# Lec 09 2.6 Inverse of a matrix

Real number: $5 \cdot 1 = 1 \cdot 5 = 5$

$1 \equiv$ identity element

$5 \cdot x = 1 \quad x = \frac{1}{5}$ (inverse element)

In general $a \in \mathbb{R} \; \exists \; a^{-1}$ st

$a \cdot a^{-1} = 1$ if $a \neq 0$.

Similarly for matrices: $\exists \; I$ st

$A \cdot I = A_{n \times n}$ and $I \cdot A = A$

So ~~does~~ there exist an ~~inverse~~ B ?

st $AB = I$

and C

$CA = I$

Is $B = C$ if it exists?

**Example:** $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} -1+2 & -1+1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = I$

$BA = \begin{bmatrix} -2+3 & 3-3 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = I$

**Theorem** If $\exists$ $B, C$ st $(n \times n)$

$$A \; B = I_n$$

$$C \; A = I_n$$

then $C = B$

**Pf:**

$$AB = CA = I$$
$$CAB = C\,CA$$
$$I\,B = C\,I \implies B = C$$

Algebra is full of weird computations like this.

**Inverse:** If $\exists$ $A^{-1}$ matrix st

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

then $A^{-1}$ is called the INVERSE of the matrix $A$.

Its worth talking about the NOTATION $A^{-1}$

and how its MERELY INSPIRED BY $a^{-1}$.

**Singular / Non singular:** If $A^{-1}$ exists then

$A$ is called NONSINGULAR

**Theorem** Consider $Ax = b$ st $A^{-1}$ exists.

$$A^{-1}Ax = A^{-1}b \implies x = A^{-1}b$$

This is the UNIQUE solution

OK Now want to answer:

When does $A^{-1}$ exist?

$$\overset{A}{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}} \begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix} = \overset{b}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$$

$$\|$$

$$\begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2} = \begin{bmatrix} A\begin{bmatrix}x_{11}\\ x_{21}\end{bmatrix} & A\begin{bmatrix}x_{12}\\ x_{22}\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix} x_{11} + 2x_{22} & x_{12} + 2x_{22}\\ 3x_{11} + 4x_{21} & 3x_{12} + 4x_{22}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

OK So we need to solve

$$A\begin{bmatrix}x_{11}\\ x_{21}\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

$$A\begin{bmatrix}x_{12}\\ x_{22}\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1\\ 3 & 4 & 0\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 0\\ 3 & 4 & 1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_{1\cancel{2}} \\ x_{21} \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ -1/2 \end{bmatrix}$$

$$= A^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & -1/2 \end{bmatrix}$$

Notice the following:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & -1/2 \end{bmatrix}$$

In general

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \sim \left[\begin{array}{c|c} RREF(A) & B \end{array}\right]$$

If $RREF(A) = I_n$ then we can solve for each column, and $B$ will be $A^{-1}$.

$RREF(A) = I_n \iff rank(A) = n$

Note that $A^{\#} = \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$

So if $rank(A) < n$ then for at least one of the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix}$, and so on we will have $rank(A^{\#}) > rank(A)$

So we will not be able to determine columns

$[x$

<u>Theorem</u>: $A$ is invertible iff $rank(A) = n$

Ex:

$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

Solve $Ax = b$ after finding the inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 5 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & -11 & 0 & -5 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -14 & -3 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{14} & \frac{2}{14} & -\frac{1}{14} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 15/14 & -16/14 & 1/14 \\ 0 & 1 & 0 & -6/14 & 10/14 & 2/14 \\ 0 & 0 & 1 & 3/14 & 2/14 & -1/14 \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad}_{A^{-1}}$$

Does this satisfy $\underbrace{A^{-1}}_{B} A = I_n$

We need $BA = I \implies ABA = A$

$A(BA - I) = 0$

$BA - I = \left[\vec{y}_1 \ \vec{y}_2 \ \cdots \ \vec{y}_n\right]$. want to solve

column vectors.

$A\vec{y}_i = 0$ since $\text{rank}(A) = n$ the RREF of

$[A \mid 0]$ is $[I \mid 0] \implies \vec{y} = 0$ is the unique

solution

## Properties of the Inverse

Let $A, B$ be nonsingular / invertible.

1) $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$ (involution)

2) $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

3) $A^{T}$ is invertible and $(A^{T})^{-1} = (A^{-1})^{T}$

Pf: $A^{-1}A = I \Rightarrow A^{-1}AB = B$

$\Rightarrow B^{-1}A^{-1}AB = B \Rightarrow B^{-1}A^{-1} = (AB)^{-1}$

$(A^{-1}A)^{T} = A^{T}(A^{-1})^{T} = I_n \Rightarrow (A^{-1})^{T} = (A^{T})^{-1}$

using $(AB)^{T} = B^{T}A^{T}$

How would you do $(ABC)^{-1}$ in terms of $A^{-1}, B^{-1}, C^{-1}$