2.5 Gauss and gauss Jordan elimination.

$$4x_1 - 3x_2 + 6x_3 = 2$$

$$x_1 - 3x_2 + 6x_3 = 5$$

$$-2x_1 + 3x_2 - 8x_3 = -6$$

$$\begin{bmatrix} 4 & -3 & 6 & 2 \\ 1 & -3 & 6 & 5 \\ -2 & 3 & -8 & -6 \end{bmatrix} \underset{P_{12}}{\sim} \begin{bmatrix} 1 & -3 & 6 & 5 \\ 4 & -3 & 6 & 2 \\ -2 & 3 & -8 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 9 & -18 & -18 \\ 0 & 3 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 3 & 4 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 10 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{REF}$$

$x_3 = 1$, $\quad x_2 - 2x_3 = -2 \Rightarrow x_2 = 0$

$x_1 - 3x_2 + 6x_3 = 5 \quad x_1 = -1 \qquad (-1, 0, 1)$ solution

lets try this in RREF.

$$\begin{bmatrix} 1 & -3 & 6 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x_3 = 1 \quad x_2 = 0 \quad x_1 = 1$

REF to solve problem: Gaussian elimination.

RREF to solve problem: Gauss-Jordan eliminatin

Compare A and $A^{\#}$ (the augmented matrix)

$$A = \begin{bmatrix} 4 & -3 & 6 \\ 1 & -3 & 6 \\ -2 & 3 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ rank $(A) = 3$ and by previous rank $(A^{\#}) = 3$

<u>Lemma</u> If rank $(A) =$ rank $(A^{\#}) =$ # of unknowns in the system, then UNIQUE solution

<u>Pf:</u> Is easy to see that the back substitution works here.

Rank (A) cannot be ~~larger~~ than

what happens if:

1) rank $(A) <$ rank $(A^{\#})$

2) ~~rank (A)~~ $=$ rank $(A^{\#}) <$ # of unknowns.

Ex:

$$x_1 + x_2 - x_3 + x_4 = 1$$
$$2x_1 + 3x_2 + x_3 = 4$$
$$3x_1 + 5x_2 + 3x_3 - x_4 = 5$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 2 & 6 & -4 & 2 \end{bmatrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

$\text{rank}(A) = 2$ $\qquad$ $\text{rank}(A^\#) = 3$

The last equation is

$$0x_1 + 0x_2 + \cdots + 0x_3 = -2$$ which is not

possible.

Lemma

So if $\text{rank}(A) < \text{rank}(A^\#)$ then INCONSISTENT.

Ex

$$\begin{bmatrix} 5 & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 9 & -9 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$ $\quad 2 = \text{rank}(A) = \text{rank}(A^\#) < 3$

$$\sim \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 3x_2 + 2x_3 = -1$$
$$x_2 - x_3 = 1$$

Set $x_3 = t$ then $x_2 = t+1$

$$x_1 - 3(t+1) + 2t = -1 \Rightarrow x_1 = -1 + 3t + 3 - 2t$$
$$= t+2$$

solution $S : \{ (t+2, t+1, t) : t \in \mathbb{R} \}$

We can write this as

$$S = \{ (2,1,0) + t(1,1,1) : t \in \mathbb{R} \}$$
$$S = \{ \vec{a} + \vec{b}t ; t \in \mathbb{R} \}$$

$t \equiv$ free variable.

Lemma: Let n be # of unknowns.
In general if rank $(A) =$ rank $(A^{\#}) = r^{\#}$

the # of free variables $= n - r^{\#}$.

DO NOT
Choose as free variables those where you have ones on the diagonal in REF.

Theorem: $Ax = b$ $A_{m \times n}$ $b_{m \times 1}$ $n = $ # of unkn.

$r =$ rank (A) $r^{\#} =$ rank $(A^{\#})$

1) $r = r^{\#} = n$ Unique soln
2) $r = r^{\#} < n$ ∞ly many soln.
3) $r < r^{\#}$ Inconsistent.

Homo. eqns

If RHS $\vec{b} = 0$.

$Ax = 0$ $\qquad \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right] = A^\#$

Lemma: $x = 0$ is always a solution.

$\text{rank}(A) = \text{rank}(A^\#)$

$0 \equiv$ trivial solution.

Ex: $A = \begin{vmatrix} 0 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 3 & 7 \end{vmatrix}$ $\qquad A\cancel{x} = 0$

$$\left[\begin{array}{cccc} 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \equiv \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$$

$\text{rank}(A) = \text{rank}(A^\#) = 2 < 3$

$\Rightarrow$ 1 free variable

So $x_1 = t$, free variable

$S = \{ (t, 0, 0) : t \in \mathbb{R} \}$

Complex # example: Ex 2.5.13