-12$$

$$= -13$$

$$M_{23} = \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = -4+3 = -1$$

Cofactors: $C_{ij} = (-1)^{i+j} M_{ij}$

$= (\text{sign})(\text{minor})$

Signs take the form

$$\begin{vmatrix} + & - & + & - & + & - \\ + & - & + & - & + & - \\ \cdot & - & - & - & & \\ \cdot & - & - & - & & \end{vmatrix}$$

Find the cofactors of M_{11}, M_{23} and M_{33} in the previous matrix.

$$C_{11} = (-1)^{1+1} M_{11} \qquad C_{23} = (-1)^{2+3} M_{23} \qquad C_{33} = (-1)^{3+3} M_{33}$$

$$= M_{11} \qquad\qquad = -M_{23} \qquad\qquad = M_{33}$$

<u>Theorem</u>: Cofactor expansion.

$$\det(A) = a_{i1} C_{i1} + a_{i2} + \cdots \quad a_{in}$$

for any row i

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 3 & 1 & 5 \end{bmatrix} \qquad \det(A) = 2C_{11} + 1C_{12} + 3C_{13}$$

$$M_{11} = \begin{vmatrix} 4 & -2 \\ 1 & 5 \end{vmatrix} = 20+2 = 22 \qquad M_{12} = \begin{vmatrix} -1 & -2 \\ 3 & 5 \end{vmatrix} = -5+6 = 1$$

$$M_{13} = \begin{vmatrix} -1 & 4 \\ 3 & 1 \end{vmatrix} = -1-12 = -13$$

$$\det(A) = 2 \cdot 22 - 1 + 3 \cdot (-13) = 44-39-1 = 4$$

$$M_{22} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10-9 = 1 \qquad M_{21} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2$$

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

$$\det(A) = 1 \cdot 2 + 4 \cdot 1 + (-2)(-1)(-1)$$

$$= 6 - 2 = 4$$